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A Combinatorial Proof for the Generating Function of Powers of a Second-Order Recurrence Sequence

Yifan Zhang and George Grossman
Department of Mathematics
Central Michigan University
Mount Pleasant, MI 48858
USA
zhang5y@cmich.edu
gross1gw@cmich.edu

Abstract

In this paper, we derive a formula for the generating function of powers of a second-order linear recurrence sequence, with initial conditions 0 and 1. As an example, we find the generating function of the powers of the nonnegative integers. We also find new formulas for computing Eulerian polynomials.

1 Introduction

Second-order linear recurrence sequences have been studied for hundreds of years. One of the earlier historical summaries was done by Dickson [1, Vol. 1, Chapter 17, pp. 393–411].

In the present paper, we use the notation $W_{n;(a,b;p,q)}$ to define the second-order linear recurrence sequence,

$$W_{n+2;(a,b;p,q)} = pW_{n+1;(a,b;p,q)} + qW_{n;(a,b;p,q)},$$

having initial conditions

$$W_{0;(a,b;p,q)} = a,\ W_{1;(a,b;p,q)} = b.$$

Example 1. The nonnegative integers are represented by $W_{n;(0,1;2,-1)}$.

Domino tiling methods have been used to find combinatorial identities and recurrence relations of certain sequences. Sellers [6] proved that the number of domino tilings of the graph $W_4 \times P_{n-1}$ equals $F_n P_n$ (sequence A001582 in Sloane's *On-Line Encyclopedia of Integer Sequences* [7]), which is the product of the n-th Fibonacci number and the n-th Pell number. Katz and Stenson [3] considered the number of tilings of a $(2 \times n)$-board A030186.

Let $F_k(x)$ be the generating function for the k-th powers of the Fibonacci numbers, defined by

$$F_k(x) = \sum_{n=0}^{\infty} F_n^k x^n, \ k \in \mathbb{N}^+.$$

For $1 \leq k \leq 10$, $F_k(x)$ can be found respectively from A000045, A007598, A056570, A056571, A056572, A056573, A056574, A056585, A056586, and A056587 in Sloane's *On-Line Encyclopedia of Integer Sequences* [7] (OEIS).

In 1962, Riordan [5] gave the following recurrence relation,

$$(1 - L_k(x) + (-1)^k x^2) F_k(x) = 1 + kx \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} (-1)^j \frac{A_{kj}}{j} F_{k-2j}\left((-1)^j x\right),$$

where L_k is the k-th Lucas number A000032. In 1999, Dujella [2] showed the doubly-indexed sequence A_{kj}, has generating function given by,

$$(1 - x - x^2)^{-j} = \sum_{k=2j}^{\infty} A_{kj} x^{k-2j}, \ j \geq 0.$$

In 2003, Stănică [8] obtained a closed form for generating function of the non-degenerate second-order recurrence relation,

$$U_{n+2} = aU_{n+1} + bU_n, \ a, \ b, \ U_0, U_1 \in \mathbb{Z},$$

such that $\delta = a^2 + 4b \neq 0$. Let $\alpha = \frac{1}{2}(a + \sqrt{a^2 + 4b})$, $\beta = \frac{1}{2}(a - \sqrt{a^2 + 4b})$ and $A = \frac{U_1 - U_0 \beta}{\alpha - \beta}$, $B = \frac{U_1 - U_0 \alpha}{\alpha - \beta}$, $V_n = \alpha^n + \beta^n$ with initial conditions $V_0 = 2$, $V_1 = a$, and let $U_k(x) = \sum_{i=0}^{\infty} U_i^k x^i$.

If k is odd, then

$$U_k(x) = \sum_{j=0}^{\frac{k-1}{2}} (-AB)^j \binom{k}{j} \frac{A^{k-2j} - B^{k-2j} + (-b)^j ((B\alpha)^{k-2j} - (A\beta)^{k-2j})x}{1 - (-b)^j V_{k-2j} x - b^k x^2}.$$

If k is even, then

$$U_k(x) = \sum_{j=0}^{\frac{k}{2}-1} (-AB)^j \binom{k}{j} \frac{B^{k-2j} + A^{k-2j} - (-b)^j ((B\alpha)^{k-2j} + (A\beta)^{k-2j})x}{1 - (-b)^j V_{k-2j} x + b^k x^2} + \binom{k}{\frac{k}{2}} \frac{(-AB)^{\frac{k}{2}}}{1 - (-b)^{\frac{k}{2}} x}.$$

In 2004, Mansour [4] obtained a formula for the generating functions of powers of second-order recurrence sequence in terms of the determinants of certain matrices, given by

$$W_{n;(a,b;p,q)}(x) = \frac{\det(\delta_k)}{\det(\Delta_k)},$$

where

$$\begin{aligned}
\Delta_k &= \begin{bmatrix} A_{1\times 1} & C_{1\times(k-1)} \\ B_{(k-1)\times 1} & D_{(k-1)\times(k-1)} \end{bmatrix} \\
\delta_k &= \begin{bmatrix} E_{1\times 1} & C_{1\times(k-1)} \\ F_{(k-1)\times 1} & D_{(k-1)\times(k-1)} \end{bmatrix} \\
A_{1\times 1} &= \begin{bmatrix} 1 - p^k x - q^k x^2 \end{bmatrix} \\
E_{1\times 1} &= \begin{bmatrix} a^k + g_k x \end{bmatrix}, \\
B_{(k-1)\times 1} &= \begin{bmatrix} -p^{k-1}x & -p^{k-2}x & \cdots & -p^1 x \end{bmatrix}^T \\
F_{(k-1)\times 1} &= \begin{bmatrix} g_{k-1}x & g_{k-2}x & \cdots & g_1 x \end{bmatrix}^T, \\
C_{1\times(k-1)} &= \begin{bmatrix} -xp^{k-1}q^1\binom{k}{1} & -xp^{k-2}q^2\binom{k}{2} & \cdots & -xp^1q^{k-1}\binom{k}{k-1} \end{bmatrix}, \\
D_{(k-1)\times(k-1)} &= \begin{bmatrix} 1 - xp^{k-2}q^1\binom{k-1}{1} & -xp^{k-3}q^2\binom{k-1}{2} & \cdots & -xq^{k-1}\binom{k-1}{k-1} \\ -xp^{k-3}q^1\binom{k-2}{1} & 1 - xp^{k-4}q^2\binom{k-2}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -xq^1\binom{1}{1} & 0 & \cdots & 1 \end{bmatrix},
\end{aligned}$$

with $g_j = (b^j - a^j p^j)a^{k-j}$, $j = 1, 2, \ldots, k$.

In 2017, Zhang and Grossman [9] used a counting method to relate the number of tilings of a $(k \times n)$-board to the generating function of the k-th power of the Fibonacci numbers.

In the present paper, we extend the previous result to the generating function of the k-th power of an arbitrary second-order linear recurrence sequence having initial conditions 0 and 1.

Eulerian polynomials, denoted in the present paper by $A_k(x)$, were introduced by Euler in 1755 and can be determined from the identity

$$\sum_{n=0}^{\infty} n^k x^n = \frac{xA_k(x)}{(1-x)^{k+1}}.$$

The coefficients of Eulerian polynomials leads to the Eulerian numbers A008292. The recurrence relation for nonnegative integers denoted $a_n,\ n \geq 0$ is given by

$$a_{n+2} = 2a_{n+1} - a_n,\ a_0 = 0,\ a_1 = 1.$$

In the next section we derive an expression for powers of $W_{n;(0,1;p,q)}$ with arbitrary $p \neq 0$, q. Sequences associated with such generating functions are Pell, Jacobsthal and the nonnegative integers. Closed-form expressions for Eulerian polynomials are also given in Corollary 23 and Remark 26.

2 Closed form for $\mathcal{W}_{k;(0,1;p,q)}(x)$

The Fibonacci number F_{n+1} gives the number of ways that 1×2 dominoes and squares can cover a $1 \times n$ checkerboard. We begin with a series of definitions.

Definition 2. Let $\mathcal{W}_{k;(0,1;p,q)}(x) = \sum_{n=0}^{\infty} W_n^k x^n$ be the generating function of k-th power of W_n where $W_{n+2} = pW_{n+1} + qW_n$ with initial conditions $W_0 = 0$ and $W_1 = 1$ with $p, q \in \mathbb{R}$, $p \neq 0$.

In this section, we use the notation (p, q) in the place of $(0, 1; p, q)$.

Definition 3. For $k \geq 1$ and $n \geq 1$, let $F_{k\times n;(p,q)}$ denote the *Fibonacci* $(k \times n; (p, q))$ *checkerboard*, which is a *checkerboard* (or simply, board) with height k and length n, covered by squares and dominoes, such that the dominoes can only be placed horizontally. Let $\mathcal{F}_{k\times n;(p,q)}$ be the set of all $F_{k\times n;(p,q)}$ boards. Define a function $V\colon \mathcal{F}_{k\times n;(p,q)} \to \mathbb{R}$ (value of $F_{k\times n;(p,q)}$) by $V(F_{k\times n;(p,q)}) = p^s q^d$, where d and s denote the number of dominoes and squares respectively in this $F_{k\times n;(p,q)}$ board. Let us also redefine the function $V\colon 2^{\mathcal{F}_{k\times n;(p,q)}} \to \mathbb{R}$ by $V(A) = \sum_{a\in A} V(a)$. Let $w_{k\times n;(p,q)} = V(\mathcal{F}_{k\times n;(p,q)})$ be the sum of all the values of the different $F_{k\times n;(p,q)}$ boards.

Note that each $F_{k\times n;(p,q)}$ board is comprised of k $F_{1\times n;(p,q)}$ boards (horizontal layers), each of which has at most n components consisting of dominoes and squares.

We define $w_{k\times(-1);(p,q)} = 0$ and $w_{k\times 0;(p,q)} = 1$ for $k \in \mathbb{N}^+$.

Definition 4. The *Fibonacci* $(k \times n; (p, q))$ *minimal checkerboard* called an $M_{k\times n;(p,q)}$ *board*, is an $F_{k\times n;(p,q)}$ board which cannot be vertically divided into two Fibonacci checkerboards. Let $\mathcal{M}_{k\times n;(p,q)}$ be the set of all $M_{k\times n;(p,q)}$ boards. Let $m_{k\times n;(p,q)} = V(\mathcal{M}_{k\times n;(p,q)})$ be the sum of all the values of the different $M_{k\times n;(p,q)}$ boards.

Note that $\mathcal{M}_{k\times n;(p,q)} \subseteq \mathcal{F}_{k\times n;(p,q)}$.

Example 5. There are total of nine different $F_{2\times 3;(p,q)}$ boards shown in Figure 1. We find that $w_{2\times 3;(p,q)} = p^6 + 4p^4 q + 4p^2 q^2$. There are two different $M_{2\times 3;(p,q)}$ boards and $m_{2\times 3;(p,q)} = 2p^2 q^2$.

Lemma 6. *For $n \geq 1$ and $k \geq 1$, $w_{1\times n;(p,q)} = W_{n+1;(p,q)}$ and $w_{k\times n;(p,q)} = W^k_{n+1;(p,q)}$.*

Proof. This lemma can be proven inductively.

If $n = 1$, then $w_{1\times 1;(p,q)} = p = W_{2;(p,q)}$. If $n = 2$, then $w_{1\times 2;(p,q)} = p^2 + q = W_{3;(p,q)}$.

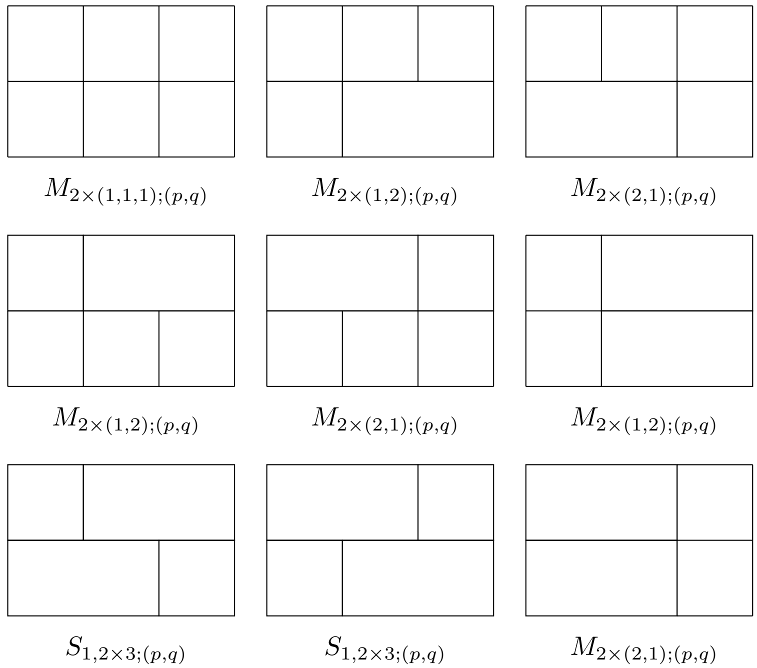


Figure 1: Nine $F_{2\times3;(p,q)}$ boards.

If $w_{1\times n;(p,q)} = W_{n+1;(p,q)}$ and $w_{1\times(n+1);(p,q)} = W_{n+2;(p,q)}$ hold, then

$$w_{1\times(n+2);(p,q)} = pw_{1\times(n+1);(p,q)} + qw_{1\times n;(p,q)} = pW_{n+2;(p,q)} + qW_{n+1;(p,q)} = W_{n+3;(p,q)}.$$

Therefore $w_{1\times n;(p,q)} = W_{n+1;(p,q)}$ for $n \geq 1$.

We know that the $F_{k\times n;(p,q)}$ board has k independent layers. Also, each independent layer is an $F_{1\times n;(p,q)}$ board which implies that the sum of all the values of each layer is $w_{1\times n;(p,q)}$. Therefore, we can conclude that $w_{k\times n;(p,q)} = W^k_{n+1;(p,q)}$. □

The generating function for powers of Pell numbers (A000129, A079291, A110272) is denoted $P_k(x) = \mathcal{W}_{k;(2,1)}(x)$; powers of Jacobsthal numbers (A001045, A139818), is denoted by $J_k(x) = \mathcal{W}_{k;(1,2)}(x)$; and powers of nonnegative integers (A001477, A000290, A000578, A000583, A000584, A001014, A001015, A001016, A001017, and A008454), is denoted by $\mathcal{W}_{k;(2,-1)}(x)$.

In Table 1 we listed values of $m_{4\times n;(2,-1)}$ for $1 \leq n \leq 7$. The smaller values were obtained by direct calculation, larger values were computed using Lemma 19. These values will be used to explain Example 10 and Example 12.

Remark 7. Any non-minimal $F_{k\times n;(p,q)}$ board can be split uniquely into at most n Fibonacci minimal boards. Specifically, there exist j, $1 < j \leq n$, and an ordered partition $n_1 + n_2 + \cdots + n_j = n$, such that the $F_{k\times n;(p,q)}$ board can be split into $M_{k\times n_1;(p,q)}$, $M_{k\times n_2;(p,q)}$, …, $M_{k\times n_j;(p,q)}$ boards (from left to right).

n	1	2	3	4	5	6	7
$m_{4\times n;(2,-1)}$	16	−175	1760	−17456	172832	−1710896	16936160

Table 1: $m_{4\times n;(2,-1)}$ for $1 \leq n \leq 7$.

Definition 8. Let an $M_{k\times(n_1,n_2,\ldots,n_j);(p,q)}$ *board* be an $F_{k\times n;(p,q)}$ board with $n_1+n_2+\cdots+n_j = n$ that can be split into $M_{k\times n_1;(p,q)}$, $M_{k\times n_2;(p,q)}$, …, $M_{k\times n_j;(p,q)}$ boards. Let $\mathcal{M}_{k\times(n_1,n_2,\ldots,n_j);(p,q)}$ be the set of all different $M_{k\times(n_1,n_2,\ldots,n_j);(p,q)}$ boards. Let

$$m_{k\times(n_1,n_2,\ldots,n_j);(p,q)} = V(\mathcal{M}_{k\times(n_1,n_2,\ldots,n_j);(p,q)})$$

be the sum of all the values of the different $M_{k\times(n_1,n_2,\ldots,n_j);(p,q)}$ boards.

The value of $m_{k\times(n_1,n_2,\ldots,n_j);(p,q)}$ is quantified in Lemma 9. Note that there are two examples in Figure 1.

Lemma 9. *For $k \geq 1$ and $n \geq 1$, $n_i \in \mathbb{N}^+$ for $i \in \{1, 2, \ldots, j\}$ we have,*

$$m_{k\times(n_1,n_2,\ldots,n_j);(p,q)} = \prod_{i=1}^{j} m_{k\times n_i;(p,q)},$$

and

$$W^k_{n+1;(p,q)} = \sum_{j=1}^{n} \sum_{n_1+n_2+\cdots+n_j=n} \prod_{i=1}^{j} m_{k\times n_i;(p,q)}.$$

Proof. Since an $M_{k\times(n_1,n_2,\ldots,n_j);(p,q)}$ board is the combination of $M_{k\times n_1;(p,q)}$, $M_{k\times n_2;(p,q)}$, …, $M_{k\times n_j;(p,q)}$ boards, we have $m_{k\times(n_1,n_2,\ldots,n_j);(p,q)} = \prod_{i=1}^{j} m_{k\times n_i;(p,q)}$.

From Remark 7, each $F_{k\times n;(p,q)}$ board is either an $M_{k\times n;(p,q)}$ board or can be uniquely divided into j minimal boards where $2 \leq j \leq n$. Then the set

$$\Big\{\mathcal{M}_{k\times n;(p,q)}, \bigcup_{n_1+n_2=n} \mathcal{M}_{k\times(n_1,n_2);(p,q)}, \ldots, \bigcup_{n_1+n_2+\cdots+n_n=n} \mathcal{M}_{k\times(n_1,n_2,\ldots,n_n);(p,q)}\Big\}$$

forms a partition of $\mathcal{F}_{k\times n;(p,q)}$. That is,

$$\mathcal{F}_{k\times n;(p,q)} = \mathcal{M}_{k\times n;(p,q)} \bigcup \left(\bigcup_{n_1+n_2=n} \mathcal{M}_{k\times(n_1,n_2);(p,q)}\right) \bigcup \cdots \bigcup \left(\bigcup_{n_1+n_2+\cdots+n_n=n} \mathcal{M}_{k\times(n_1,n_2,\ldots,n_n);(p,q)}\right).$$

Therefore, from Lemma 6,

$$\begin{aligned}
W^k_{n+1;(p,q)} &= w_{k\times n;(p,q)} \\
&= V(\mathcal{F}_{k\times n;(p,q)}) \\
&= V(\mathcal{M}_{k\times n;(p,q)}) + V\left(\bigcup_{n_1+n_2=n} \mathcal{M}_{k\times(n_1,n_2);(p,q)}\right) + \cdots \\
&\quad + V\left(\bigcup_{n_1+n_2+\cdots+n_n=n} \mathcal{M}_{k\times(n_1,n_2,\ldots,n_n);(p,q)}\right) \\
&= m_{k\times n;(p,q)} + \sum_{n_1+n_2=n} m_{k\times(n_1,n_2);(p,q)} + \cdots + \sum_{n_1+n_2+\cdots+n_n=n} m_{k\times(n_1,n_2,\ldots,n_n);(p,q)} \\
&= m_{k\times n;(p,q)} + \sum_{n_1+n_2=n}\prod_{i=1}^{2} m_{k\times n_i;(p,q)} + \cdots + \sum_{n_1+n_2+\cdots+n_n=n}\prod_{i=1}^{n} m_{k\times n_i;(p,q)} \\
&= \sum_{j=1}^{n}\sum_{n_1+n_2+\cdots+n_j=n}\prod_{i=1}^{j} m_{k\times n_i;(p,q)}.
\end{aligned}$$

□

Example 10.

$$\begin{aligned}
W^4_{4;(2,-1)} &= m_{4\times 3;(2,-1)} + m_{4\times(2,1);(2,-1)} + m_{4\times(1,2);(2,-1)} + m_{4\times(1,1,1);(2,-1)} \\
&= 1760 + (-175)\cdot 16 + 16\cdot(-175) + 16\cdot 16\cdot 16 \\
&= 256 \\
&= 4^4.
\end{aligned}$$

Lemma 11. *For $k \in \mathbb{N}^+$, let $e_{k;(p,q)} = \sum_{n=1}^{\infty} m_{k\times n;(p,q)} x^n$. Then,*

$$\mathcal{W}_{k;(p,q)}(x) = \frac{x}{1 - e_{k;(p,q)}}.$$

Proof. For $k, n \in \mathbb{N}^+$, the coefficient of x^n in $e_{k;(p,q)} + e^2_{k;(p,q)} + \cdots + e^n_{k;(p,q)} + \cdots$ is

$$\begin{aligned}
& m_{k\times n;(p,q)} + \sum_{\substack{n_1+n_2=n\\ n_i\in\mathbb{N}^+}}\prod_{i=1}^{2} m_{k\times n_i;(p,q)} + \sum_{\substack{n_1+n_2+n_3=n\\ n_i\in\mathbb{N}^+}}\prod_{i=1}^{3} m_{k\times n_i;(p,q)} + \cdots \\
& + \sum_{\substack{n_1+n_2+\cdots+n_n=n\\ n_i\in\mathbb{N}^+}}\prod_{i=1}^{n} m_{k\times n_i;(p,q)} \\
= & \sum_{j=1}^{n}\sum_{\substack{n_1+n_2+\cdots+n_j=n\\ n_i\in\mathbb{N}^+}}\prod_{i=1}^{j} m_{k\times n_i;(p,q)}.
\end{aligned}$$

Also, $e_{k;(p,q)} + e^2_{k;(p,q)} + \cdots + e^n_{k;(p,q)} + \cdots = \frac{1}{1-e_{k;(p,q)}} - 1$. Therefore,

$$\sum_{n=1}^{\infty}\left(\left(\sum_{j=1}^{n}\sum_{\substack{n_1+n_2+\cdots+n_j=n\\ n_i\in\mathbb{N}^+}}\prod_{i=1}^{j} m_{k\times n_i;(p,q)}\right)x^n\right) = \frac{1}{1-e_{k;(p,q)}} - 1.$$

Employing Lemmas 6 and 9, we obtain,

$$\begin{aligned}
\mathcal{W}_{k;(p,q)}(x) &= \sum_{n=0}^{\infty} W^k_{n;(p,q)}x^n \\
&= W^k_{0;(p,q)} + W^k_{1;(p,q)}x + \sum_{n=2}^{\infty} W^k_{n;(p,q)}x^n \\
&= x + \sum_{n=1}^{\infty} W^k_{n+1;(p,q)}x^{n+1} \\
&= x + x\sum_{n=1}^{\infty} W^k_{n+1;(p,q)}x^{n} \\
&= x + x\sum_{n=1}^{\infty}\left(\left(\sum_{j=1}^{n}\sum_{\substack{n_1+n_2+\cdots+n_j=n\\ n_i\in\mathbb{N}^+}}\prod_{i=1}^{j} m_{k\times n_i;(p,q)}\right)x^n\right) \\
&= x + x\left(\frac{1}{1-e_{k;(p,q)}} - 1\right) \\
&= \frac{x}{1-e_{k;(p,q)}}.
\end{aligned}$$

□

Example 12. From Table 1, we have $e_{4;(2,-1)} = \sum_{n=1}^{\infty} m_{4\times n;(2,-1)}x^n = 16x - 175x^2 + 1760x^3 - \cdots$ and $\mathcal{W}_{4;(2,-1)}(x) = \frac{x}{1-e_{4;(2,-1)}}$.

Now we determine the closed form of $e_{k;(p,q)}$.

Lemma 13. *Suppose $S_{1,n}$, $S_{2,n}$, $\ldots$, $S_{m,n}$ are real-valued sequences such that $S_{i,n+1} = \sum_{j=1}^{m} a_{i,j}S_{j,n}$ for $1 \le i \le m$ and $n \ge 0$.*

Let $[a_{i,j}]_{m\times m} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,m} \end{bmatrix}_{m\times m}$, $[S_{j,n}]_{m\times 1} = \begin{bmatrix} S_{1,n} \\ S_{2,n} \\ \vdots \\ S_{m,n} \end{bmatrix}_{m\times 1}$, $B_{m+1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}^T_{1\times m}$.

Then $\sum_{j=1}^{m} S_{j,n} = B_{m+1}[a_{i,j}]^{n-l}_{m\times m}[S_{j,l}]_{m\times 1}$.

Proof. Since

$$[a_{i,j}]_{m\times m}^{n-l}[S_{j,l}]_{m\times 1} = [a_{i,j}]_{m\times m}^{n-l-1}[a_{i,j}]_{m\times m}[S_{j,l}]_{m\times 1} = [a_{i,j}]_{m\times m}^{n-l-1}[S_{j,l+1}]_{m\times 1} = \cdots = [S_{j,n}]_{m\times 1},$$

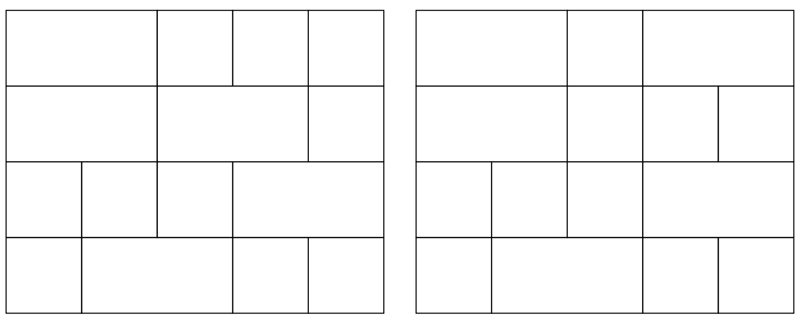
then

$$\sum_{j=1}^{m} S_{j,n} = B_{m+1}[S_{j,n}]_{m\times 1} = B_{m+1}[a_{i,j}]_{m\times m}^{n-l}[S_{j,l}]_{m\times 1}.$$

□

Definition 14. Let an $S_{j,k\times n;(p,q)}$ *board* ($j \in \{1, 2, 3, \ldots, k\}, n \geq 2$) be an $M_{k\times n;(p,q)}$ board with j dominoes in last two columns. Let $\mathcal{S}_{j,k\times n;(p,q)}$ be the set of all $S_{j,k\times n;(p,q)}$ boards. Let $s_{j,k\times n;(p,q)} = V(\mathcal{S}_{j,k\times n;(p,q)})$ be the sum of all the values of the different $S_{j,k\times n;(p,q)}$ boards.

Example 15. In Figure 2, the board on the left is an $S_{1,4\times 5;(p,q)}$ board and the board on the right is a combination of an $S_{1,4\times 3;(p,q)}$ board and an $S_{2,4\times 2;(p,q)}$ board.



$M_{4\times 5;(p,q)}$, also $S_{1,4\times 5;(p,q)}$ $\quad$ $M_{4\times(3,2);(p,q)}$

Figure 2: Two different boards.

Lemma 16. *For* $k \in \mathbb{N}^+$, $s_{j,k\times 2;(p,q)} = p^{2(k-j)}q^j\binom{k}{j}$ *for* $1 \leq j \leq k$ *and*

$$m_{k\times n;(p,q)} = \begin{cases} p^k, & \text{if } n = 1; \\ q^k + \sum_{j=1}^{k-1} s_{j,k\times 2;(p,q)}, & \text{if } n = 2; \\ \sum_{j=1}^{k-1} s_{j,k\times n;(p,q)}, & \text{if } n \geq 3. \end{cases}$$

Proof. For $n \geq 3$, $\mathcal{S}_{1,k\times n;(p,q)}$, $\mathcal{S}_{2,k\times n;(p,q)}$, $\ldots$, $\mathcal{S}_{k-1,k\times n;(p,q)}$ forms a partition of $\mathcal{M}_{k\times n;(p,q)}$. Therefore,

$$m_{k\times n;(p,q)} = V(\mathcal{M}_{k\times n;(p,q)}) = \sum_{j=1}^{k-1} V(\mathcal{S}_{j,k\times n;(p,q)}) = \sum_{j=1}^{k-1} s_{j,k\times n;(p,q)}.$$

For $n = 2$ and $j = k$, there is only one $S_{k,k\times 2;(p,q)}$ board and $V(\mathcal{S}_{k,k\times 2;(p,q)}) = q^k$. For $1 \leq j \leq k-1$, there are $\binom{k}{j}$ different $S_{j,k\times 2;(p,q)}$ boards. Then $s_{j,k\times 2;(p,q)} = p^{2(k-j)} q^j \binom{k}{j}$. Therefore,

$$m_{k\times 2;(p,q)} = q^k + \sum_{j=1}^{k-1} s_{j,k\times 2;(p,q)} = q^k + \sum_{j=1}^{k-1} p^{2(k-j)} q^j \binom{k}{j}.$$

□

Lemma 17. *For $n \geq 2$ and $1 \leq j \leq k-1$,*

$$s_{j,k\times(n+1);(p,q)} = \sum_{i=1}^{k-j} p^{k-2j} q^j \binom{k-i}{j} s_{i,k\times n;(p,q)}.$$

Proof. An $S_{j,k\times(n+1);(p,q)}$ board, $1 \leq j \leq k-1$, can be obtained from an $S_{i,k\times n;(p,q)}$ board, $1 \leq i \leq k-j$, by the following procedure. Suppose we start with an $S_{i,k\times n;(p,q)}$ board, $1 \leq i \leq k-j$, then this board has $k-i$ squares in the last column. Subsequently, choose j squares from this column, replace the chosen squares with dominoes. For each replacement, since each square has weight p and each domino has weight q, we multiply $s_{i,k\times n;(p,q)}$ by $\left(\frac{q}{p}\right)^j$.

So now we obtain an $S_{j,k\times(n+1);(p,q)}$ board by filling the remaining $(k-j)$ empty positions in the $n+1$ column with squares. Thus, we need to multiply $s_{i,k\times n;(p,q)} \left(\frac{q}{p}\right)^j$ by p^{k-j}. Therefore,

$$s_{j,k\times(n+1);(p,q)} = \sum_{i=1}^{k-j} p^{k-2j} q^j \binom{k-i}{j} s_{i,k\times n;(p,q)}$$

for $1 \leq j \leq k-1$. □

In Figure 3, there is an example of Lemma 17.

Definition 18. For $k \geq 2$, define matrices $A_{k;(p,q)} = [a_{ji}]_{(k-1)\times(k-1)}$ where $a_{ji} = p^{k-2j} q^j \binom{k-i}{j}$; $B_k = [b_{1i}]_{1\times(k-1)}$ such that $b_{1i} = 1$; $C_{k;(0,1;p,q)} = [c_{j1}]_{(k-1)\times 1}$ where $c_{j1} = p^{2(k-j)} q^j \binom{k}{j}$, and I_k is the $(k-1)\times(k-1)$ identity matrix.

If $k = 1$, let $A_{1;(p,q)} = B_1 = C_{1;(0,1;p,q)} = 0$, $I_1 = 1$.

Lemma 19. *For $k \in \mathbb{N}^+$,*

$$m_{k\times n;(p,q)} = \begin{cases} q^k + B_k A^0_{k;(p,q)} C_{k;(0,1;p,q)}, & \textit{if } n = 2; \\ B_k A^{n-2}_{k;(p,q)} C_{k;(0,1;p,q)}, & \textit{if } n \geq 3. \end{cases}$$

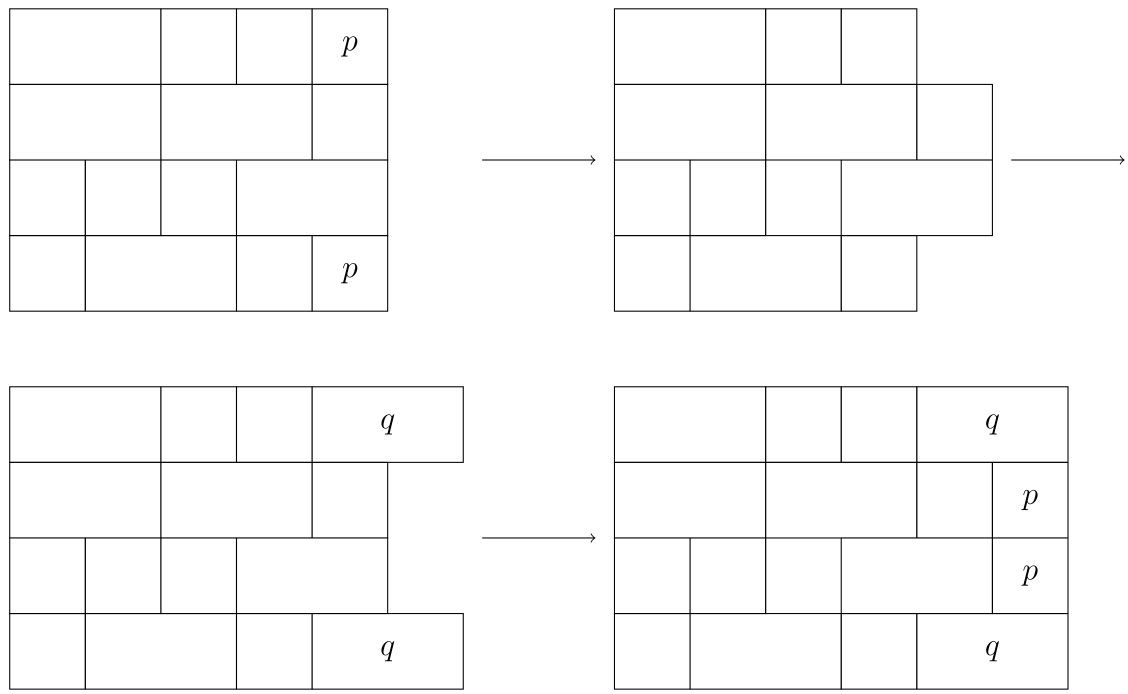


Figure 3: How to transform $S_{1,4\times 5;(p,q)}$ to $S_{2,4\times 6;(p,q)}$.

Proof. Employing Lemmas 13, 17, and 16, with $k \in \mathbb{N}^+$ and $n \geq 3$, we have,

$$\begin{aligned} m_{k\times n;(p,q)} &= \sum_{j=1}^{k-1} s_{j,k\times n;(p,q)} \\ &= B_k A_{k;(p,q)}^{n-2} \begin{bmatrix} s_{1,k\times 2;(p,q)} \\ s_{2,k\times 2;(p,q)} \\ \vdots \\ s_{k-1,k\times 2;(p,q)} \end{bmatrix}_{(k-1)\times 1} \\ &= B_k A_{k;(p,q)}^{n-2} \begin{bmatrix} p^{2(k-1)} q^1 \binom{k}{1} \\ p^{2(k-2)} q^2 \binom{k}{2} \\ \vdots \\ p^2 q^{k-1} \binom{k}{k-1} \end{bmatrix}_{(k-1)\times 1} \\ &= B_k A_{k;(p,q)}^{n-2} C_{k;(0,1;p,q)}. \end{aligned}$$

For $n = 2$, $m_{k\times 2;(p,q)} = q^k + \sum_{j=1}^{k-1} s_{j,k\times 2;(p,q)} = q^k + B_k A_{k;(p,q)}^0 C_{k;(0,1;p,q)}$. □

Table 2 contains $s_{j,4\times n;(2,-1)}$ for $2 \leq n \leq 7$, with $j = 1, 2, 3$. We have that, $s_{1,4\times(n+1);(2,-1)} = (-12) \cdot s_{1,4\times n;(2,-1)} + (-8) \cdot s_{2,4\times n;(2,-1)} + (-4) \cdot s_{3,4\times n;(2,-1)}$,

$s_{2,4\times(n+1);(2,-1)} = 3 \cdot s_{1,4\times n;(2,-1)} + 1 \cdot s_{2,4\times n;(2,-1)} + 0 \cdot s_{3,4\times n;(2,-1)}$, and
$s_{3,4\times(n+1);(2,-1)} = (-\frac{1}{4}) \cdot s_{1,4\times n;(2,-1)} + 0 \cdot s_{2,4\times n;(2,-1)} + 0 \cdot s_{3,4\times n;(2,-1)}$.

n	1	2	3	4	5	6	7
$m_{4\times n;(2,-1)}$	16	−175	1760	−17456	172832	−1710896	16936160
$s_{1,4\times n;(2,-1)}$		−256	2368	−23296	230464	−2281216	22581568
$s_{2,4\times n;(2,-1)}$		96	−672	6432	−63456	627936	−6215712
$s_{3,4\times n;(2,-1)}$		−16	64	−592	5824	−57616	570304

Table 2: $s_{j,4\times n;(2,-1)}$, $j = 1, 2, 3$, for $2 \le n \le 7$.

Example 20. From Lemma 19, and for $n \ge 3$,

$$\begin{aligned} m_{4\times n;(2,-1)} &= s_{1,4\times n;(2,-1)} + s_{2,4\times n;(2,-1)} + s_{3,4\times n;(2,-1)} \\ &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -12 & -8 & -4 \\ 3 & 1 & 0 \\ -\frac{1}{4} & 0 & 0 \end{bmatrix}^{n-2} \begin{bmatrix} -256 \\ 96 \\ -16 \end{bmatrix}. \end{aligned}$$

Theorem 21. *For $k \in \mathbb{N}^+$,*

$$\mathcal{W}_{k;(p,q)}(x) = \frac{x}{1 - p^k x - q^k x^2 - x^2 B_k (I_k - x A_{k;(p,q)})^{-1} C_{k;(0,1;p,q)}}.$$

Proof. For $k \ge 2$, according to Lemmas 11, 16, and 19 we obtain,

$$\begin{aligned} e_{k;(p,q)} &= \sum_{n=1}^{\infty} m_{k\times n;(p,q)} x^n \\ &= p^k x + q^k x^2 + \sum_{n=2}^{\infty} B_k A_{k;(p,q)}^{n-2} C_{k;(0,1;p,q)} x^n \\ &= p^k x + q^k x^2 + x^2 \sum_{n=0}^{\infty} B_k A_{k;(p,q)}^{n} C_{k;(0,1;p,q)} x^n \\ &= p^k x + q^k x^2 + x^2 B_k \left(\sum_{n=0}^{\infty} (x A_{k;(p,q)})^n \right) C_{k;(0,1;p,q)} \\ &= p^k x + q^k x^2 + x^2 B_k \left(I_k - x A_{k;(p,q)} \right)^{-1} C_{k;(0,1;p,q)}. \end{aligned}$$

Thus,

$$\mathcal{W}_{k;(p,q)}(x) = \frac{x}{1-e_{k;(p,q)}} = \frac{x}{1-p^kx-q^kx^2-x^2B_k(I_k-xA_{k;(p,q)})^{-1}C_{k;(0,1;p,q)}}.$$

Since $(B_1(I_1-xA_{1;(p,q)})^{-1}C_{1;(0,1;p,q)}=0$, the theorem is also true for $k=1$. □

Example 22. Let $k=4$, $p=2$ and $q=-1$. Then

$$(I_4-xA_{4;(2,-1)})^{-1} = \frac{1}{1+11x+11x^2+x^3}\begin{bmatrix} 1-x & -8x & 4x(1-x) \\ 3x & 1+12x-x^2 & -12x^2 \\ \frac{x(x-1)}{4} & 2x^2 & 1+11x+12x^2 \end{bmatrix},$$

$$B_4(I_4-xA_{4;(2,-1)})^{-1}C_{4;(0,1,2,-1)} = \frac{-16(2x^2+11x+11)}{1+11x+11x^2+x^3},$$

therefore,

$$\mathcal{W}_{4;(2,-1)}(x) = \frac{x}{1-16x-x^2-x^2\frac{-16(2x^2+11x+11)}{1+11x+11x^2+x^3}} = \frac{x(1+11x+11x^2+x^3)}{(1-x)^5}.$$

The polynomial $1+11x+11x^2+x^3$ is the Eulerian polynomial $A_4(x)$.

Corollary 23. $\det(I_k-xA_{k;(2,-1)})=A_k(x)$.

Proof. By Theorem 21,

$$\begin{aligned}
\mathcal{W}_{k;(p,q)}(x) &= \frac{x}{1-p^kx-q^kx^2-x^2B_k(I_k-xA_{k;(p,q)})^{-1}C_{k;(0,1;p,q)}} \\
&= \frac{x}{1-p^kx-q^kx^2-x^2B_k\frac{\operatorname{adj}(I_k-xA_{k;(p,q)})}{\det(I_k-xA_{k;(p,q)})}C_{k;(0,1;p,q)}} \\
&= \frac{x\det(I_k-xA_{k;(p,q)})}{(1-p^kx-q^kx^2)\det(I_k-xA_{k;(p,q)})-x^2B_k\operatorname{adj}(I_k-xA_{k;(p,q)})C_{k;(0,1;p,q)}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{xA_k(x)}{(1-x)^{k+1}} &= \sum_{n=0}^{\infty} n^kx^n \\
&= \mathcal{W}_{k;(2,-1)}(x) \\
&= \frac{x\det(I_k-xA_{k;(2,-1)})}{(1-2^kx-(-1)^kx^2)\det(I_k-xA_{k;(2,-1)})-x^2B_k\operatorname{adj}(I_k-xA_{k;(2,-1)})C_{k;(0,1;2,-1)}}.
\end{aligned}$$

Since $A_k(1)=k!\neq 0$, $(1-x)\nmid A_k(x)$ and $\deg\left(\det(I_k-xA_{k;(2,-1)})\right)\leq k-1$, then $\deg\left(\det(I_k-xA_{k;(2,-1)})\right)=k-1$. Thus, $\det(I_k-xA_{k;(2,-1)})=A_k(x)$. □

Example 24. If $k = 6$,

$$\begin{aligned}
\det(I_6 - xA_{6;(2,-1)}) &= \det \begin{bmatrix} 1+80x & 64x & 48x & 32x & 16x \\ -40x & 1-24x & -12x & -4x & 0 \\ 10x & 4x & 1+x & 0 & 0 \\ \frac{-5x}{4} & \frac{-x}{4} & 0 & 1 & 0 \\ \frac{x}{16} & 0 & 0 & 0 & 1 \end{bmatrix} \\
&= 1 + 57x + 302x^2 + 302x^3 + 57x^4 + x^5 \\
&= A_6(x).
\end{aligned}$$

Corollary 25. *Let* $D_{k;(p,q)} = [d_{ji}]_{(k-1)\times(k-1)}$ *be the diagonal matrix with* $d_{jj} = p^{2j-k}q^{-j}$ *and* $d_{ji} = 0$ *if* $j \neq i$. *Then*

$$\det(D_{k;(p,q)} - xA_{k;(1,1)}) = \det(I_k - xA_{k;(p,q)}).$$

Proof. Let $\lambda_{ji} = 1$ if $j = i$, otherwise $\lambda_{ji} = 0$. Then

$$\begin{aligned}
\det(I_k - xA_{k;(p,q)}) &= \det \left[\lambda_{ji} - p^{k-2j}q^j \binom{k-i}{j} x \right]_{(k-1)\times(k-1)} \\
&= \det \left[p^{2j-k}q^{-j}\lambda_{ji} - \binom{k-i}{j} x \right]_{(k-1)\times(k-1)} \\
&= \det \left[d_{ji} - \binom{k-i}{j} x \right]_{(k-1)\times(k-1)} \\
&= \det(D_{k;(p,q)} - xA_{k;(1,1)}).
\end{aligned}$$

□

Remark 26. From Corollaries 23 and 25, we have

$$\det(D_{k;(p,q)} - xA_{k;(1,1)}) = A_k(x).$$

Remark 27. From Corollary 23, Remark 26, and $A_k(1) = k!$, we have

$$\det(I_k - A_{k;(2,-1)}) = \det(D_{k;(2,-1)} - A_{k;(1,1)}) = k!.$$

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