Journal of Lie Theory
Volume **10** (2000) 455–461

Geodesic loops

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Communicated by K. Strambach

Abstract. Using new tangential objects, which we call Λ-algebras, we obtain for geodesic loops (L, ∇, e) statements that are analogous to Lie's theorems. We prove that any geodesic loop L defines in the tangential space $T_e L$ a unique Λ-algebra, and that to any finite dimensional real Λ-algebra F there exists a geodesic loop L, the Λ-algebra of which is isomorphic to F. Geodesic loops which have isomorphic Λ-algebras, are locally isomorphic.

If the local loop L is diassociative then to the Λ-algebra of L there corresponds a subseries of the Hausdorff-Campbell formula with respect to the binary Lie algebra belonging to L.

Introduction

A.I. Malcev perceived that the fundamental ideas of Sophus Lie to assign to any local Lie group G a tangential object -its Lie algebra- which determines G in a unique way can be extended to non-associative structures. In particular with any differentiable Moufang loop L is associated a Malcev algebra which determines L uniquely up to coverings (cf. [9] Theorem 4.4). Moreover, in the mean time the theory of Lie Moufang loops reached the same level as the theory of Lie groups (cf. [3], [5], [8]). The most general class of local analytical loops for which we know that they are determined by their tangential objects -by their Bol algebras- in a unique way are the Bol loops (cf. [7] pp. 424-425). In general, the differentiable loops have as tangential objects the Akivis algebras, but an Akivis algebra corresponds to a variety of local differentiable loops the algebraic properties of which show greate differences (cf. [2] pp. 240-241).

The aim of this paper is to show that there exist suitable tangential objects for very wide classes of local analytical loops L which determines L in a unique way; concretely we demonstrate this for the class of geodesic loops which respect to linear connections the curvature of which is zero; these tangential objects have in general infinitely many binary operations. Since a differentiable local loop is strongly left alternative if and only if it can be represented as the opposite loop of a geodesic loop with respect to a linear connection with curvature zero, the local

*Partially supported by a Grant 323 PPP-Ungarn of DAAD

ISSN 0949–5932 / \$2.50

analytic Bol loops are a proper subclass of geodesic loops considered in our paper (cf. [11] Proposition 4.7).

The tangential object T_eL of a geodesic loop L with respect to a linear connection with vanishing curvature is called a Λ-algebra. The infinitely many binary operations $\Lambda_{(p,1)}(x,y) : T_eL \times T_eL \to T_eL$ are linear in y and p-homogeneous in x; moreover, they satisfy certain symmetry conditions. In the case that the local loop L is diassociative to the Λ-algebra of L there corresponds a subseries of the Hausdorff-Campbell formula with respect to the binary Lie algebra belonging to L.

I would like to express my gratitude to my teacher P.T. Nagy for his guidance and encouragement.

1. Geodesic loops and their Λ-algebras

Definition 1.1. Let L be an analytic manifold and B an open neighbourhood of the identity e of L. Let $*, /, \backslash : B \times B \to L$ be analytic mappings. The manifold L is called a local analytic loop if the analytic mappings $*, /, \backslash : B \times B \to L$ satisfy the following identities:

1. $x * e = e * x = x/e = e\backslash x = x$ for all $x \in B$.
2. $(x/y) * y = x, \quad y * (y\backslash x) = x,$
   $(x * y)/y = x, \quad y\backslash(y * x) = x$ for all $x, y \in B$.

provided the left sides of these equations are defined.

Definition 1.2. Let L be a differentiable manifold equipped with an affine connection ∇ and let e be a point in L. A local analytic loop (L, ∇, e) is called a geodesic loop if in a normal neighbourhood U of $e \in L$ the multiplication of L is given by

$$x * y = exp_y \circ \tau_{e,y} \circ exp_e^{-1}(x)$$

for all $x, y \in U$, where $\tau_{e,y}$ denotes the parallel translation $T_eL \to T_yL$ along the unique geodesic segment, contained in U, between e and y (cf. [4], p. 160 and [7], p. 369).

Definition 1.3. Let V be an n-dimensional real or complex vector space. V is called a Λ-algebra if there exists for every natural number p an algebraic map $\Lambda_{(p,1)} : V \times V \to V$ such that the following properties are satisfied:

1. $\Lambda_{(p,1)}(x, \lambda y) = \lambda \Lambda_{(p,1)}(x,y)$ for all $\lambda \in K$ and $x, y \in V$.
2. $\Lambda_{(p,1)}(\lambda x, y) = \lambda^p \Lambda_{(p,1)}(x,y)$ for all $\lambda \in K$ and $x, y \in V$.
3. $\sum_{p=1}^{\infty} \Lambda_{(p,1)}(x,y)$ converges on a neigbourhood N of 0 in V.

Remark 1.4. The coordinate functions $\Lambda^i_{(p,1)}(x,y)$, $i = 1, \dots, n$, of $\Lambda_{(p,1)}(x,y)$ are homogeneous polynomials of degree $p+1$ which are linear in the coordinates of y and homogeneous polynomials of degree p in the coordinates of x.

The Λ-algebras are suitable tangential objects for geodesic loops (L, ∇, e) for which the curvature tensor is 0, in order to obtain analogous statements to Lie's theorems.

Theorem 1.5. (Lie's First Theorem) *Any geodesic loop L defines in the tangential space T_eL a unique Λ-algebra.*

Proof. The 1-parameter subgroups of a geodesic loop (L, ∇, e) are geodesic lines through the identity e of L with respect to the linear connection ∇. Hence the canonical coordinate system for L (which is determined by the 1-parameter subgroups) is a normal coordinate system of the connection ∇ in a normal neighbourhood $U \subset \mathbb{R}^n$. Since the curvature of ∇ is zero every parallel translation with respect to ∇ can be given in U by the independent parallel vector fields $A_1(x), \dots, A_n(x)$, $(i = 1, \dots, n)$, where $A_i(0) = e_i$ yield the canonical basis of $\mathbb{R}^n$. The parallel translation $\tau_{e,y}$ is determined by $e_i \mapsto A_i(y)$ $(i = 1, \dots, n)$. Hence the map $\sum\limits_{i=1}^{n} \xi^i e_i \mapsto \sum_{i=1}^n A_i(y)\xi^i$ gives the tangent map $T_e\rho_y$ for the right translation $\rho_y : x \to x * y$. The differential equations of the coordinate functions $y^i(t)$ for the geodesic line $y(t)$ with the initial values $y(0) = y_0$ and $y'(0) = \sum\limits_{j=1}^{n} A_j(y_0)\xi^j$ have the form

$$y'^i(t) = \sum_{j=1}^{n} A_j^i(y(t))\xi^j, \tag{1}$$

where A_j^i denote the coordinate functions of the vector fields A_j and ξ^j are the coordinates with respect to the basis $A_j(y_0)$. Since in the coordinate system of U the geodesic lines are the euclidean lines $(\xi^1, \dots, \xi^n)t$, $t \in \mathbb{R}$, equation (1) implies that the coordinate functions $A_j^i(x)$ of A_j satisfy the identities

$$\xi^i = \sum_{j=1}^{n} A_j^i((\xi^1, \dots, \xi^n)t)\xi^j.$$

The power series of the coordinate functions $A_j^i(x)$ through the point $(0, \dots, 0)$ can be written as

$$\xi^i = \sum_{j=1}^{n} [A_j^i(0) + \sum_{p=1}^{\infty} \frac{1}{p!} \sum_{j_1, \dots, j_p = 1}^{n} \frac{\partial^p A_j^i}{\partial x^{j_1} \dots \partial x^{j_p}}(0)\xi^{j_1} \dots \xi^{j_p}]\xi^j.$$

Since $A_j^i(0) = \delta_j^i$ we obtain

$$\xi^i = \xi^i + \sum_{p=1}^{\infty} \frac{1}{p!} \sum_{j_1, \dots, j_p = 1}^{n} \frac{\partial^p A_j^i}{\partial x^{j_1} \dots \partial x^{j_p}}(0)\xi^{j_1} \dots \xi^{j_p}\xi^{j_{p+1}}.$$

Consequently the coefficients of this expansion satisfy

$$\sum_{\sigma \in \mathbb{Z}_{p+1}} \frac{\partial^p A^i_{j(p+1)\sigma}}{\partial x^{j_{1\sigma}} \dots \partial x^{j_{p\sigma}}}(0) = 0, \tag{2}$$

where $\mathbb{Z}_{p+1}$ is the cyclic permutation group of order $p+1$.

In the tangential space of the identity $e \in L$ we define now the binary operations $\Lambda_{(p,1)}$ as follows:

$$\Lambda_{(1,1)}(x, y) = (\sum_{j=1, h_1=1}^{n} \frac{\partial A_j^i}{\partial x^{h_1}}(0)x^{h_1}y^j)_{i=1}^n$$

$$\Lambda_{(2,1)}(x,y) = (\sum_{j=1,h_1=1,h_2=1}^{n} \frac{1}{2!}\frac{\partial^2 A^i_j}{\partial x^{h_1}\partial x^{h_2}}(0)x^{h_1}x^{h_2}y^j)^n_{i=1}$$

$$\Lambda_{(p,1)}(x,y) = (\sum_{j=1,h_1=1,\dots,h_p=1}^{n} \frac{1}{p!}\frac{\partial^p A^i_j}{\partial x^{h_1}\dots\partial x^{h_p}}(0)x^{h_1}\dots x^{h_p}y^j)^n_{i=1}.$$

The i-th coordinate of the vector $\Lambda_{(p,1)}(x,y)$ is

$$\Lambda^i_{(p,1)}(x,y) = \sum_{j=1,h_1=1,\dots,h_p=1}^{n} \frac{1}{p!}\frac{\partial^p A^i_j}{\partial x^{h_1}\dots\partial x^{h_p}}(0)x^{h_1}\dots x^{h_p}y^j.$$

This function is linear in the coordinates of y since it is depending only of the sum of the coordinates of y. Hence $\Lambda_{(p,1)}(x,y)$ is linear in y. We have

$$\Lambda^i_{(p,1)}(\lambda x,y) = \sum_{j=1,h_1=1,\dots,h_p=1}^{n} \frac{1}{p!}\frac{\partial^p A^i_j}{\partial x^{h_1}\dots\partial x^{h_p}}(0)\lambda x^{h_1}\dots\lambda x^{h_p}y^j =$$

$$\lambda^p\Lambda^i_{(p,1)}(x,y).$$

Hence the second condition for a Λ-algebra is satisfied. The coordinate functions of the product $(x^1,\dots,x^n)*(y^1,\dots,y^n)$ are given by $A^i_j(y)$ and its partial derivaties (cf.[10] pp.309-312). Since L is analytic loop and the series $\sum_{p=1}^{\infty}\Lambda_{(p,1)}(x,y)$ is a subseries of the power expansion of the product $x*y$ the condition 3. in the definition of a Λ-algebra is valid. ■

Theorem 1.6. (Lie's Third Theorem) *To any finite dimensional real Λ-algebra F there exists a geodesic loop L, the Λ-algebra of which is isomorphic to F. The geodesic loops which have isomorphic Λ-algebras, are locally isomorphic.*

Proof. Let F be an n-dimensional real Λ-algebra. The convergent series $\sum_{p=1}^{\infty}\Lambda_{(p,1)}(x,y)$ determines a function $f=(f^1,\dots,f^n)$ from $F\times F$ into F. The function f is linear in y since the operations $\Lambda_{(p,1)}(x,y)$ are linear in y. Hence the i-th coordinate function $f^i(x,y)$ has the form

$$f^i(x^1,\dots,x^n,y^1,\dots,y^n) = \sum_{j=1}^{n} f^i_j(x)y^j$$

with suitable real analytic functions f^i_j. If we develop the functions f^i_j into Taylor series in the point zero we obtain

$$f^i_j(x) = \sum_{p=1}^{\infty}\frac{1}{p!}\sum_{h_1=1,\dots,h_p=1}^{n}\frac{\partial^p f^i_j}{\partial x^{h_1}\dots\partial x^{h_p}}(0)x^{h_1}\dots x^{h_p}.$$

Since the real analytic function f is described uniquely by a convergent series we can write

$$\Lambda_{(p,1)}(x,y) = (\sum_{j=1,h_1=1,\dots,h_p=1}^{n} \frac{1}{p!}\frac{\partial^p A^i_j}{\partial x^{h_1}\dots\partial x^{h_p}}(0)x^{h_1}\dots x^{h_p}y^j)^n_{i=1},$$

where $A^i_j = f^i_j$. The coordinate functions A^i_j determine the vector fields A_j which are defined as $A_j : y \to (A^1_j(y), \dots, A^n_j(y))$. Considering F as the affine space $\mathbb{R}^n$ we can assign to any point y of F a tangent space $T_yF = \sum_{i=1}^{n} \eta^i A_i(y), \eta_i \in \mathbb{R}$ and form the tangent bundle TF. Any vector $v \in T_0F$, $v = \sum_{i=1}^{n} \eta^i e_i$, determines a vector field Y_v putting $Y_v(y) = \sum_{i=1}^{n} \eta^i A_i(y)$.

We define now a covariant derivation ∇, such that the vector fields A_i become parallel vector fields with respect to ∇. Hence in the expression

$$\nabla_X Y = \nabla_X(\sum_{i=1}^{n} Y^i A_i) = \sum_{i=1}^{n}(X(Y)^i)A_i + \sum_{i=1}^{n} Y^i \nabla_X A_i$$

we put $\sum_{i=1}^{n} Y^i \nabla_X A_i = 0$ for any vector field X. For the curvature tensor $R(X,Z)Y$ with respect to ∇ we have

$$R(X,Z)Y = \nabla_X \nabla_Z(\sum_{i=1}^{n} Y^i A_i) - \nabla_Z \nabla_X(\sum_{i=1}^{n} Y^i A_i) - \nabla_{[X,Z]}(\sum_{i=1}^{n} Y^i A_i) =$$

$$\nabla_X(\sum_{i=1}^{n}(Z(Y^i))A_i) - \nabla_Z(\sum_{i=1}^{n}(X(Y^i))A_i) - \sum_{i=1}^{n}([X,Z](Y^i))A_i =$$

$$X(\sum_{i=1}^{n}(Z(Y^i))A_i) - Z(\sum_{i=1}^{n}(X(Y^i))A_i) - \sum_{i=1}^{n}([X,Z](Y^i))A_i.$$

Since $[X,Z] = XZ - ZX$ we can write $[X,Z](Y^i) = X(Z(Y^i) - Z(X(Y^i))$. It follows

$$R(X,Z)Y = X(\sum_{i=1}^{n}(Z(Y^i))A_i) - Z(\sum_{i=1}^{n}(X(Y^i))A_i) - \sum_{i=1}^{n}([X,Z](Y^i))A_i = 0.$$

The parallel translations $\tau_{0,y}$ are determined by $\tau_{0,y}(e_i) = A_i(y)$. Since the curvature tensor R with respect to ∇ is zero the parallel translations do not depend on the path. Let exp be the exponential map belonging to the linear connection ∇. We have $exp_y : T_yF \to F : exp_y(v_y) = \gamma(1)$ where γ is the geodesic curve on F with respect to ∇ satisfying the conditions $\gamma(0) = y$ and $\gamma'(0) = v_y$. The multiplication $x * y = exp_y \circ \tau_{e,y} \circ exp_e^{-1}(x)$ for $x, y \in F$ defines on F a geodesic loop L with 0 as the identity. The Λ-algebra of L is uniquely determined by the parallel translations $\tau_{0,y}$ and hence by the vector fields A_i, $i = 1, \dots, n$, as well as by the derivaties of their coordinate functions. ■

A real algebra $(C, [.,.])$ is called a binary Lie algebra if any two elements of C generate a Lie subalgebra.

To the Λ-algebras of Lie algebras and of binary Lie algebras there correspond subseries of the Campbell-Hausdorff-series. These subseries are determined by the Lie algebras or binary Lie algebras.

Theorem 1.7. *Let C a binary Lie algebra and $(b_{2n})_{n\in\mathbb{N}}$ the sequence of the even Bernoulli numbers. The operations $\Lambda_{(p,1)} : C^2 \to C$ given by*

$$\begin{aligned}
\Lambda_{(1,1)}(X,Y) &= \frac{1}{2}[X,Y] \\
\Lambda_{(2,1)}(X,Y) &= \frac{1}{12}[X,[X,Y]] \\
\Lambda_{(2n,1)}(X,Y) &= \frac{1}{(2n)!}b_{2n}[X,\dots,[X,Y],\dots] \\
\Lambda_{(2n+1,1)}(X,Y) &= 0
\end{aligned}$$

define on the vector space C a Λ-algebra $\Lambda(C)$.

Proof. The operations $\Lambda_{(p,1)}$ are linear in Y. Moreover, we have

$$\Lambda_{(p,1)}(\lambda X,Y) = \lambda^p \Lambda_{(p,1)}(X,Y)$$

for any $\lambda \in \mathbb{R}$. The series $\sum_{p=1}^{\infty} \Lambda_{(p,1)}(X,Y)$ converges since it is a subseries of the Hausdorff-Campbell series (cf. [1] pp. 201-202). ■

An analytic loop L is called diassociative if every two elements generate a subgroup of L.

Theorem 1.8. *The Λ-algebra F of a local diassociative analytic loop $L = exp(F)$ coincides with the Λ-algebra $\Lambda(C)$, where C is the binary Lie algebra belonging to L.*

Proof. According to Malcev, theorems 1 and 2 (cf. [6] pp. 570-571), the tangential algebra of L is a binary Lie algebra C. The power series of the multiplication of L is the classical Hausdorff-Campbell series. Let $\Lambda_{(p,1)}, p \in \mathbb{N}$, be the operations of the Λ-algebra of L and $\Lambda^C_{(p,1)}, p \in \mathbb{N}$, be the operations for the Λ-algebra $\Lambda(C)$. Since $\sum_{p=1}^{\infty} \Lambda_{(p,1)}(x,y)$ is a subseries of the loop multiplication and

$$\sum_{p=1}^{\infty} \Lambda_{(p,1)}(x,y) = \sum_{p=1}^{\infty} \Lambda^C_{(p,1)}(x,y)$$

we have $\Lambda_{(p,1)} = \Lambda^C_{(p,1)}$. ■

Theorem 1.9. *The local analytic loop L the Λ-algebra of which is $\Lambda(C)$ for a binary Lie algebra C is diassociative and C is the tangential algebra of L in sense of Malcev.*

Proof. Since $\Lambda(C)$ is the Λ-algebra of L, the series

$$\sum_{p=1}^{\infty} \Lambda^C_{(p,1)}(x,y) \tag{3}$$

is a subseries of the series of the multiplication $(x,y)\mapsto x*y : L\times L\to L$ such that $x*y = x+y+\sum_{p,q=1}^{\infty}\Lambda^C_{(p,q)}(x,y)$ and the operations $\Lambda^C_{(p,q)}$ are uniquely determined by $\Lambda^C_{(p,1)}$ through a recursive formula (G) in (cf. [10] p. 311). Since (3) is a subseries of the classical Hausdorff-Campbell series too, hence the multiplication $(x,y)\mapsto x*y$ is described by the classical Hausdorff-Campbell formula and there exists a diassociative loop having the same subseries (3) as L. The multiplication of this loop coincides with the multiplication of L and C is the binary Lie algebra in sense of Malcev (cf. [6] pp. 570-571). ■

Remark 1.10. Theorems 4 and 5 remain valid also for monoalternative analytic loops since any such loop is diassociative (cf. [12]).

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Received October 21, 1999
and in final form January 12, 2000