Journal of Integer Sequences, Vol. 22 (2019),
Article 19.6.4

Some New Restricted n-Color Composition Functions

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Abstract

An n-color composition is one in which a part of size m can come in m colors (denoted by subscripts). Let $\mathcal{C}(\nu)$ denote the set of n-color compositions of the positive integer ν. In this paper, we consider further modular restrictions on the subscripts of the parts within members of $\mathcal{C}(\nu)$. We first count members of $\mathcal{C}(\nu)$ in which all parts have subscripts of the form $\ell a + b$, where b and ℓ are fixed and $a \geq 0$ is arbitrary. Generating function and explicit formulas are found for general b and ℓ which extend earlier results when $\ell = 2$ and $b \leq 3$. We study the case $\ell = b - 1$ in further detail and find that the corresponding subset of $\mathcal{C}(\nu)$ is in bijection with various classes of compositions. Finally, we consider two related problems: one where the subscript restriction applies only to parts within a given modular class and another where the subscript of a part belongs to the same modular class mod ℓ as the part where ℓ is fixed.

1 Introduction

A *composition* of a positive integer ν is a sequence of positive integers $\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_r\}$ such that $\sigma_1 + \sigma_2 + \cdots + \sigma_r = \nu$. The summands σ_i are called the *parts* of σ and ν is the *weight* of σ. For example, the compositions of 4 are

$$\{4\},\ \{3,1\},\ \{1,3\},\ \{2,2\},\ \{2,1,1\},\ \{1,2,1\},\ \{1,1,2\},\ \{1,1,1,1\}.$$

Agarwal [1] introduced a generalization of the concept of a composition known as an *n-color composition* wherein a part of size $m \geq 1$ can come in one of m different colors. The colors of the part m are denoted by subscripts $m_1, m_2, \ldots, m_m$. For example, the n-color compositions of 4 are

$$\{4_1\},\ \{4_2\}, \{4_3\}, \{4_4\}, \{3_1, 1_1\}, \{3_2, 1_1\}, \{3_3, 1_1\}, \{1_1, 3_1\}, \{1_1, 3_2\}, \{1_1, 3_3\}, \{2_1, 2_1\},$$
$$\{2_1, 2_2\}, \{2_2, 2_1\}, \{2_2, 2_2\}, \{2_1, 1_1, 1_1\}, \{2_2, 1_1, 1_1\}, \{1_1, 2_1, 1_1\}, \{1_1, 2_2, 1_1\}, \{1_1, 1_1, 2_1\},$$
$$\{1_1, 1_1, 2_2\}, \{1_1, 1_1, 1_1, 1_1\}.$$

It is well-known that the total number of n-color compositions of ν is given by the Fibonacci number $F_{2\nu}$. Moreover, the number of n-color compositions of ν with exactly m parts is the binomial coefficient $\binom{\nu+m-1}{2m-1}$. For further results about n-color compositions, see, e.g., [1, 2, 4, 6, 7, 9, 10, 11, 13, 14, 15]. In this paper, we study some new restrictions on n-color compositions that generalize previous results given by Sachdeva and Agarwal [13].

The organization of this paper is as follows. In the next section, we count the members of $\mathcal{C}(\nu)$ in which the subscripts on all parts are of the form $\ell a + b$ for some $a \geq 0$, where $b, \ell \geq 1$ are fixed, providing generating function and explicit formulas. This extends recent work [13] in the case $\ell = 2$. We consider further the case $\ell = b - 1$, which yields several previously studied sequences from [16], and find bijections between various restricted classes of binary words and compositions and the corresponding subset of $\mathcal{C}(\nu)$. In the third section, we count members of $\mathcal{C}(\nu)$ in which only parts of the form $\ell a + b$ for some $a \geq 0$ satisfy a similar modular requirement with respect to their subscripts. An explicit formula for the generating function is found which extends prior results [13]. Finally, a comparable formula can be given which counts members of $\mathcal{C}(\nu)$ in which parts of the form $\ell a + b$ where $a \geq 0$ and $1 \leq b \leq \ell$ must have subscripts of the same form.

2 Generalized restricted n-color compositions

Given positive integers ℓ and b, let $\mathcal{C}_{\ell a+b}(\nu)$ denote the number of n-color compositions of ν into parts with subscripts of the form $\ell a + b$ for some integer $a \geq 0$. We also denote by $\mathcal{C}_{\ell a+b}(m, \nu)$ the number of n-color compositions of ν into m parts with subscripts of the form $\ell a + b$.

For example, $\mathcal{C}_{3a+1}(4) = 9$, the compositions being

$$\{4_1\}, \{4_4\}, \{3_1, 1_1\}, \{1_1, 3_1\}, \{2_1, 2_1\}, \{2_1, 1_1, 1_1\}, \{1_1, 2_1, 1_1\}, \{1_1, 1_1, 2_1\}, \{1_1, 1_1, 1_1, 1_1\}.$$

Theorem 1. *Let $\mathcal{GC}_{\ell a+b}(m, x)$ and $\mathcal{GC}_{\ell a+b}(x)$ denote the generating functions for the sequences $\mathcal{C}_{\ell a+b}(m, \nu)$ and $\mathcal{C}_{\ell a+b}(\nu)$, respectively. Then we have*

$$\mathcal{GC}_{\ell a+b}(m, x) = \left(\frac{x^b}{(1-x)(1-x^\ell)}\right)^m,$$

$$\mathcal{GC}_{\ell a+b}(x) = \frac{x^b}{1 - x - x^\ell + x^{\ell+1} - x^b}.$$

Proof. Let $\sigma = \sigma_1 \cdots \sigma_m$ be a non-empty n-color composition having m parts where each subscript is of the form $\ell a + b$ for some $a \geq 0$. If $\sigma_j = i$ with $i \geq b$, then σ_j contributes to the generating function the term $w_i x^i$, where

$$w_i = \left\lfloor \frac{i - b + \ell}{\ell} \right\rfloor,$$

while if $i < b$, then it fails to contribute.

Note that the generating function of the sequence

$$\{w_i\}_{i\geq 0} = \left\{ \underbrace{0, \ldots, 0}_{b}, \underbrace{1, \ldots, 1}_{\ell}, \underbrace{2, \ldots, 2}_{\ell}, \ldots \right\}$$

is given by

$$\frac{x^b}{(1-x)(1-x^\ell)}.$$

Therefore,

$$\mathcal{GC}_{\ell a+b}(m, x) = \left(\sum_{i\geq 0} w_i x^i\right)^m = \left(\frac{x^b}{(1-x)(1-x^\ell)}\right)^m.$$

Finally, summing the last expression over $m \geq 1$, we get

$$\mathcal{GC}_{\ell a+b}(x) = \frac{\frac{x^b}{(1-x)(1-x^\ell)}}{1 - \frac{x^b}{(1-x)(1-x^\ell)}} = \frac{x^b}{1 - x - x^\ell + x^{\ell+1} - x^b}.$$

□

We have the following combinatorial formula for the sequence $\mathcal{C}_{\ell a+b}(m, \nu)$.

Theorem 2. *The sequence $\mathcal{C}_{\ell a+b}(m, \nu)$ is given by the expression*

$$\mathcal{C}_{\ell a+b}(m, \nu) = \sum_{i=0}^{\lfloor \frac{\nu - bm}{\ell} \rfloor} \binom{m+i-1}{m-1} \binom{\nu - \ell i + m(1-b) - 1}{m-1}.$$

Moreover, $\mathcal{C}_{\ell a+b}(\nu) = \mathcal{C}_{\ell a+b}(\nu - 1) + \mathcal{C}_{\ell a+b}(\nu - \ell) - \mathcal{C}_{\ell a+b}(\nu - \ell - 1) + \mathcal{C}_{\ell a+b}(\nu - b)$ *when* $\nu > \max\{\ell + 1, b\}$.

Proof. By Theorem 1, we have

$$\begin{aligned}\mathcal{GC}_{\ell a+b}(m, x) &= \left(\frac{x^b}{(1-x)(1-x^\ell)} \right)^m \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \binom{m+i-1}{i} \binom{m+j-1}{j} x^{j+i\ell+bm}.\end{aligned}$$

Taking $t = j + \ell i + bm$ gives

$$\mathcal{GC}_{\ell a+b}(m, x) = \sum_{i=0}^{\infty} \sum_{t=i\ell+bm}^{\infty} \binom{m+i-1}{m-1} \binom{t - \ell i + m(1-b) - 1}{m-1} x^t.$$

By comparing the ν-th coefficient of both sides of the last equation, we obtain the desired result. The recurrence relation follows from the generating function formula for $\mathcal{GC}_{\ell a+b}(x)$ given in Theorem 1. □

Remark 3. Setting $\ell = b = 1$ in Theorem 2, and using the binomial identity [5, Formula 5.26], recovers the fact that there are $\binom{\nu+m-1}{2m-1}$ n-color compositions of ν with exactly m parts and thus $F_{2\nu}$ altogether with no restriction as to the number of parts.

By setting $\ell = 2$ and $b = 1$, we have the following corollary (see Theorem 2.1 of [13]).

Corollary 4. *The generating functions for the number of n-color compositions of ν into m parts with odd subscripts and for the total number of n-color compositions of ν with odd subscripts are*

$$\begin{aligned}\mathcal{GC}_{2a+1}(m, x) &= \left(\frac{x}{(1-x)(1-x^2)} \right)^m = \left(\frac{x}{(1+x)(1-x)^2} \right)^m, \\ \mathcal{GC}_{2a+1}(x) &= \frac{x}{1 - 2x - x^2 + x^3}.\end{aligned}$$

Moreover,

$$\mathcal{C}_{2a+1}(m,\nu) = \sum_{i=0}^{\lfloor \frac{\nu-m}{2} \rfloor} \binom{m+i-1}{m-1}\binom{\nu-2i-1}{m-1}$$

and $\mathcal{C}_{2a+1}(\nu) = 2\mathcal{C}_{2a+1}(\nu-1) + \mathcal{C}_{2a+1}(\nu-2) - \mathcal{C}_{2a+1}(\nu-3)$ *for* $\nu > 3$*, with the initial values* $\mathcal{C}_{2a+1}(1) = 1, \mathcal{C}_{2a+1}(2) = 2, \mathcal{C}_{2a+1}(3) = 5$*.*

Letting $\ell = 2$ and $b = 2$ yields the following corollary (see Theorem 2.3 of [13]).

Corollary 5. *The generating functions for the number of n-color compositions of* ν *into* m *parts with even subscripts and for the total number of n-color compositions of* ν *with even subscripts are*

$$\mathcal{GC}_{2a+2}(m,x) = \left(\frac{x^2}{(1-x)(1-x^2)}\right)^m = \left(\frac{x^2}{(1+x)(1-x)^2}\right)^m,$$
$$\mathcal{GC}_{2a+2}(x) = \frac{x^2}{1-x-2x^2+x^3}.$$

Moreover,

$$\mathcal{C}_{2a+2}(m,\nu) = \sum_{i=0}^{\lfloor \frac{\nu-2m}{2} \rfloor} \binom{m+i-1}{m-1}\binom{\nu-2i-m-1}{m-1}$$

and $\mathcal{C}_{2a+2}(\nu) = \mathcal{C}_{2a+2}(\nu-1) + 2\mathcal{C}_{2a+2}(\nu-2) - \mathcal{C}_{2a+2}(\nu-3)$ *for* $\nu > 3$*, with the initial values* $\mathcal{C}_{2a+2}(1) = 0, \mathcal{C}_{2a+2}(2) = 1, \mathcal{C}_{2a+2}(3) = 1$*.*

Letting $\ell = 2$ and $b = 3$ yields the further corollary (see Theorem 2.2 of [13]).

Corollary 6. *The generating functions for the number of n-color compositions of* ν *into* m *parts with odd subscripts* > 1 *and for the total number of n-color compositions of* ν *with odd subscripts* > 1 *are*

$$\mathcal{GC}_{2a+3}(m,x) = \left(\frac{x^3}{(1+x)(1-x)^2}\right)^m,$$
$$\mathcal{GC}_{2a+3}(x) = \frac{x^3}{1-x-x^2}.$$

Moreover,

$$\mathcal{C}_{2a+3}(m,\nu) = \sum_{i=0}^{\lfloor \frac{\nu-3m}{2} \rfloor} \binom{m+i-1}{m-1}\binom{\nu-2i-2m-1}{m-1}$$

and $\mathcal{C}_{2a+3}(\nu) = \mathcal{C}_{2a+3}(\nu-1) + \mathcal{C}_{2a+3}(\nu-2)$ *for* $\nu > 3$*, with the initial values* $\mathcal{C}_{2a+3}(1) = 0, \mathcal{C}_{2a+3}(2) = 0, \mathcal{C}_{2a+3}(3) = 1$*.*

ℓ	b	Sequence $\mathcal{C}_{\ell a+b}(\nu)$	A-Sequence
3	1	1, 2, 4, 9, 19, 40, 85, 180, 381, 807, 1709, 3619, 7664, 16230, 34370	A052908
3	2	1, 1, 2, 4, 6, 11, 19, 32, 56, 96, 165, 285, 490, 844, 1454, 2503, 4311	A116732
3	3	1, 1, 1, 3, 4, 5, 10, 15, 21, 36, 56, 83, 134, 210, 320, 505, 791, 1221	A176848

Table 1: Some particular cases for $\ell = 3$.

When $\ell = 3$, we obtain some known sequences from the OEIS [16]. In Table 1, we give the first several non-zero values.

Note that the sequence A052908 does not have a combinatorial interpretation listed. For the sequence A116732, our combinatorial interpretation differs from the one given. Let $\mathcal{A}$ be the set of compositions with parts in $\{1, 2, 3\}$ such that the order of adjacent 1's and 3's is unimportant. Let $a(n)$ be the number of elements in $\mathcal{A}$ of weight n. For example, $a(6) = 19$, where the compositions are

$\{3, 3\}, \{3, 2, 1\}, \{3, 1, 2\}, \{2, 3, 1\}, \{1, 2, 3\}, \{3, 1, 1, 1\}, \{2, 2, 2\}, \{2, 2, 1, 1\}, \{2, 1, 2, 1\}, \{2, 1, 1, 2\}, \{1, 2, 2, 1\}, \{1, 2, 1, 2\}, \{1, 1, 2, 2\}, \{2, 1, 1, 1, 1\}, \{1, 2, 1, 1, 1\}, \{1, 1, 2, 1, 1\}, \{1, 1, 1, 2, 1\}, \{1, 1, 1, 1, 2\}, \{1, 1, 1, 1, 1, 1\}$.

Theorem 7. *For $n \geq 0$, $a(n) = \mathcal{C}_{3a+2}(n + 2)$.*

Proof. Let w be a composition in $\mathcal{A}$. Then w is either an integer partition (non-ordered composition) with parts in $\{1, 3\}$ or can be factorized as $p\,2\,w'$, where p is a partition with parts in $\{1, 3\}$ and $w' \in \mathcal{A}$. Thus, the generating function $A(x)$ of the sequence $a(n)$ satisfies the relation

$$A(x) = P_{1,3}(x) + P_{1,3}(x)x^2A(x),$$

where $P_{1,3}(x)$ counts integer partitions with parts in $\{1, 3\}$. Since

$$P_{1,3}(x) = \frac{1}{(1 - x)(1 - x^3)},$$

we have

$$A(x) = \frac{1}{1 - x - x^2 - x^3 + x^4}.$$

Finally, by Theorem 1,

$$\mathcal{GC}_{3a+2}(x) = x^2A(x),$$

which yields the desired result upon comparing n-th coefficients. □

Let $b(n)$ be the number of compositions of n where each part of size j for $j \geq 1$ comes in $\lfloor j/3 \rfloor$ kinds (sequence A176848). For example, $b(7) = 4$, the enumerated compositions being $\{7_x\}, \{7_y\}, \{3_x, 4_x\}, \{4_x, 3_x\}$. It is clear from the definitions that $b(n) = \mathcal{C}_{3a+3}(n)$ for $n \geq 1$.

We now give a bijective proof of the prior theorem.

Combinatorial proof of Theorem 7.

Let $\mathcal{A}_n$ and $\mathcal{C}_n$ denote the set of compositions enumerated by $a(n)$ and $\mathcal{C}_{3a+2}(n)$, respectively. We will define a bijection between $\mathcal{A}_n$ and $\mathcal{C}_{n+2}$ for $n \geq 0$. Let us assume that 3 always precedes 1 whenever there is an adjacency of the two letters within a member of $\mathcal{A}_n$. Let $\lambda \in \mathcal{A}_n$. First assume λ contains no 2's. Then we may write $\lambda = 3^i 1^j$, where $i, j \geq 0$ with $3i + j = n$. In this case, we map λ to the colored composition $\lambda' = (3i + j + 2)_{3i+2}$ of $n + 2$ containing a single part. So assume λ contains at least one 2, in which case we may write

$$\lambda = 3^{i_0} 1^{j_0} 2^{a_1} 3^{i_1} 1^{j_1} 2^{a_2} 3^{i_2} 1^{j_2} \cdots 2^{a_r} 3^{i_r} 1^{j_r},$$

where all exponents are non-negative, $r \geq 1$, $a_1, \ldots, a_r \geq 1$, and $i_k + j_k \geq 1$ for $1 \leq k \leq r-1$. In this case, we let

$$\lambda' = (3i_0 + j_0 + 2)_{3i_0+2}, (2_2)^{a_1-1}, (3i_1 + j_1 + 2)_{3i_1+2}, \ldots, (2_2)^{a_r-1}, (3i_r + j_r + 2)_{3i_r+2},$$

where $(2_2)^t$ denotes a run of the part 2_2 of length t.

Note that λ' contains $r + 1$ parts and indeed belongs to $\mathcal{C}_{n+2}$. Also, while it is possible for the first or the last part of λ' to be 2_2, all parts of the form $(3i_k + j_k + 2)_{3i_k+2}$ where $1 \leq k \leq r - 1$ are greater than 2. Furthermore, since $j_k \geq 0$ for $0 \leq k \leq r$, arbitrary differences can occur between the part sizes and subscripts. Thus, the mapping $\lambda \mapsto \lambda'$ may be reversed and hence is a bijection between $\mathcal{A}_n$ and $\mathcal{C}_{n+2}$, as desired, upon decomposing members of $\mathcal{C}_{n+2}$ in the same way λ' was above. □

2.1 The case $\ell = b - 1$

In this subsection, we provide additional combinatorial interpretations for the sequence $\mathcal{C}_{\ell a+\ell+1}(n)$, where $\ell \geq 1$. In Table 2, we give the first several non-zero values of these sequences for $2 \leq \ell \leq 6$.

ℓ	b	Sequence $\mathcal{C}_{\ell a+b}(\nu)$	A-Sequence
2	3	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597	A000045
3	4	1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88, 129, 189, 277, 406, 595	A000930
4	5	1, 1, 1, 1, 2, 3, 4, 5, 7, 10, 14, 19, 26, 36, 50, 69, 95, 131, 181, 250	A003269
5	6	1, 1, 1, 1, 1, 2, 3, 4, 5, 6, 8, 11, 15, 20, 26, 34, 45, 60, 80, 106, 140	A003520
6	7	1, 1, 1, 1, 1, 1, 2, 3, 4, 5, 6, 7, 9, 12, 16, 21, 27, 34, 43, 55, 71, 92	A005708

Table 2: Some particular cases of $\ell = b - 1$.

Let $F_\ell(n) := \mathcal{C}_{\ell a+\ell+1}(n)$. By Theorem 1, we have

$$F_\ell(x) := \sum_{n=0}^{\infty} F_\ell(n) x^n = \frac{x^{\ell+1}}{1 - x - x^\ell}.$$

Moreover, $F_\ell(n) = F_\ell(n-1) + F_\ell(n-\ell)$ for $n > \ell+1$, with the initial values $F_\ell(\ell+1) = 1$ and $F_\ell(n) = 0$ for $n \in [\ell] = \{1, 2, \ldots, \ell\}$. For $\ell = 2$, it is clear that the sequence $F_2(n)$ coincides with the Fibonacci numbers, i.e., $F_2(n) = F_{n-2}$ for $n \geq 2$. Moreover, $F_3(n)$ is seen to correspond to the Narayana sequence (cf. [12]).

Let $\mathcal{E}_\ell$ be the set of compositions into parts 1 and ℓ, where $\ell \geq 2$. Let $e_\ell(n)$ denote the number of elements in $\mathcal{E}_\ell$ of weight n. Chinn and Heubach [3] studied this family of compositions and, in particular, found

$$E_\ell(x) := \sum_{n=0}^{\infty} e_\ell(n)x^n = \frac{1}{1-x-x^\ell}.$$

Then $x^{\ell+1}E_\ell(x) = F_\ell(x)$ and we have the following result.

Theorem 8. *For $n \geq 0$, $F_\ell(n+\ell+1) = e_\ell(n)$.*

Let $\mathcal{H}_\ell$ be the set of compositions into parts greater than or equal to ℓ. Let $h_\ell(n)$ be the number of elements in $\mathcal{H}_\ell$ of weight n. It is not difficult to show that (see, for example, [8, Theorem 3.13])

$$H_\ell(x) := \sum_{n=0}^{\infty} h_\ell(n)x^n = \frac{1}{1-(x^\ell + x^{\ell+1} + \cdots)} = \frac{1-x}{1-x-x^\ell}.$$

Therefore, we have the following relation.

Theorem 9. *For $n \geq 1$, $F_\ell(n+1) = h_\ell(n)$.*

Let $\mathcal{G}_\ell$ be the set of binary words such that between any two successive ones there are at least $\ell-1$ zeros. Let $g_\ell(n)$ be the number of words in $\mathcal{G}_\ell$ of length n. Let w be a binary word in $\mathcal{G}_\ell$ of length $n > \ell$. Then w can be decomposed as $w = 0w_1$ or $w = 1\underbrace{0\cdots0}_{\ell-1}w_2$, where $w_1, w_2 \in \mathcal{G}_\ell$, which implies $g_\ell(n) = g_\ell(n-1) + g_\ell(n-\ell)$ for all $n > \ell$. Thus, this sequence satisfies the same recurrence relation as $F_\ell(n)$. Note that $g_\ell(n) = n+1$ if $n \in [\ell]$, which follows from the definitions. Since $F_\ell(n+\ell) = 1$ if $n \in [\ell]$, applying the recurrence for $F_\ell(n)$ implies $F_\ell(n+2\ell) = n+1$ for $n \in [\ell]$. Comparing the recurrences and initial values gives the following relation.

Theorem 10. *For $n \geq 0$, $F_\ell(n+2\ell) = g_\ell(n)$.*

We conclude this section by providing bijective proofs of the last three results.

Combinatorial proofs of Theorems 8 and 9.

Let $\mathcal{E}_\ell(n)$ denote the set of compositions of n with parts 1 and ℓ and $\mathcal{F}_\ell(n)$ the set of colored compositions enumerated by $F_\ell(n)$. We define a mapping $f : \mathcal{E}_\ell(n) \to \mathcal{F}_\ell(n+\ell+1)$ as follows. If $\lambda = 1^{n-b\ell}\ell^b$, where $0 \leq b \leq \lfloor n/\ell \rfloor$, then let $f(\lambda) = ((b+1)\ell + n - b\ell + 1)_{(b+1)\ell+1}$. Otherwise, we have

$$\lambda = 1^{a_0}\ell^{b_1}1^{a_1}\cdots\ell^{b_r}1^{a_r}\ell^{b_{r+1}},$$

where $r \geq 1$, $a_0 \geq 0$, $a_i, b_i \geq 1$ if $1 \leq i \leq r$ and $b_{r+1} \geq 0$. In this case, let

$$f(\lambda) = (b_1\ell + a_0 + 1)_{b_1\ell+1}, (b_2\ell + a_1)_{b_2\ell+1}, \ldots, (b_r\ell + a_{r-1})_{b_r\ell+1}, ((b_{r+1}+1)\ell + a_r)_{(b_{r+1}+1)\ell+1}.$$

Note that $f(\lambda)$ contains $r+1$ parts and indeed belongs to $\mathcal{F}_\ell(n+\ell+1)$ (a 1 not accounted for by λ occurs in the first part and there is an extra ℓ in the last part). Observe further that the last part of $f(\lambda)$ has subscript greater than or equal to $\ell+1$ depending on whether the last part of λ is ℓ or 1. Upon considering the number of parts in a member of $\mathcal{F}_\ell(n+\ell+1)$, the mapping f is seen to be reversible and hence yields the desired bijection.

To show Theorem 9, let $\mathcal{H}_\ell(n)$ denote the set of compositions of n having parts of size ℓ or more. We define $g : \mathcal{H}_\ell(n) \to \mathcal{F}_\ell(n+1)$ for $n \geq 1$ as follows. If $n \in [\ell-1]$, then both sets are empty, so assume $n \geq \ell$. Then we may express $\lambda \in \mathcal{H}_\ell(n)$ as

$$\lambda = x_1\ell^{a_1}x_2\ell^{a_2}\cdots x_r\ell^{a_r},$$

where $r \geq 1$, $x_1 \geq \ell$, $x_i \geq \ell+1$ if $i > 1$ and $a_i \geq 0$ for all i. Let

$$g(\lambda) = (a_1\ell + x_1 + 1)_{(a_1+1)\ell+1}, (a_2\ell + x_2)_{(a_2+1)\ell+1}, \ldots, (a_r\ell + x_r)_{(a_r+1)\ell+1}.$$

One may verify that the mapping g is a bijection, which completes the proof. □

Combinatorial proof of Theorem 10.

Let $\mathcal{G}_\ell(n)$ denote the set of binary words enumerated by $g_\ell(n)$. We define a mapping $f : \mathcal{G}_\ell(n) \to \mathcal{F}_\ell(n+2\ell)$ in several steps as follows. Let $\lambda = \lambda_1\lambda_2\cdots\lambda_n \in \mathcal{G}_\ell(n)$ and first assume $n \in [\ell]$. In this case, let

$$f(\lambda) = \begin{cases} (n+2\ell)_{\ell+1}, & \text{if } \lambda = 0^n; \\ (n-s+\ell)_{\ell+1}, (s+\ell)_{\ell+1}, & \text{if } \lambda = 0^s10^{n-1-s}, \text{ where } 1 \leq s \leq n-1; \\ (n+2\ell)_{2\ell+1}, & \text{if } \lambda = 10^{n-1}. \end{cases}$$

Henceforth, assume $n > \ell$. We will also assume $\ell > 1$, as the adjustments necessary in the $\ell = 1$ case will be apparent. Note that $\lambda \in \mathcal{G}_\ell(n)$ may start with an initial (possibly empty) run of 0's with the remainder of λ being decomposed into sections of the form $u = 10^{\ell-1}$ (1 followed by $\ell-1$ 0's) and $v = 10^{m-1}$ where $m \geq \ell+1$ is arbitrary (to be specified). Furthermore, it is possible for λ to end in a section w of the form $w = 10^p$, where $0 \leq p \leq \ell-2$.

First assume λ contains no section of the form v above. Then either

$$\lambda = 0^{n-i\ell}u^i, \qquad 0 \leq i \leq \lfloor n/\ell \rfloor, \tag{1}$$

or

$$\lambda = 0^{n-p-1-i\ell}u^iw, \qquad 0 \leq p \leq \ell-2 \text{ and } 0 \leq i \leq \lfloor (n-p-1)/\ell \rfloor, \tag{2}$$

where $w = 10^p$. We define f in this case by considering whether or not n is divisible by ℓ. If ℓ divides n, then let $f(\lambda) = (n+2\ell)_{(i+1)\ell+1}$, if λ is of the form (1), and let

$$f(\lambda) = (\ell+p+1)_{\ell+1}, ((i+1)\ell + n - p - 1 - i\ell)_{(i+1)\ell+1},$$

if of form (2). If ℓ does not divide n, then we define $f(\lambda)$ the same way as before provided λ is not of the form (2) with $n-p-1=i\ell$. Note that $n-p-1=i\ell$ corresponds to exactly one λ in (2) since $0\le p\le \ell-2$. We set $f(\lambda)=(n+2\ell)_{q\ell+1}$ in this case where $q=\lfloor n/\ell\rfloor+2$ (note that $q\ell+1\le n+2\ell$ if and only if ℓ does not divide n). Observe that in either case f maps the members of $\mathcal{G}_\ell(n)$ not containing a v section in a one-to-one manner to the subset of $\mathcal{F}_\ell(n+2\ell)$ whose members either have one part or have two parts where the first part is less than 2ℓ.

Assume henceforth that λ contains at least one section of the form v above. Then we may write

$$\lambda = 0^j u^{i_1} v_1 \cdots u^{i_r} v_r u^{i_{r+1}}, \tag{3}$$

where $r\ge 1$, $j,i_1,\ldots,i_{r+1}\ge 0$, and $v_i=10^{m_i-1}$ with $m_i\ge \ell+1$ for $1\le i\le r$, or

$$\lambda = 0^j u^{i_1} v_1 \cdots u^{i_r} v_r u^{i_{r+1}} w, \tag{4}$$

with all the same restrictions as before and $w=10^p$ for some $0\le p\le \ell-2$. If λ is of the form (3), then let

$$f(\lambda)=((i_1+2)\ell+j)_{(i_1+1)\ell+1},(i_2\ell+m_1)_{(i_2+1)\ell+1},\ldots,(i_{r+1}\ell+m_r)_{(i_{r+1}+1)\ell+1}.$$

Observe that $r\ge 1$ implies $f(\lambda)$ contains at least two parts in this case and $m_i\ge \ell+1$ for all i implies the size of the part always exceeds the size of the subscript (with the first part of size at least 2ℓ).

Now suppose λ is of form (4). To define f, we consider cases on j. If $j\ge 1$ in (4), then let

$$f(\lambda)=(\ell+p+1)_{\ell+1},((i_1+1)\ell+j)_{(i_1+1)\ell+1},(i_2\ell+m_1)_{(i_2+1)\ell+1},\ldots,(i_{r+1}\ell+m_r)_{(i_{r+1}+1)\ell+1}.$$

Note $f(\lambda)$ here must contain at least three parts and therefore this covers the remaining cases where the first part is less than 2ℓ. If $j=0$ in (4), then let

$$f(\lambda)=((i_1+2)\ell+p+1)_{(i_1+2)\ell+1},(i_2\ell+m_1)_{(i_2+1)\ell+1},\ldots,(i_{r+1}\ell+m_r)_{(i_{r+1}+1)\ell+1}.$$

Notice that this covers the remaining $\rho\in\mathcal{F}_\ell(n+2\ell)$ in which the first part of ρ is at least 2ℓ with ρ containing at least two parts. The inverse of f can then be constructed (we leave the details to the reader) in a composite manner in much the same way as f was above upon considering the number of parts and whether or not the first part is at least 2ℓ. □

3 Subscript restrictions only on certain parts

Given integers $\ell,\ell',b,b'\ge 1$, let $\mathcal{D}_{\ell a+b}^{\ell' a'+b'}(\nu)$ denote the number of n-color compositions of ν such that the parts of the form $\ell a+b$ for some $a\ge 0$ have only subscripts of the form $\ell' a'+b'$ for some $a'\ge 0$. Additionally, we denote by $\mathcal{D}_{\ell a+b}^{\ell' a'+b'}(m,\nu)$ the number of such n-color compositions of ν that have exactly m parts.

For example, $\mathcal{D}_{4a+3}^{3a'+1}(3)=6$, the compositions being

$$\{3_1\},\{2_1,1_1\},\{2_2,1_1\},\{1_1,2_1\},\{1_1,2_2\},\{1_1,1_1,1_1\}.$$

Theorem 11. *Let* $\mathcal{GD}_{\ell a+b}^{\ell' a'+b'}(m,x)$ *and* $\mathcal{GD}_{\ell a+b}^{\ell' a'+b'}(x)$ *denote the generating functions for the sequences* $\mathcal{D}_{\ell a+b}^{\ell' a'+b'}(m,\nu)$ *and* $\mathcal{D}_{\ell a+b}^{\ell' a'+b'}(\nu)$*, respectively. Then we have*

$$\mathcal{GD}_{\ell a+b}^{\ell' a'+b'}(m,x) = \left(x^b H(x^\ell) + P(x) + \sum_{i=1,\ i\not\equiv b \pmod{\ell}}^{\ell} \frac{i+(\ell-i)x^\ell}{(1-x^\ell)^2} x^i\right)^m,$$

$$\mathcal{GD}_{\ell a+b}^{\ell' a'+b'}(x) = \frac{x^b H(x^\ell) + P(x) + \sum_{i=1,\ i\not\equiv b \pmod{\ell}}^{\ell} \frac{i+(\ell-i)x^\ell}{(1-x^\ell)^2} x^i}{1-\left(x^b H(x^\ell) + P(x) + \sum_{i=1,\ i\not\equiv b \pmod{\ell}}^{\ell} \frac{i+(\ell-i)x^\ell}{(1-x^\ell)^2} x^i\right)},$$

where $H(x)$ *is the generating function of the sequence*

$$h_n = \begin{cases} \lfloor \frac{\ell n+b-b'}{\ell'} \rfloor + 1, & \text{if } \ell n + b \geq b' \text{ and } n \geq 0; \\ 0, & \text{otherwise,} \end{cases}$$

and $P(x)$ *is the polynomial given by*

$$P(x) = \sum_{\substack{i\equiv b \pmod{\ell} \\ 0\leq i<b}} i x^i.$$

Proof. Summing the first expression over $m \geq 1$ gives the second, so we need only prove the first. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_m$ be a non-empty n-color composition having m parts such that parts of the form $\ell a + b$ where $a \geq 0$ have only subscripts of the form $\ell' a' + b'$ where $a' \geq 0$. First assume $\sigma_j \equiv b \pmod{\ell}$ and suppose $\sigma_j = r = \ell a + b$. If $a \geq 0$ and $r \geq b'$, then σ_j contributes to the generating function a $w_a x^r$ term, where

$$w_a = \left\lfloor \frac{\ell a + b - b'}{\ell'} \right\rfloor + 1.$$

If $a \geq 0$ and $r < b'$, then there are no possible such parts for otherwise the index would exceed the part (note that this case can occur only if $b < b'$).

If $a < 0$, then $\sigma_j = r < b$ and there is a contribution to the generating function of rx^r per the definitions, and combining all such r yields the polynomial $P(x)$ defined above. If $\sigma_j \not\equiv b \pmod{\ell}$, then there is again a contribution of rx^r. Thus, for each $i \in [\ell]$ such that $i \not\equiv b \pmod{\ell}$, we have a total contribution of

$$\begin{aligned} &ix^i + (\ell+i)x^{\ell+i} + (2\ell+i)x^{2\ell+i} + \cdots \\ &= ix^i(1 + x^\ell + x^{2\ell} + \cdots) + \ell x^i(x^\ell + 2x^{2\ell} + 3x^{3\ell} + \cdots) \\ &= \frac{ix^i}{1-x^\ell} + \ell x^i \left[\frac{y}{(1-y)^2}\right]_{y=x^\ell} = \frac{ix^i}{1-x^\ell} + \frac{\ell x^{i+\ell}}{(1-x^\ell)^2}, \end{aligned}$$

which gives the final part of the formula for $\mathcal{GD}_{\ell a+b}^{\ell' a'+b'}(m,x)$ above. □

For example, the generating function for the sequence $\mathcal{D}_{3a+7}^{4a'+3}(m,\nu)$ is given by

$$\mathcal{GD}_{3a+7}^{4a'+3}(m,x) = \left(x^7 H(x^3) + x + 4x^4 + \frac{(2+x^3)}{(1-x^3)^2}x^2 + \frac{3}{(1-x^3)^2}x^3\right)^m,$$

where $H(x) = \frac{2+x^2+x^3-x^4}{(1-x)^2(1+x+x^2+x^3)}$. Note that $H(x)$ is the generating function for the sequence

$$\{2,2,3,4,5,5,6,7,8,8,9,10,11,11,12,13,14,14,15,16,17,17,18,19,20,20,21,22,\dots\}.$$

Moreover,

$$\begin{aligned}
&\mathcal{GD}_{3a+7}^{4a'+3}(x)\\
&= \frac{-3x^{19}+2x^{16}-x^{14}-3x^{12}-3x^{11}-3x^9-3x^8+2x^7-3x^6-3x^5-3x^4-3x^3-2x^2-x}{3x^{19}-2x^{16}-x^{15}+x^{14}+4x^{12}+3x^{11}+3x^9+3x^8-2x^7+3x^6+3x^5+3x^4+4x^3+2x^2+x-1}\\
&= x+3x^2+8x^3+21x^4+55x^5+144x^6+372x^7+977x^8+2549x^9+6647x^{10}+\cdots.
\end{aligned}$$

For example, $\mathcal{D}_{3a+7}^{4a'+3}(7) = 372$, as all n-color compositions of $n=7$ are counted except

$$\{7_1\},\{7_2\},\{7_4\},\{7_5\},\{7_6\}.$$

Remark 12. Taking all of the relevant parameters to be one in Theorem 11 gives

$$\mathcal{GD}_{a+1}^{a'+1}(m,x) = \frac{x^m}{(1-x)^{2m}}, \qquad m \ge 1,$$

and

$$\mathcal{GD}_{a+1}^{a'+1}(x) = \frac{x}{1-3x+x^2},$$

which are the generating functions for the number with m parts and the total number of n-color compositions of ν for $\nu \ge 1$, respectively.

By setting $\ell = 2 = \ell'$ in Theorem 11, we have the following corollaries.

Corollary 13 (Theorem 2.4 of [13])**.** *The generating functions for the number of n-color compositions of ν into m parts such that the odd parts have only even subscripts and for the total number of n-color compositions of ν such that the odd parts have only even subscripts are*

$$\mathcal{GD}_{2a+1}^{2a'+2}(m,x) = \left(\frac{2x^2+x^3}{(1-x^2)^2}\right)^m,$$

$$\mathcal{GD}_{2a+1}^{2a'+2}(x) = \frac{2x^2+x^3}{1-4x^2-x^3+x^4}.$$

Corollary 14 (Theorem 2.5 of [13])**.** *The generating functions for the number of n-color compositions of ν into m parts such that the odd parts have only odd subscripts and for the* *total number of n-color compositions of ν such that the odd parts have only odd subscripts are*

$$\mathcal{GD}_{2a+1}^{2a'+1}(m,x) = \left(\frac{x+2x^2}{(1-x^2)^2}\right)^m,$$
$$\mathcal{GD}_{2a+1}^{2a'+1}(x) = \frac{x+2x^2}{1-x-4x^2+x^4}.$$

Corollary 15 (Theorem 2.6 of [13])**.** *The generating functions for the number of n-color compositions of ν into m parts such that the even parts have only even (odd) subscripts and for the total number of n-color compositions of ν such that the even parts have only even (odd) subscripts are*

$$\mathcal{GD}_{2a+2}^{2a'+2}(m,x) = \mathcal{GD}_{2a+2}^{2a'+1}(m,x) = \left(\frac{x+x^2+x^3}{(1-x^2)^2}\right)^m,$$
$$\mathcal{GD}_{2a+2}^{2a'+2}(x) = \mathcal{GD}_{2a+2}^{2a'+1}(x) = \frac{x+x^2+x^3}{1-x-3x^2-x^3+x^4}.$$

4 A further related restriction

Given $\ell \geq 1$, let $\mathcal{T}_\ell(\nu)$ denote the number of n-color compositions of ν such that any part of the form $\ell a + b$ for some $a \geq 0$ and $1 \leq b \leq \ell$ has a subscript of the same form. Additionally, we denote by $\mathcal{T}_\ell(m,\nu)$ the number of such n-color compositions of ν that have m parts.

For example, $\mathcal{T}_4(5) = 17$, the compositions being

$$\{5_1\}, \{5_5\}, \{4_4, 1_1\}, \{1_1, 4_4\}, \{3_3, 2_2\}, \{2_2, 3_3\}, \{3_3, 1_1, 1_1\}, \{1_1, 3_3, 1_1\}, \{1_1, 1_1, 3_3\},$$
$$\{2_2, 2_2, 1_1\}, \{2_2, 1_1, 2_2\}, \{1_1, 2_2, 2_2\}, \{2_2, 1_1, 1_1, 1_1\}, \{1_1, 2_2, 1_1, 1_1\}, \{1_1, 1_1, 2_2, 1_1\},$$
$$\{1_1, 1_1, 1_1, 2_2\}, \{1_1, 1_1, 1_1, 1_1, 1_1\}.$$

Similar to the proof of Theorems 1 and 2 above, we have the following result.

Theorem 16. *Let $\mathcal{GT}_\ell(m,x)$ and $\mathcal{GT}_\ell(x)$ denote the generating functions for the sequences $\mathcal{T}_\ell(m,\nu)$ and $\mathcal{T}_\ell(\nu)$, respectively. Then we have*

$$\mathcal{GT}_\ell(m,x) = \left(\frac{x}{(1-x)(1-x^\ell)}\right)^m,$$
$$\mathcal{GT}_\ell(x) = \frac{x}{1-2x-x^\ell+x^{\ell+1}}.$$

Moreover, the sequence $\mathcal{T}_\ell(m,\nu)$ for $1 \leq m \leq \nu$ is given explicitly by

$$\mathcal{T}_\ell(m,\nu) = \sum_{i=0}^{\left\lfloor \frac{\nu-m}{\ell} \right\rfloor} \binom{m+i-1}{m-1}\binom{\nu-\ell i-1}{m-1},$$

with $\mathcal{T}_\ell(\nu) = 2\mathcal{T}_\ell(\nu-1) + \mathcal{T}_\ell(\nu-\ell) - \mathcal{T}_\ell(\nu-\ell-1)$ for $\nu > \ell+1$.

Note that the sequences $\mathcal{T}_\ell(\nu)$ and $\mathcal{C}_{\ell a+1}(\nu)$ are the same which can be shown using the definitions.

We now describe a statistic on n-color compositions which accounts for the expression given for $\mathcal{T}_\ell(m,\nu)$ above. More precisely, let $\mathcal{S}_\ell(m,\nu)$ denote the set of n-color compositions enumerated by $\mathcal{T}_\ell(m,\nu)$ and we determine a statistic σ on $\mathcal{S}_\ell(m,\nu)$ such that

$$|\{\pi \in \mathcal{S}_\ell(m,\nu) : \sigma(\pi) = i\}| = \binom{m+i-1}{m-1}\binom{\nu-\ell i-1}{m-1}.$$

Given a part α_β of $\pi \in \mathcal{S}_\ell(m,\nu)$, let $\sigma(\alpha_\beta) = \lfloor(\beta-1)/\ell\rfloor$. Define $\sigma(\pi)$ to be the sum of the σ-values of its individual parts. For example, if $\ell = 3$ and $\pi = 5_2, 7_7, 8_5, 12_9, 3_3 \in \mathcal{S}_3(5,35)$, then $\sigma(\pi) = 0+2+1+2+0 = 5$. Note that if β corresponds to the i-th smallest possible subscript on a part of π of size α, then α_β contributes $i-1$ towards the $\sigma(\pi)$ statistic value. If $\ell = 1$, then it is seen that $\sigma(\pi)$ is simply the sum of the subscripts of all the parts minus the number of parts of π. Define

$$t_{\nu,m}^{(\ell)}(q) = \sum_{\pi \in \mathcal{S}_\ell(m,\nu)} q^{\sigma(\pi)}, \qquad \nu \geq m \geq 1,$$

where q is an indeterminate. We have the following explicit formula for $t_{\nu,m}^{(\ell)}(q)$.

Theorem 17. *If $\nu \geq m \geq 1$ and $\ell \geq 1$, then*

$$t_{\nu,m}^{(\ell)}(q) = \sum_{i=0}^{\lfloor \frac{\nu-m}{\ell} \rfloor} \binom{m+i-1}{m-1}\binom{\nu-\ell i-1}{m-1} q^i. \tag{5}$$

Proof. Let σ' be the statistic defined on $\pi \in \mathcal{S}_\ell(m,\nu)$ as follows. Given a part α_β of π, let $\sigma'(\alpha_\beta) = \frac{\alpha-\beta}{\ell}$ and define $\sigma'(\pi)$ to be the sum of the σ' values of its parts. For example, if $\pi \in \mathcal{S}_3(5,35)$ is as before, then $\sigma'(\pi) = 3$. We first show that σ and σ' are identically distributed on $\mathcal{S}_\ell(m,\nu)$. To do so, we change the subscripts on each part of $\pi \in \mathcal{S}_\ell(m,\nu)$ as follows. Let r_s be a part of π. First assume r is not divisible by ℓ. Then $r = \ell a + b$ where $a \geq 0$ and $1 \leq b \leq \ell - 1$ and $s = \ell a' + b$ for some $0 \leq a' \leq a$. In this case, we replace r_s with r_t, where $t = \ell(a-a') + b$. If r is divisible be ℓ, then $r = \ell a$ and $s = \ell a'$ for some $1 \leq a' \leq a$, in which case we replace the part r_s with r_t, where $t = \ell(a-a'+1)$. Let π' denote the resulting member of $\mathcal{S}_\ell(m,\nu)$. One may verify that the mapping $\pi \mapsto \pi'$ is a bijection with $\sigma(\pi) = \sigma'(\pi')$ for all π.

We now count members $\pi \in \mathcal{S}_\ell(m,\nu)$ such that $\sigma'(\pi) = i$ where $0 \leq i \leq \lfloor(\nu-m)/\ell\rfloor$. We denote these π by

$$\pi = (a_1 + \ell b_1)_{a_1}, \ldots, (a_m + \ell b_m)_{a_m},$$

where $a_j \geq 1$ and $b_j \geq 0$ for all j. Then $b_1 + \cdots + b_m = i$ implies there are $\binom{m+i-1}{m-1}$ possibilities for the b_j. Thus, $a_1 + \cdots + a_m = \nu - \ell i$ so that there are $\binom{\nu-\ell i-1}{m-1}$ possibilities for the a_j. Since the a_j and b_j may be chosen independently of one another, it follows that there are $\binom{m+i-1}{m-1}\binom{\nu-\ell i-1}{m-1}$ such π, which completes the proof of (5). $\square$

Let $t_\nu^{(\ell)}(q,u) = \sum_{m=1}^{\nu} t_{\nu,m}^{(\ell)}(q)u^m$ for $\nu \geq 1$ and define the generating function

$$T^{(\ell)}(x;q,u) = \sum_{\nu\geq 1} t_\nu^{(\ell)}(q,u)x^\nu.$$

Using (5) and interchanging summation yields the following result.

Corollary 18. *We have*

$$T^{(\ell)}(x;q,u) = \frac{xu}{1 - x(1+u) - x^\ell q + x^{\ell+1}q} \tag{6}$$

and thus

$$t_\nu^{(\ell)}(q,u) = (1+u)t_{\nu-1}^{(\ell)}(q,u) + qt_{\nu-\ell}^{(\ell)}(q,u) - qt_{\nu-\ell-1}^{(\ell)}(q,u), \qquad \nu > \ell + 1. \tag{7}$$

Formulas (6) and (7) reduce, respectively, to the generating function and recurrence formulas for $\mathcal{T}_\ell(\nu)$ in Theorem 16 when $q = u = 1$. Note that the $\ell = u = 1$ case of recurrence (7) was previously considered in [9]. A combinatorial proof may be given for (7) by considering whether or not the last part is 1_1, and if not, whether or not the last part is equal to its subscript. Finally, taking $\ell = 2$ in the preceding yields a polynomial generalization of the problem of counting n-color compositions of a given size in which each part and its respective subscript always have the same parity.

5 Acknowledgments

The fourth author thanks an invitation from the Department of Mathematics and Statistics of Universidad del Tolima, Tolima, Colombia, where the presented work was initiated. Additionally, the research of the fourth author was partially supported by Universidad Nacional de Colombia, Project No. 46240.

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2010 *Mathematics Subject Classification*: Primary 05A15; Secondary 05A19.
Keywords: n-color composition, generating function, combinatorial identity.

(Concerned with sequences A000045, A000930, A003269, A003520, A005708, A052908, A116732, A176848.)

Received January 26 2019; revised version received August 15 2019. Published in *Journal of Integer Sequences*, August 24 2019.

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