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Embeddings of almost homogeneous Heisenberg groups

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Abstract. We study injective continuous homomorphisms between simply connected Lie groups that are nilpotent of class 2 and whose group of automorphisms acts with 3 orbits.

1. Introduction

A topological group G is called *almost homogeneous* if the group $\mathrm{Aut}(G)$ of all topological automorphisms has 3 orbits on G. In the general case, there are different constructions leading to almost homogeneous groups (including $\mathrm{Sym}(3)$, $\mathrm{Alt}(4)$, additive groups of free modules over the ring $\mathbb{Z}/p^2\mathbb{Z}$ for a prime p, Suzuki 2-groups, semidirect products $Q \ltimes V$ with respect to a suitable linear representation of a cyclic group Q on a vector space V, and certain generalized Heisenberg groups), compare [2].

In [5], it has been shown that every almost homogeneous locally compact connected group is a simply connected Lie group which is nilpotent of class 2: that is, a *Heisenberg group.* The Heisenberg groups are obtained by the following

Construction. Let V and Z be vector spaces of finite dimension over $\mathbb{R}$, and let $\beta: V \times V \to Z$ be a symplectic map (that is, a skew-symmetric bilinear map). Then the vector space $V \times Z$ becomes a Lie algebra $\mathfrak{gh}\,(V, Z, \beta)$ with Lie bracket[†] $[(v,x),(w,y)] = \big(0,(v,w)^\beta\big)$. The corresponding simply connected Lie group $\mathrm{GH}\,(V,Z,\beta)$ is then the space $V \times Z$ with (Campbell–Hausdorff) multiplication $(v,x) * (w,y) = \big(v+w, x+y+\frac{1}{2}(v,w)^\beta\big)$. If $H := \mathrm{GH}\,(V,Z,\beta)$ is not abelian then the set

$$C := \left\{ \big(0,(v,w)^\beta\big) \;\middle|\; v,w \in V \right\}$$

[†] Maps will be applied from the right, and we use exponential notation. Linear maps, however, will be denoted by juxtaposition; this should remind of multiplication of row vectors by matrices.

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of commutators is a nontrivial characteristic subset, and it is contained in the center of H which is a proper characteristic subset of H. In an almost homogeneous group H, we thus have that C and the center of H coincide; this entails $C = \{0\} \times Z$. Note also that β is nondegenerate if $\mathrm{GH}(V, Z, \beta)$ is almost homogeneous.

2. Classification of Almost Homogeneous Heisenberg Groups

The almost homogeneous ones among the Heisenberg groups have been determined in [4]. They are characterized by the following result, see [4] 2.2 and 4.5.

Proposition 2.1. *Let $\beta: V \times V \to Z$ be a nonzero symplectic map. The Heisenberg group $\mathrm{GH}(V, Z, \beta)$ is almost homogeneous if, and only if, there is a compact group Φ inducing groups of orthogonal transformations on V and on Z that act transitively on the respective spheres, and such that the equation $(v\varphi, w\varphi)^\beta = (v, w)^\beta \varphi$ holds for all $v, w \in V$ and all $\varphi \in \Phi$.* ■

Transitive actions on spheres are well understood, compare [3] 96.16–24. The symplectic map β is somewhat hard to deal with. Standard methods of multi-linear algebra lead us to consider the exterior product $V \wedge V$ (that is, the space of skew-symmetric tensors) and the linearization $\bar\beta: V \wedge V \to Z$ of β. The action of Φ on V then yields an action of Φ on $V \wedge V$, and the map $\bar\beta$ is a homomorphism of $\mathbb{R}[\Phi]$-modules. The $\mathbb{R}[\Phi]$-modules V and Z are both simple because of the transitive action of Φ on the respective sphere. As an almost homogeneous Heisenberg group $\mathrm{GH}(V, Z, \beta)$ is generated by $V \times \{0\}$, we may (and will) assume that Φ acts effectively on V; the reader should be aware, however, that this does not imply that Φ acts effectively on $V \wedge V$. As Φ is compact, the $\mathbb{R}[\Phi]$-module $V \wedge V$ is semi-simple, and the surjective homomorphism $\bar\beta$ may be considered as projection onto some suitable simple submodule of $V \wedge V$. We will freely use the representation theory of compact groups, cf. [1].

The $\mathbb{R}[\Phi]$-module $V \wedge V$ has a convenient model: we identify $V \wedge V$ with the Lie algebra $\mathfrak{o}_d$ of all skew-symmetric $d \times d$ matrices over $\mathbb{R}$, where $d = \dim V$. The universal bilinear map from $V \times V$ to $V \wedge V$ is then given by mapping (v, w) to the matrix $v^{\mathrm{tr}}w - w^{\mathrm{tr}}v$, where we consider the elements of V as row vectors, and v^{tr} denotes the transpose of v. The action of the group Φ on $V \wedge V$ is obtained by identifying Φ with a subgroup of the orthogonal group $\mathrm{O}_d\mathbb{R}$ and restricting the adjoint action of $\mathrm{O}_d\mathbb{R}$ on its Lie algebra $\mathfrak{o}_d$: that is, a matrix $A \in \Phi$ maps $X \in \mathfrak{o}_d$ to $A^{\mathrm{tr}}XA = A^{-1}XA$. This interpretation makes it easy to find certain submodules of the $\mathbb{R}[\Phi]$-module $\mathfrak{o}_d$: for instance, we have the Lie algebra $\mathfrak{f}$ of Φ, and its centralizer (which is a submodule on which Φ acts trivially).

The results of [4] amount to a complete classification of almost homogeneous Heisenberg groups. There are three infinite series:

Examples 2.2. First of all, one has nondegenerate symplectic $\mathbb{F}$-bilinear forms $\beta^n_{\mathbb{F}}: \mathbb{F}^{2n} \times \mathbb{F}^{2n} \to \mathbb{F}$, for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and each positive integer n. This leads to almost homogeneous Heisenberg groups $\mathrm{H}_{\mathbb{R}}^{2n} := \mathrm{GH}\left(\mathbb{R}^{2n}, \mathbb{R}, \beta_{\mathbb{R}}^{n}\right)$ and $\mathrm{H}_{\mathbb{C}}^{4n} := \mathrm{GH}\left(\mathbb{C}^{2n}, \mathbb{C}, \beta_{\mathbb{C}}^{n}\right)$, respectively. The skewfield $\mathbb{H}$ of Hamilton's quaternions does not admit nonzero $\mathbb{H}$-*bi*linear forms. However, each anisotropic hermitian form $\gamma_{\mathbb{H}}^{n}\colon \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H}$ leads to a symplectic $\mathbb{R}$-bilinear map $\beta_{\mathbb{H}}^{n}\colon \mathbb{H}^n \times \mathbb{H}^n \to \mathrm{Pu}(\mathbb{H})$ via $(v,w)^{\beta_{\mathbb{H}}^{n}} := \mathrm{Pu}\left((v,w)^{\gamma_{\mathbb{H}}^{n}}\right)$. Thus we obtain another infinite series of almost homogeneous Heisenberg groups $\mathrm{H}_{\mathbb{H}}^{4n} := \mathrm{GH}\left(\mathbb{H}^n, \mathrm{Pu}(\mathbb{H}), \beta_{\mathbb{H}}^{n}\right)$.

The proof that these series consist of almost homogeneous groups uses the fact that, for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, the symplectic group $\mathrm{Sp}_{2n}\mathbb{F}$ contains a compact subgroup $\mathrm{U}_n\mathbb{K}$ (where $\mathbb{K}$ is the skewfield extension of degree 2 over $\mathbb{F}$) acting transitively on the unit sphere of $\mathbb{F}^{2n}$, and transitivity of $\mathrm{U}_n\mathbb{H}$ on the unit sphere in $\mathbb{H}^n$. It remains to enlarge these compact groups in order to obtain transitivity on the spheres in $\mathbb{R}$, $\mathbb{C}$, and $\mathrm{Pu}(\mathbb{H})$, respectively.

In the case of $\mathrm{H}_{\mathbb{R}}^{2n}$, one may take a finite extension of $\mathrm{U}_n\mathbb{C}$ for the group Φ. For $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}\}$ one uses $\Phi = \mathbb{S}_{\mathbb{F}} \cdot \mathrm{U}_n\mathbb{H}$ to show that $\mathrm{H}_{\mathbb{F}}^{4n}$ is almost homogeneous; here $\mathbb{S}_{\mathbb{F}}$ is the group consisting of scalar multiplications by elements of the unit sphere in $\mathbb{F}$.

Apart from these infinite series, there are 6 *exceptional* almost homogeneous Heisenberg groups. The group H_z^d is obtained using a subgroup Φ of $\mathrm{SO}_d\mathbb{R}$, the $\mathbb{R}[\Phi]$-module $\mathfrak{o}_d$, and the projection onto some simple submodule Z of dimension z in $\mathfrak{o}_d$. More specifically:

Examples 2.3: Exceptional almost homogeneous Heisenberg groups.

- The group H_3^3 is obtained from $\Phi = \mathrm{SO}_3\mathbb{R}$, with $Z = \mathfrak{o}_3$.
- The group H_6^6 is obtained from $\Phi = \mathrm{SU}_3\mathbb{C}$, with Z the complement to the Lie algebra of $\mathrm{U}_3\mathbb{C}$ in the $\mathbb{R}[\mathrm{SU}_3\mathbb{C}]$-module $\mathfrak{o}_6$.
- The group H_7^7 is obtained from $\Phi = \mathrm{G}_2$, with Z the complement of the Lie algebra of G_2 in the $\mathbb{R}[\mathrm{G}_2]$-module $\mathfrak{o}_7$.
- The group H_5^8 is obtained from $\Phi = \mathrm{U}_2\mathbb{H}$, with Z one of the five-dimensional submodules in the complement of the sum of the Lie algebra of $\mathrm{U}_2\mathbb{H}$ and its 3-dimensional centralizer in the $\mathbb{R}[\mathrm{U}_2\mathbb{H}]$-module $\mathfrak{o}_8$.
- The group H_6^8 is obtained from $\Phi = \mathrm{SU}_4\mathbb{C}$, with Z one of the simple 6-dimensional submodules of the 12-dimensional complement of the Lie algebra of $\mathrm{U}_4\mathbb{C}$ in the $\mathbb{R}[\mathrm{SU}_4\mathbb{C}]$-module $\mathfrak{o}_8$.
- The group H_7^8 is obtained from $\Phi = \mathrm{Spin}_7$, with Z the complement of the Lie algebra of Spin_7 in the $\mathbb{R}[\mathrm{Spin}_7]$-module $\mathfrak{o}_8$.

For details, see the discussion in [4]. The main result of that paper is the following (see [4] 7.5):

Theorem 2.4. *Let $H = \mathrm{GH}(V, Z, \beta)$ be an almost homogeneous real Heisenberg group. Then H is isomorphic to an element of the series $\mathrm{H}_{\mathbb{R}}^{2n}$, $\mathrm{H}_{\mathbb{C}}^{4n}$, or $\mathrm{H}_{\mathbb{H}}^{4n}$, where n runs over the positive integers; or H is isomorphic to one of the groups H_3^3, H_6^6, H_7^7, H_5^8, H_6^8, and H_7^8.* ■

As we are interested in embeddings, the following alternative descriptions of the groups H_3^3, H_6^6, and H_7^7 are helpful. For $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, let V be a 3-dimensional vector space over $\mathbb{F}$. Then the exterior product $V \wedge V$, formed over $\mathbb{F}$, is a 3-dimensional vector space over $\mathbb{F}$ which may be identified with the orthogonal Lie algebra $\mathfrak{o}_3\mathbb{F}$. Mapping (v,w) to $v \wedge w$ (that is, the commutator map in $\mathfrak{o}_3\mathbb{F}$) yields a symplectic map $\wedge: V \times V \to V \wedge V$ and an almost homogeneous Heisenberg group $\mathrm{S}_\mathbb{F} := \mathrm{GH}(V, V \wedge V, \wedge)$. A second method to construct almost homogeneous Heisenberg groups starts with a noncommutative alternative division algebra $\mathbb{F}$ over $\mathbb{R}$ (that is, with $\mathbb{F} \in \{\mathbb{H}, \mathbb{O}\}$). The commutator map induces a symplectic map from $\mathrm{Pu}(\mathbb{F}) \times \mathrm{Pu}(\mathbb{F})$ onto $\mathrm{Pu}(\mathbb{F})$, leading to almost homogeneous Heisenberg groups $\mathrm{C}_\mathbb{H}$ and $\mathrm{C}_\mathbb{O}$, respectively. The classification 2.4 yields isomorphisms $\mathrm{H}_3^3 \cong \mathrm{S}_\mathbb{R} \cong \mathrm{C}_\mathbb{H}$, $\mathrm{H}_6^6 \cong \mathrm{S}_\mathbb{C}$, and $\mathrm{H}_7^7 \cong \mathrm{C}_\mathbb{O}$.

3. Embeddings

We already noticed the isomorphisms $\mathrm{S}_\mathbb{R} \cong \mathrm{C}_\mathbb{H} \cong \mathrm{H}_3^3$ and $\mathrm{C}_\mathbb{O} \cong \mathrm{H}_7^7$. There are obvious inclusions $\mathrm{H}_\mathbb{R}^{2n} \le \mathrm{H}_\mathbb{C}^{4n}$, for each natural number n. Restricting ourselves to the subspace $\mathbb{C}^n$ of $\mathbb{H}^n$, we also find inclusions of $\mathrm{H}_\mathbb{R}^{2n}$ in $\mathrm{H}_\mathbb{H}^{4n}$; note that the set of commutators of elements from $\mathbb{C}^n \times \mathbb{C}^n$ is a vector space of dimension 1 since we take the pure part of a complex number. Moreover, we have an inclusion $\mathrm{H}_3^3 \cong \mathrm{S}_\mathbb{R} \le \mathrm{S}_\mathbb{C} \cong \mathrm{H}_6^6$, and an inclusion $\mathrm{H}_3^3 \cong \mathrm{C}_\mathbb{H} \le \mathrm{C}_\mathbb{O} \cong \mathrm{H}_7^7$. It appears reasonable to ask for other embeddings.

In general, an injective continuous group homomorphism need not be an embedding (that is, a homeomorphism onto its image). In the present situation, we are dealing with connected, simply connected, nilpotent Lie groups, where this problem does not occur: every injective continuous homomorphism between such groups is an embedding. However, we will not use that fact in the sequel, because we reduce our problems to $\mathbb{R}$-linear maps. Recall that continuous group homomorphisms between vector groups (that is, additive groups of vector spaces of finite dimension over $\mathbb{R}$) are $\mathbb{R}$-linear. Thus it makes sense to consider dimensions of the vector groups H' and H/H', for any Heisenberg group H.

There is one Heisenberg group that appears to be omni-present:

Proposition 3.1. *Let $G = \mathrm{GH}(V, Z, \beta)$ be any noncommutative Heisenberg group. Then there is an embedding of $\mathrm{H}_\mathbb{R}^2$ into G.*

Proof. We pick elements $x, y \in G$ such that $z := [x,y] \ne 1$. Putting $S := \mathbb{R}x + \mathbb{R}y$ and $Y := \mathbb{R}z$ we obtain the subgroup $\mathrm{GH}(S, Y, \beta|_{S\times S}) \cong \mathrm{H}_\mathbb{R}^2$ in G. ∎

Lemma 3.2. *Let $G = \mathrm{GH}(S, Y, \gamma)$ and $H = \mathrm{GH}(V, Z, \beta)$ be almost homogeneous Heisenberg groups. Assume that $\varphi: G \to H$ is an injective continuous homomorphism. Then the following hold.*

a. *The commutator groups satisfy $G'^{\varphi} = H' \cap G^{\varphi}$.*

b. *There are injective linear maps* $\varphi_1 : S \to V$ *and* $\varphi_2 : Y \to Z$ *such that the following diagram commutes:*

$$\begin{array}{ccc} S \times S & \xrightarrow{\ \gamma\ } & Y \\ {\scriptstyle(\varphi_1,\varphi_1)}\big\downarrow & & \big\downarrow{\scriptstyle\varphi_2} \\ V \times V & \xrightarrow[\ \beta\ ]{} & Z \end{array}$$

c. *We have inequalities* $\dim G' \leq \dim H'$ *and* $\dim(G/G') \leq \dim(H/H')$.

d. *If* $\dim(G/G') = \dim(H/H')$ *then* φ_1, φ_2 *and* φ *are isomorphisms.*

Proof. First of all, we note that $G'^{\varphi} = (G^{\varphi})' \leq H'$ yields that φ induces an injection of $G' = \{0\} \times Y$ into $H' = \{0\} \times Z$. This gives the injective linear map φ_2 from Y into Z. Moreover, we obtain an injective homomorphism from $G^{\varphi}/(H' \cap G^{\varphi})$ into H/H'. As H' coincides with the center of H, we have that $H' \cap G^{\varphi}$ is contained in the center of G^{φ}. But the latter is G'^{φ}, and we have proved assertion a. Moreover, we obtain an inclusion $\bar{\varphi} : G/G' \to H/H'$, and a corresponding linear injection $\varphi_1 : S \to V$.

For each $s \in S$ we write $g_s := (s, 0) \in G$. Then we have $(G'g_s)^{\bar{\varphi}} = H'(s^{\varphi_1}, 0)$, and $[g_s^{\varphi}, g_t^{\varphi}] = \big(0, (s^{\varphi_1}, t^{\varphi_1})^{\beta}\big)$ leads to $(0, (s,t)^{\gamma\varphi_2}) = [g_s, g_t]^{\varphi} = [g_s^{\varphi}, g_t^{\varphi}] = \big(0, (s^{\varphi_1}, t^{\varphi_1})^{\beta}\big)$. This completes the proof of assertion b. We obtain a commutative diagram

$$\begin{array}{ccccccc} S \wedge S & \longrightarrow & G/G' \wedge G/G' & \xrightarrow{\ \bar{\gamma}\ } & G' & \longrightarrow & Y \\ {\scriptstyle\varphi_1 \wedge \varphi_1}\big\downarrow & & {\scriptstyle\bar{\varphi} \wedge \bar{\varphi}}\big\downarrow & & \big\downarrow{\scriptstyle\tilde{\varphi}} & & \big\downarrow{\scriptstyle\varphi_2} \\ V \wedge V & \longrightarrow & H/H' \wedge H/H' & \xrightarrow[\ \bar{\beta}\ ]{} & H' & \longrightarrow & Z \end{array}$$

where $\tilde{\varphi}$ is obtained by restriction of φ. Equality $\dim(G/G') = \dim(H/H')$ yields that φ_1 is an isomorphism. Then surjectivity of $\bar{\beta}$ entails that the embedding φ_2 is also surjective, and we obtain the last assertion. ∎

Corollary 3.3. *There are no injective continuous homomorphisms between the groups* $\mathrm{H}^{4n}_{\mathbb{R}}$, $\mathrm{H}^{4n}_{\mathbb{C}}$ *and* $\mathrm{H}^{4n}_{\mathbb{H}}$, *nor between* $\mathrm{H}^{6}_{\mathbb{R}}$ *and* H^{6}_{6}; *and there are no injective continuous homomorphisms between the groups* $\mathrm{H}^{8}_{\mathbb{R}}$, $\mathrm{H}^{8}_{\mathbb{C}}$, $\mathrm{H}^{8}_{\mathbb{H}}$, H^{8}_{5}, H^{8}_{6}, *and* H^{8}_{7}. ∎

Proposition 3.4. *There is no injective homomorphism from* $\mathrm{H}^{4}_{\mathbb{C}}$ *into* $\mathrm{H}^{8}_{\mathbb{H}}$.

Proof. Assume, to the contrary, that there is an injective homomorphism φ from $\mathrm{H}^{4}_{\mathbb{C}}$ into $\mathrm{H}^{8}_{\mathbb{H}}$. We note that φ may also be considered as an (injective) homomorphism of Lie algebras, because the exponential map is bijective (in fact, the identity) for these groups. We consider the elements

$$w := \big((1,0),0\big), \quad x := \big((i,0),0\big), \quad y := \big((0,1),0\big), \quad \text{and} \quad z := \big((0,i),0\big)$$

in $\mathrm{H}^{4}_{\mathbb{C}}$. Our assumption that φ is injective entails that none of these elements is mapped into the center of $\mathrm{H}^{8}_{\mathbb{H}}$. Since $\mathrm{H}^{8}_{\mathbb{H}}$ is almost homogeneous, we may assume $w^\varphi = \big((1,0),0\big)$. As w and x commute with each other, we have $x^\varphi = \big((\rho,\sigma),p\big)$ with $(\rho,\sigma) \in \mathbb{R}\times\mathbb{H}$ and $p \in \mathrm{Pu}(\mathbb{H})$. Using a suitable element in the group $\mathrm{U}_2\mathbb{H}$ (which acts as a group of automorphisms on $\mathrm{H}^8_{\mathbb{H}}$), we achieve $\sigma \in \mathbb{R}$, as well. We write $\big((\alpha+a,\beta+b),q\big) := y^\varphi$ and $\big((\gamma+c,\delta+d),u\big) := z^\varphi$ with $(\alpha,a),(\beta,b),(\gamma,c),(\delta,d) \in \mathbb{R}\times\mathrm{Pu}(\mathbb{H})$ and $q,u \in \mathrm{Pu}(\mathbb{H})$. The commutator relations for w,x,y,z in the Lie algebra of $\mathrm{H}^4_{\mathbb{C}}$ are translated by φ into the relations

$$[w^\varphi,y^\varphi] = -[x^\varphi,z^\varphi], \quad [w^\varphi,z^\varphi] = [x^\varphi,y^\varphi], \quad \text{and} \quad [y^\varphi,z^\varphi] = 0.$$

The first two of these relations yield the equations

$$\sigma d = -a-\rho c \quad \text{and} \quad \sigma b = c-\rho a.$$

As φ is injective, we also have that $[w^\varphi,y^\varphi] = \big((0,0,),-a\big)$ and $[w^\varphi,z^\varphi] = \big((0,0,),-c\big)$ are linearly independent over $\mathbb{R}$. This implies $\sigma \neq 0$, and the third commutator relation now gives

$$0 \;=\; \mathrm{Pu}\big((\alpha+a)(\gamma-c)+(\beta+b)(\delta-d)\big) \;\in\; \mathbb{R}a+\mathbb{R}c-\frac{\sigma^2+1+\rho^2}{\sigma^2}\mathrm{Pu}(ac).$$

However, since the elements $a,c \in \mathrm{Pu}(\mathbb{H})$ are linearly independent, the three elements a, c, and $\mathrm{Pu}(ac) = a\times c$ are linearly independent, as well. This means that the positive real number $\sigma^2+1+\rho^2$ has to be zero, a contradiction. ■

Proposition 3.5. *There is an embedding of* $\mathrm{H}^4_{\mathbb{C}}$ *into* $\mathrm{H}^{12}_{\mathbb{H}}$.

Proof. Resuming the notation from the proof of 3.4, we notice that $w^\varphi := \big((1,0,0),0\big)$, $x^\varphi := \big((0,1,0),0\big)$, $y^\varphi := \big((i,j,1),0\big)$, $z^\varphi := \big((j,-i,-2k),0\big)$ defines an embedding $\varphi\colon \mathrm{H}^4_{\mathbb{C}} \to \mathrm{H}^{12}_{\mathbb{H}}$, as claimed. ■

Theorem 3.6. *Let k and n be positive integers, let $\mathbb{F},\mathbb{K} \in \{\mathbb{R},\mathbb{C},\mathbb{H}\}$ and assume that there is an injective continuous homomorphism $\varphi\colon \mathrm{H}^{2k}_{\mathbb{F}} \to \mathrm{H}^{4n}_{\mathbb{K}}$. Then $k>n$ implies $\mathbb{F}=\mathbb{K}$.*

Proof. From 3.2 we know $\mathbb{F} \le \mathbb{K}$. For $\mathbb{K}=\mathbb{R}$, there is nothing left to prove. Now assume $k>n$. We have $\mathrm{H}^{2k}_{\mathbb{F}} = \mathrm{GH}(V,Z,\beta)$ with $\dim_{\mathbb{R}} V = 2k > 2n$. The image of V under $\bar\varphi$ will be denoted by S.

In the case $\mathbb{K}=\mathbb{C}$, we have $\dim_{\mathbb{R}} S > \frac{1}{2}\dim_{\mathbb{R}}\mathbb{C}^{2n}$, and the intersection $S\cap iS$ is not trivial. Therefore, we find $u,v \in V\setminus\{0\}$ with $v^{\bar\varphi} = iu^{\bar\varphi}$. Commutators $[(x,0),(y,0)]$ in $\mathrm{H}^{2d}_{\mathbb{R}}$ or $\mathrm{H}^{2d}_{\mathbb{C}}$ are both given as $(0,\langle x|y\rangle)$, where $\langle\cdot|\cdot\rangle$ denotes the (essentially unique) nondegenerate symplectic form on $\mathbb{R}^{2d}$ and $\mathbb{C}^{2d}$, respectively. Picking $w \in V$ such that $\langle u|w\rangle \neq 0$, we compute $0 \neq (0,\langle u^{\bar\varphi}|w^{\bar\varphi}\rangle) = [(u,0),(w,0)]^\varphi$ and $[(v,0),(w,0)]^\varphi = (0,\langle v^{\bar\varphi}|w^{\bar\varphi}\rangle) = (0,i\langle u^{\bar\varphi}|w^{\bar\varphi}\rangle)$. This means that the commutator of $(\mathrm{H}^{2k}_{\mathbb{F}})^\varphi \cong \mathrm{H}^{2k}_{\mathbb{F}}$ has dimension 2, and $\mathbb{F}=\mathbb{C}$ follows.

Now consider the case $\mathbb{K}=\mathbb{H}$. The commutators in $\mathrm{H}^{4n}_{\mathbb{H}}$ are given by $[(x,0),(y,0)] = \big(0,\mathrm{Pu}((x|y))\big)$ for some anisotropic hermitian form $(\cdot|\cdot)$ on $\mathbb{H}^n$. We consider a pure quaternion $h \in \mathrm{Pu}(\mathbb{H})$. As before, we have $S\cap hS \neq \{0\}$, for dimension reasons, and find $u,v \in V$ such that $v^{\bar\varphi} = hu^{\bar\varphi}$. Computing $[(v,0),(u,0)]^\varphi = (0,\mathrm{Pu}((v^{\bar\varphi}|u^{\bar\varphi}))) = (0,h(u^{\bar\varphi}|u^{\bar\varphi})) \in (0,h)\mathbb{R}$ we find that the commutator of $(\mathrm{H}^{2k}_{\mathbb{F}})^\varphi \cong \mathrm{H}^{2k}_{\mathbb{F}}$ contains $\{0\}\times\mathrm{Pu}(\mathbb{H})$, and $\mathbb{F}=\mathbb{H}$. ■

Theorem 3.7. *There is an embedding of* H_3^3 *into* $\mathrm{H}_{\mathbb{H}}^4$.

Proof. We construct the symplectic map $\beta\colon \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^3$ corresponding to $\mathrm{H}_{\mathbb{H}}^4$, as follows: identify the $\mathbb{R}[\mathrm{SO}_4\mathbb{R}]$-module $\mathbb{R}^4 \wedge \mathbb{R}^4$ with the Lie algebra $\mathfrak{o}_4$, and let $\bar\beta$ be the projection onto some 3-dimensional ideal Z.

Now restrict the $\mathbb{R}[\mathrm{SO}_4\mathbb{R}]$-module $\mathbb{R}^4$ to an $\mathbb{R}[\mathrm{SO}_3\mathbb{R}]$-module, and let S be the (unique) submodule of dimension 3. Then the submodule $S \wedge S$ is not invariant under the action of the larger group $\mathrm{SO}_4\mathbb{R}$. Therefore, it is not one of the ideals, and its image under $\bar\beta$ is a nontrivial $\mathbb{R}[\mathrm{SO}_3\mathbb{R}]$-submodule of Z. This means $(S \wedge S)\bar\beta = Z$, and we have found the subgroup $\mathrm{GH}\,(S, Z, \beta|_{S\times S}) \cong \mathrm{H}_3^3$ inside $\mathrm{GH}\,(\mathbb{R}^4, Z, \beta) \cong \mathrm{H}_{\mathbb{H}}^4$. ■

Theorem 3.8. *There are embeddings of* $\mathrm{H}_{\mathbb{C}}^4$ *into* H_6^6 *and of* $\mathrm{H}_{\mathbb{H}}^4$ *into* H_7^7. *However, there are no injective continuous homomorphisms from* $\mathrm{H}_{\mathbb{R}}^4$ *or* $\mathrm{H}_{\mathbb{H}}^4$ *into* H_6^6, *and no injective continuous homomorphisms from* $\mathrm{H}_{\mathbb{R}}^4$ *or* $\mathrm{H}_{\mathbb{C}}^4$ *into* H_7^7.

Proof. For $z \in \{6, 7\}$, we model H_z^z by the group $\mathrm{GH}\,(\mathbb{R}^z, Z, \beta_z)$ obtained from a suitable projection $\bar\beta_z\colon \mathfrak{o}_z \to Z$, where $\mathfrak{o}_z$ is considered as an $\mathbb{R}[\Delta]$-module with $\Delta \in \{\mathrm{SU}_3\mathbb{C}, \mathrm{G}_2\}$, according to the value of z.

Guided by 3.2, we search for a vector subspace S of $\mathbb{R}^z$ such that $Y := (S\wedge S)\bar\beta_z$ has dimension 1, 2 or 3. The restriction $\gamma\colon S\times S \to Y$ of β_z then gives a Heisenberg group $\mathrm{GH}\,(S, Y, \gamma)$. We will find S together with a group of automorphisms of $\mathrm{GH}\,(S, Y, \gamma)$ in such a way that we also see that $\mathrm{GH}\,(S, Y, \gamma)$ is almost homogeneous; together with the parameters, this determines $\mathrm{GH}\,(S, Y, \gamma)$ up to isomorphism.

Let $S < \mathbb{R}^z$ be a vector subspace of dimension 4. Then, under our usual identification of $\mathbb{R}^z \wedge \mathbb{R}^z$ with $\mathfrak{o}_z$, the subspace $S \wedge S$ corresponds to the Lie algebra of the pointwise stabilizer of the orthogonal complement $S^\perp$ of S in $\mathbb{R}^z$. This allows to determine $\mathfrak{k} := (S \wedge S) \cap \ker\bar\beta_z$, as follows.

For $z = 6$ we have that $\ker\bar\beta_6$ is the Lie algebra of $\mathrm{U}_3\mathbb{C}$, and $\mathfrak{k}$ either is the Lie algebra of $\mathrm{U}_2\mathbb{C}$ (if S is a complex subspace), or of $\mathrm{U}_1\mathbb{C}$. In the first case, we obtain $\dim Y = 2$, and the action of $\mathrm{U}_2\mathbb{C}$ shows $\mathrm{GH}\,(S, Y, \gamma) \cong \mathrm{H}_{\mathbb{C}}^4$. The second case, where $\dim Y = 5$, does not lead to an almost homogeneous subgroup.

For $z = 7$ we have to consider stabilizers in the group G_2. We regard $\mathbb{R}^7$ as the space $\mathrm{Pu}\,(\mathbb{O})$ of pure octonions in such a way that G_2 acts as the group $\mathrm{Aut}(\mathbb{O})$. If $S^\perp$ generates $\mathbb{O}$, the pointwise stabilizer of $S^\perp$ in $\mathrm{Aut}(\mathbb{O})$ is of course trivial. This means $\dim Y = 6$, and we do not obtain an almost homogeneous group. The situation is different if $S^\perp$ generates a proper subalgebra of $\mathbb{O}$: this subalgebra is isomorphic to $\mathbb{H}$, and the pointwise stabilizer of $S^\perp$ in $\mathrm{Aut}(\mathbb{O})$ is a group isomorphic to $\mathrm{U}_1\mathbb{H} \cong \mathrm{SU}_2\mathbb{C}$, while a subgroup $\Sigma \cong \mathrm{SO}_4\mathbb{R}$ of $\mathrm{Aut}(\mathbb{O})$ leaves $S^\perp$ and S invariant, compare [3] 11.31. This yields $\dim Y = 3$, and the action of Σ shows that $\mathrm{GH}\,(S, Y, \gamma)$ is almost homogeneous. ■

Theorem 3.9. *There is an embedding of* $\mathrm{H}_{\mathbb{R}}^4$ *into* H_5^8.

Proof. We consider $\mathrm{H}_5^8 = \mathrm{GH}\,(\mathbb{H}^2, Z, \beta)$ as in 2.3: the symplectic map β is obtained from the projection of the $\mathbb{R}[\mathrm{U}_2\mathbb{H}]$-module $\mathbb{H}^2 \wedge \mathbb{H}^2$ onto a five-dimensional submodule Z.

We claim that $S := \mathbb{H} \times \{0\}$ generates a subgroup $\mathrm{GH}(S, Y, \gamma) \cong \mathrm{H}^4_{\mathbb{R}}$ in H^8_5 where, of course, we have $\gamma := \beta|_{S\times S}$. We know that $S \wedge S$ is the Lie algebra of the pointwise stabilizer Γ of $S^\perp = \{0\} \times \mathbb{H}$ in $\mathrm{SO}_8\mathbb{R}$. The intersection $\Delta := \Gamma \cap \mathrm{U}_2\mathbb{H}$ is a group isomorphic to $\mathrm{U}_1\mathbb{H}$. As the central involution of $\mathrm{U}_2\mathbb{H}$ is not contained in Δ, we have that Δ acts effectively on Z, and the $\mathbb{R}[\Delta]$-module Z splits as the sum of $F := \mathrm{Fix}_Z(\Delta)$ and some irreducible module of dimension 4. The central involution of Δ acts trivially on Y because it induces $-\mathrm{id}$ on S. Thus Y is contained in F. In order to see that equality holds, we consider the projection π from the $\mathbb{R}[\mathrm{U}_2\mathbb{H}]$-module $\mathfrak{o}_8\mathbb{R}$ onto the complement C of the Lie algebra $\mathfrak{g}$ of $\mathbb{S}_{\mathbb{H}} \cdot \mathrm{U}_2\mathbb{H}$. The centralizer of Δ in $\mathbb{S}_{\mathbb{H}} \cdot \mathrm{U}_2\mathbb{H}$ has dimension 6. Since Δ has 9-dimensional centralizer in $\mathrm{SO}_8\mathbb{R}$, this implies that $(\mathrm{Fix}_{\mathfrak{o}_8}(\Delta))\,\pi = \mathrm{Fix}_C(\Delta)$ has dimension 3. The kernel of the restriction $\pi|_{S\wedge S}$ is the intersection of $S \wedge S$ with $\mathfrak{g}$, and thus 3-dimensional. We obtain the equalities $(S\wedge S)\pi = \mathrm{Fix}_C(\Delta)$ and $(S \wedge S)\bar\beta = F$. ∎

I did not find any embeddings from $\mathrm{H}^4_{\mathbb{F}}$ into H^8_5, for $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}\}$. According to 3.2, the only other almost homogeneous Heisenberg group admitting an embedding into H^8_5 would be $\mathrm{H}^6_{\mathbb{R}}$.

The main problem is that the $\mathbb{R}[\mathrm{U}_2\mathbb{H}]$-module $\mathfrak{o}_8$ has lots of submodules: it splits as the sum of the 10-dimensional Lie algebra of $\mathrm{U}_2\mathbb{H}$, its 3-dimensional centralizer, and the 15-dimensional sum of (pairwise isomorphic, and thus not uniquely determined) 5-dimensional modules. The search for embeddings of $\mathrm{H}^4_{\mathbb{F}}$ (or for proofs of nonexistence of such embeddings) would have to consider subalgebras $S \wedge S \cong \mathfrak{o}_4$ corresponding to 4-dimensional subspaces S of $\mathbb{R}^8$. For such subalgebras, one needs to know the intersection with the kernel of the projection onto one of the 5-dimensional submodules.

For all $(v, z) \neq (8, 5)$, the exceptional almost homogeneous Heisenberg group H^v_z is described by a rather simple factorization of the module $\mathfrak{o}_v$. We use this in the sequel to obtain a fairly complete understanding of embeddings.

Theorem 3.10. *There are no injective continuous homomorphisms from* $\mathrm{H}^6_{\mathbb{R}}$ *or* H^6_6 *into* H^7_7.

Proof. Let $V = \mathrm{Pu}(\mathbb{O}) = Z$, and define $\beta\colon V \times V \to Z$ by $(v,w)^\beta := [v,w] := vw - wv$. Then $H := \mathrm{GH}(V, Z, \beta) = \mathrm{C}_{\mathbb{O}} \cong \mathrm{H}^7_7$. Assume that $G < H$ is a subgroup satisfying $G \cong \mathrm{H}^6_{\mathbb{R}}$ or $G \cong \mathrm{H}^6_6$. From 3.2a we know $G' = H' \cap G$. The commuting diagram

$$\begin{array}{ccc} H/H' \times H/H' & \xrightarrow{\beta} & H' \\ \uparrow & & \uparrow \\ G/G' \times G/G' & \xrightarrow[\beta]{} & G' \end{array}$$

shows that there is a hyperplane T of $\mathrm{Pu}(\mathbb{O}) \cong H/H'$ such that $\dim[T,T] = 1$ or $\dim[T,T] = 6$, respectively. Every hyperplane in $\mathrm{Pu}(\mathbb{O})$ is the orthogonal space $x^\perp$ of some nonzero element $x \in \mathrm{Pu}(\mathbb{O})$. Using transitivity properties of $\mathrm{Aut}(\mathbb{O})$, we may assume $T = i^\perp$. Now we compute $2i = [j,k] \in [i^\perp, i^\perp]$ and $-2j = [jl, l] \in [i^\perp, i^\perp]$. The orbit of j under the stabilizer of i in $\mathrm{Aut}(\mathbb{O})$ is the unit sphere in $i^\perp$, see [3] 11.30. Therefore, we obtain $\mathrm{Pu}(\mathbb{O}) = [i^\perp, i^\perp]$, contradicting our assumptions. This shows that there is no subgroup isomorphic to $\mathrm{H}^6_{\mathbb{R}}$ or H^6_6 in H^7_7. ■

Theorem 3.11. *There is an embedding of H^6_6 into H^8_6, but there is no continuous injective homomorphism from $\mathrm{H}^6_{\mathbb{R}}$ into H^8_6.*

Proof. Choose a vector subspace S of $\mathbb{R}^8$, and assume that $\dim S = 6$. Then $S \wedge S$ corresponds to the Lie algebra $\mathfrak{g}$ of a subgroup $\Gamma \cong \mathrm{SO}_6\mathbb{R}$ in $\mathrm{SO}_8\mathbb{R}$, and $\Gamma \cap \mathrm{U}_4\mathbb{C} \cong \mathrm{U}_3\mathbb{C}$ (if $\mathbb{C}S = S$) or $\Gamma \cap \mathrm{U}_4\mathbb{C} \cong \mathrm{U}_2\mathbb{C}$ (if S is not a complex subspace). The $\mathbb{R}[\mathrm{SU}_4\mathbb{C}]$-module $\mathfrak{o}_8$ has two 6-dimensional simple submodules Z_1 and Z_2 whose sum $Z_1 + Z_2$ forms a complement to the Lie algebra $\mathfrak{u}$ of $\mathrm{U}_4\mathbb{C}$. Restricting the projection $\pi\colon \mathfrak{o}_8 \to Z_1 + Z_2$ modulo $\mathfrak{u}$ to the 15-dimensional space $S \wedge S = \mathfrak{g}$, we find $\dim(\mathfrak{g} \cap \mathfrak{u}) \in \{9, 4\}$. In the latter case, the projection of $\mathfrak{g}$ to Z_l has dimension at least $11 - 6 = 5$. Surely, this does not yield an embedding of $\mathrm{H}^6_{\mathbb{R}}$.

It remains to consider the case where S is a complex subspace. Then there is at least one simple $\mathbb{R}[\mathrm{SU}_4\mathbb{C}]$-submodule Z of $Z_1 + Z_2$ such that the projection onto this submodule has nontrivial restriction $\bar\beta$ to $S \wedge S$. As an $\mathbb{R}[\mathrm{SU}_3\mathbb{C}]$-submodule, the module Z has to be simple (note that all the trivial $\mathbb{R}[\mathrm{SU}_3\mathbb{C}]$-submodules of $\mathfrak{o}_8$ are contained in $\mathfrak{u}$). We have thus found a subgroup $\mathrm{GH}(S, Z, \beta)$ of H^8_6 such that $\mathrm{GH}(S, Z, \beta) \cong \mathrm{H}^6_6$. ■

Theorem 3.12. *There is an embedding of H^7_7 into H^8_7. Moreover, up to automorphisms of H^8_7, any almost homogeneous Heisenberg group properly contained in H^8_7 is contained in the image of this embedding. Consequently, there are no continuous injective homomorphisms from $\mathrm{H}^4_{\mathbb{R}}$, $\mathrm{H}^4_{\mathbb{C}}$, $\mathrm{H}^6_{\mathbb{R}}$ or H^6_6 into H^8_7.*

Proof. The group Spin_7 contains a subgroup isomorphic to G_2 (this is just the stabilizer in the transitive action of Spin_7 on $\mathbb{S}_7$). Restricting the simple $\mathbb{R}[\mathrm{Spin}_7]$-module $\mathbb{R}^8$ to an $\mathbb{R}[\mathrm{G}_2]$-module, we obtain a (unique) submodule S of dimension 7. Under the identification of $\mathbb{R}^8 \wedge \mathbb{R}^8$ with $\mathfrak{o}_8$, the exterior product $S \wedge S$ corresponds to the Lie algebra of the stabilizer of a vector. Thus it does not coincide with the Lie algebra of Spin_7. This means that the projection $\bar\beta$ from the $\mathbb{R}[\mathrm{Spin}_7]$-module $\mathfrak{o}_8$ onto its 7-dimensional submodule Z (leading to the Heisenberg group $\mathrm{GH}(\mathbb{R}^8, Z, \bar\beta) \cong \mathrm{H}^8_7$) restricts to a nontrivial map from $S \wedge S$ to Z. But the group G_2 acts irreducibly on Z, and we obtain that $\mathrm{GH}(S, Z, \beta|_{S\times S})$ is a subgroup of $\mathrm{GH}(\mathbb{R}^8, Z, \bar\beta)$ with the property $\mathrm{GH}(S, Z, \beta|_{S\times S}) \cong \mathrm{H}^7_7$.

Now assume that $H < \mathrm{H}^8_7$ is an almost homogeneous Heisenberg group. Then $H \neq \mathrm{H}^8_7$ implies $H + Z \neq \mathrm{H}^8_7$, see 3.2d. Thus $(H + Z)/Z$ is a proper subspace of the vector space H^8_7/Z, and its orthogonal complement contains a nonzero vector v. As the orbit of v under Spin_7 meets the orthogonal complement of $(S + Z)/Z$, we may assume $(H + Z)/Z \le (S + Z)/Z$, and $H \le S + Z$, as required. The rest follows from 3.8 and 3.10. ■

Open Problems. It remains to answer the following questions:

(a) Is there an embedding of $\mathrm{H}^4_{\mathbb{R}}$ into H^8_6 ?
(b) Is there an embedding of $\mathrm{H}^4_{\mathbb{C}}$ into H^8_5 ?
(c) Is there an embedding of $\mathrm{H}^4_{\mathbb{H}}$ into H^8_5, or into H^8_6 ?

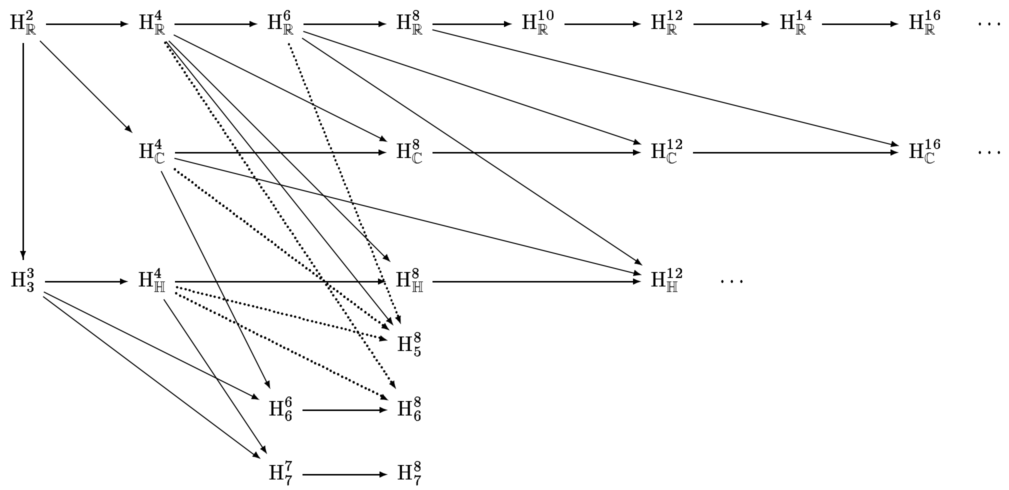
(d) Is there an embedding of $H^6_{\mathbb{R}}$ into H^8_5 ?

Note that in each of the open cases the larger group has a dimension which is considerably larger than that of the subgroup in question.
Apart from these special problems, there are the following, more general ones:

(e) For which pairs (k,n) of positive integers is there an embedding of $H^{4k}_{\mathbb{C}}$ into $H^{4n}_{\mathbb{H}}$? (Partial answers have been given in 3.4, 3.5, and 3.6.)

(f) In those cases where embeddings exist: what can be said about uniqueness?

The following diagram attempts to visualize the embeddings of almost homogeneous Heisenberg groups, as far as we have found them. The diagram contains all almost homogeneous Heisenberg groups of dimension at most 18; in particular, all exceptional ones. Absence of (paths along) arrows indicates that no embedding exists. There remain the Open Problems, as stated above (and indicated by dotted arrows in the diagram). See, however, the results in 3.3, 3.8, 3.10, and 3.11.



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