

## A Connected Lie Group Equals the Square of the Exponential Image

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**Abstract.** It is shown that in every connected real Lie group every element is the product of two elements in the exponential image.

More than one hundred years ago, F. ENGEL and E. STUDY considered the question of the surjectivity of the exponential function of Lie groups ([2],[3], compare also the remarks in [14]). Interestingly, a general solution of this problem is not found yet though there exist equivalent criteria for the surjectivity of special classes of Lie groups (compare [12]). M. MOSKOWITZ and R. SACKSTEDER first formulated the fact that every real connected Lie group is equal to  $(\exp \mathfrak{g})^2$  ([7]) and provided a proof in the final version of their paper using control theory like methods. Using partly well-known decomposition results on Lie groups, we shall provide a short proof for this result.

In the whole article,  $G$  denotes a real connected Lie group. A subgroup  $K \subseteq G$  is called *compactly embedded* if  $\text{Ad}(K)$  is relatively compact in  $\text{GL}(\mathfrak{g})$ . It is shown in [10] that there exist maximal compactly embedded subgroups and that all maximal compactly embedded subgroups are closed and connected. The last property ensures that all maximal compactly embedded subgroups are conjugate to each other. Moreover, it is well-known that every compactly embedded connected subgroup  $K$  has surjective exponential function: Indeed,  $K = K'Z_0$ , where  $Z_0$  denotes the identity component of the center. Moreover,  $K'$  is compact, hence has surjective exponential function. This implies the surjectivity of the exponential function of  $K$ .

First, let us assume that  $G$  is semisimple. The Iwasawa decomposition says that every semisimple Lie group can be written as  $KAM$  where  $K$  is maximal compactly embedded,  $A$  is abelian connected and consists of  $\text{Ad}$ -semisimple elements with  $\text{spec ad}(a) \subseteq \mathbb{R}$  for every  $a \in \mathfrak{a}$ ,  $M$  is nilpotent connected, and  $AM$  is solvable ([4]). By DIXMIER's and SAITO's Theorem ([1],[9]),  $S := AM = \exp(\mathfrak{a} + \mathfrak{m})$ . Thus,  $G = \exp \mathfrak{k} \exp \mathfrak{s}$ .

Now we recall the Levi decomposition: If  $G$  is a real connected Lie group, then for every maximal semisimple subgroup (called *Levi factor*)  $L$  we have  $G = L \text{Rad}(G)$  and  $L \cap \text{Rad}(G)$  discrete. All Levi factors are conjugate to each other.

This follows directly from the Levi decomposition of real finite-dimensional Lie algebras (see e.g. Korollar II.4.8 of [5]).

A *Cartan subalgebra*  $\mathfrak{h} \subset \mathfrak{g}$  of a real finite dimensional Lie algebra  $\mathfrak{g}$  is a nilpotent subalgebra which equals its own normalizer  $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$ . Cartan subgroups are defined in various ways which in case of connected Lie groups are all equivalent (compare [8]). We will use the following definition:  $H \subseteq G$  is a Cartan subgroup if  $\mathfrak{h} := \mathbf{L}(H)$  is a Cartan subalgebra and for all  $g \in H$  and all  $x \in \mathfrak{h}$  we have  $\text{Ad}(g)\text{ad}(x)_s = \text{ad}(x)_s \text{Ad}(g)$  where  $\text{ad}(x)_s$  is the semisimple part of the Jordan decomposition of  $\text{ad}(x)$ .

Theorem 1.9 (ii) of [13] states that for all Cartan subgroups  $H$  the intersection  $H \cap \text{Rad}(G)$  is connected. Since  $H \cap \text{Rad}(G)$  is nilpotent, we have  $H \cap \text{Rad}(G) = \exp(\mathfrak{h} \cap \text{rad}(\mathfrak{g}))$ . Theorem 1.11 of [13] says that for each Cartan subgroup  $H$  of  $G$  there exists a Levi factor  $L$  such that  $H \cap L$  is a Cartan subgroup of  $L$ ,  $H = (H \cap L)(H \cap \text{Rad}(G))$ , and  $H \cap \text{Rad}(G) \subseteq Z_G(L)$ . This leads to the following:

**Lemma 1.** *If  $\mathfrak{h}$  is a Cartan subalgebra of a real finite-dimensional Lie algebra  $\mathfrak{g}$  and  $\mathfrak{n}$  the nilradical of  $\mathfrak{g}$ , then  $\text{rad}(\mathfrak{g}) = (\mathfrak{h} \cap \text{rad}(\mathfrak{g})) + \mathfrak{n}$ . If  $H$  is a Cartan subgroup of a real Lie group  $G$  and  $N$  the nilradical, then  $\text{Rad}(G) = (H \cap \text{Rad}(G))N$ .*

**Proof.** We choose  $\mathfrak{l}$  such that  $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{l}) \oplus (\mathfrak{h} \cap \text{rad}(\mathfrak{g}))$  (compare Theorem 1.8 of [13]). We observe that  $\mathfrak{g}' \subseteq \mathfrak{l} + \mathfrak{n}$ . Moreover,  $\mathfrak{g} = \mathfrak{h} + \mathfrak{g}' = (\mathfrak{h} \cap \text{rad}(\mathfrak{g})) + \mathfrak{n} + \mathfrak{l}$ , hence  $\text{rad}(\mathfrak{g}) = (\mathfrak{h} \cap \text{rad}(\mathfrak{g})) + \mathfrak{n}$ . The proof for Lie groups works similarly. ■

Thus, with the above notation we obtain  $G = KS(H \cap \text{Rad}(G))N = K(H \cap \text{Rad}(G))SN = \exp \mathfrak{k} \exp(\mathfrak{h} \cap \text{rad}(\mathfrak{g})) \exp \mathfrak{s} \exp \mathfrak{n}$ . Since  $[\mathfrak{k}, \mathfrak{h} \cap \text{rad}(\mathfrak{g})] = \{0\}$ , we get  $\exp \mathfrak{k} \exp(\mathfrak{h} \cap \text{rad}(\mathfrak{g})) = \exp(\mathfrak{k} + (\mathfrak{h} \cap \text{rad}(\mathfrak{g})))$ . Moreover,  $SN$  is a solvable connected Lie group and  $\text{spec ad}_{\mathfrak{g}}(s) \subseteq \mathbb{R}$  for each  $s \in \mathfrak{s}$  by Proposition 3.7 of [11]. Again by Dixmier's and Saito's Theorem we get  $SN = \exp(\mathfrak{s} + \mathfrak{n})$  because by Proposition 3.7 of [11], every element  $x \in \mathfrak{s} + \mathfrak{n}$  satisfies  $\text{spec ad}(x) \subseteq \mathbb{R}$ .

**Definition 2.** An element  $x$  of a real finite-dimensional Lie algebra is called *exp-regular* if  $\text{spec ad } x \cap 2\pi i\mathbb{Z} = \{0\}$ . The set of all exp-regular elements of  $\mathfrak{g}$  is denoted by  $\text{reg exp}$ .

Taken together, we have proved the following:

**Theorem 3.** *If  $G$  is a real connected Lie group, then  $G = (\text{exp } \mathfrak{g})^2$ . Moreover, for every element  $g \in G$  there is an exp-regular element  $x$  and an element  $y$  in  $\mathfrak{g}$  with  $g = \exp x \exp y$ .*

**Theorem 4.** *If  $G$  is a connected real Lie group and for every  $g \in G$  there is an exp-regular  $x \in \mathfrak{g}$  and a  $y \in \mathfrak{g}$  with  $g = \exp x \exp y$ , then there are exp-regular  $u, w \in \mathfrak{g}$  with  $g = \exp u \exp w$ .*

**Proof.** Since  $\text{reg exp}$  is dense in  $\mathfrak{g}$  (for example by Lemma 2 of [6]), the image  $\exp(\text{reg exp})$  is dense in  $\exp \mathfrak{g}$ . Since  $\exp$  is regular at each  $x \in \text{reg exp}$ , the set  $\exp(\text{reg exp})$  is open in  $G$ , hence also in  $\exp \mathfrak{g}$ . The openness of  $\exp(\text{reg exp})$  implies that there is a symmetric 1-neighborhood  $U$  such that  $\exp x \cdot U \subseteq \exp(\text{reg exp})$ . The density implies that  $\exp y \cdot U \cap \exp \text{reg exp} \neq \emptyset$ . So there is a  $u \in U$  such that  $(\exp x)u^{-1}$  and  $u \exp y$  are in  $\exp(\text{reg exp})$ . This implies the assertion. ■

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