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Words and Linear Recurrences

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Abstract

In previous papers, we defined functions f_m and c_m based on an arithmetical function f_0, and determined numbers of restricted words over a finite alphabet counted by these functions. In this paper, we examine the reverse problem: for each of the five specific types of restricted words, we find the initial function f_0 such that f_m and c_m enumerate these words. We derive explicit formulas for f_m and c_m.

Fibonacci, Mersenne, Pell, Jacosthal, Tribonacci, and Padovan numbers all appear as values of f_m. We derive their new combinatorial interpretations and the explicit formulas.

1 Introduction

We continue the investigation of restricted word enumeration from previous papers Janji¸[2, 3, 4], where functions f_m and c_m were defined as follows. For an initial arithmetic function f_0 and $m \geq 1$, the function f_m is the m^{th} invert transform of f_0. The function $c_m(n,k)$ was defined as

$$c_m(n,k) = \sum_{i_1+i_2+\cdots+i_k=n} f_{m-1}(i_1) \cdot f_{m-1}(i_2) \cdots f_{m-1}(i_k), \tag{1}$$

where the sum is over positive integers $i_1, i_2, \ldots, i_k$.

The functions f_m and c_m depend only on the initial function f_0 and are related to the enumeration of weighted compositions. Namely, if the value of $f_{m-1}(i)$ is the weight of i, then the value of $f_m(n)$ is the number of weighted compositions of n, and the value of $c_m(n,k)$ is the number of weighted compositions of n into k parts.

In [2, 3, 4] weighted compositions were related to restricted words over a finite alphabet. For a given initial function f_0, we investigated restricted words counted by f_m and c_m. In this paper, we consider the reverse problem. For each of the five types of restricted words, we first find the initial function f_0 which counts these words. We then derive formulas for f_m and c_m and give their combinatorial meanings in term of restricted words.

To begin with, we restate the results from papers [2, 3, 4] that we will use in this work.

(A) [2, Theorem 6] Let f_0 be an arithmetic function and let k be a positive integer. Assume that there exist constants $a_0(1), a_0(2), \dots, a_0(k)$ such that

$$f_0(n+k;k) = \sum_{i=1}^{k} a_0(i) f_0(n+k-i;k), (n \geq 1),$$

where $f_0(1;k), f_0(2;k), \dots, f_0(k;k)$ are arbitrary numbers. Then, we have

$$\begin{aligned} f_1(i;k) &= \sum_{j=1}^{i} f_0(j;k) f_1(i-j;k), (i = 1, 2, \dots, k), \\ f_1(n+k;k) &= \sum_{i=1}^{k} a_1(i) f_1(n+k-i;k), (n \geq 1), \end{aligned}$$

where

$$\begin{aligned} a_1(1) &= a_0(1) + f_0(1;k), \\ a_1(i) &= a_0(i) + f_0(i;k) - \sum_{j=1}^{i-1} a_0(j) f_0(i-j;k), (2 \leq i \leq k). \end{aligned}$$

(B) [2, Corollary 9] If $f_0(1), f_0(2), a_0(1), a_0(2)$ are arbitrary, and

$$f_0(n+2) = a_0(1) f_0(n+1) + a_0(2) f_0(n),$$

then

$$\begin{aligned} f_m(1) = f_0(1),\ f_m(2) &= m f_0(1)^2 + f_0(2), \\ f_m(n+2) &= a_m(1) f_m(n+1) + a_m(2) f_m(n), \end{aligned}$$

where

$$\begin{aligned} a_m(1) &= a_0(1) + m f_0(1), \\ a_m(2) &= a_0(2) - m a_0(1) f_0(1) + m f_0(2). \end{aligned}$$

(C) [2, Proposition 23] Assume that $f_0(1) = 0$ and $f_0(i) = 1, (i > 1)$. Then, we have

$$f_m(1) = 0, f_m(2) = 1,$$
$$f_m(n+2) = f_m(n+1) + m f_m(n).$$

(D) [3, Corollary 2] The following formula holds:

$$f_m(n) = \sum_{k=1}^{n} c_m(n,k).$$

(E) [4, Proposition 6] The following formula holds:

$$c_m(n,k) = \sum_{i=k}^{n} (m-1)^{i-k} \binom{i-1}{k-1} c_1(n,i), \ (1 \le k \le n).$$

(F) [4, Propositions 12] Assume that $f_0(1) = 1$, and that $m > 1$. Assume next that, for $n \ge 1$, we have $f_{m-1}(n)$ words of length $n-1$ over a finite alphabet α. Let x be a letter which is not in α. Then, the value of $c_m(n,k)$ is the number of words of length $n-1$ over the alphabet $\alpha \cup \{x\}$ in which x appears exactly $k-1$ times.

We proceed to consider the following five types of restricted words over a finite alphabet:

1. Words over $\{0, 1, \ldots, a-1, \ldots\}$ such that no two adjacent letters from $\{0, 1, \ldots, a-1\}$ are the same (Property $\mathcal{P}_1$).

2. Words over $\{0, 1, \ldots, a-1, \ldots\}$ such that letters $0, 1, \ldots, a-1$ avoid a run of odd length (Property $\mathcal{P}_2$).

3. Words over $\{0, 1, \ldots, a, \ldots\}$ avoiding subwords of the form $0i, (i = 1, \ldots, b)$ for $b < a$ (Property $\mathcal{P}_3$).

4. Words over $\{0, 1, \ldots\}$ such that 0 and 1 appear only as subwords of the form $1i$, where i is a run of zeros (Property $\mathcal{P}_4$).

5. Words over $\{0, 1, \ldots\}$ in which 0 appears only in a run of even length, and 1 appears only in a run of a length divisible by 3 (Property $\mathcal{P}_5$).

We also note that, in all types, the initial function f_0 is defined by a linear homogenous recurrence.

2 Type 1

In this case, we consider the following linear recurrence:

$$f_0(1) = 1, f_0(2) = a,$$
$$f_0(n+2) = (a-1)f_0(n+1), (n \geq 1),$$

where $a > 0$. It is easy to see that

$$f_0(n) = a(a-1)^{n-2}, (n \geq 2).$$

Remark 1. This formula appears in Birmajer at al. [1, Example 17]. Also, the case $a = 1$ is considered in [4, Example 18].

The function f_0 has the following combinatorial interpretation.

Proposition 2. *The value of $f_0(n)$ is the number of words of length $n-1$ over $\{0, 1, \dots, a-1\}$ satisfying $\mathcal{P}_1$.*

Proof. We have $f_0(1) = 1$ since only the empty word has length 0. Also, $f_0(2) = a$ since a word of length 1 may consist of an arbitrary letter. To obtain a word of length $n+2$ for $n > 0$, we need to insert $a-1$ letters in front of each word of length $n+1$. □

As an immediate consequence of (B), we obtain the following result.

Corollary 3. *For $m \geq 0$, the following recurrence holds:*

$$f_m(1) = 1, f_m(2) = m + a,$$
$$f_m(n+2) = (m+a-1)f_m(n+1) + mf_m(n), (n \geq 1).$$

We now describe words counted by f_m.

Proposition 4. *The number of words of length $n-1$ over the alphabet $\{0, 1, \dots, a-1, a, \dots, m+a-1\}$ satisfying $\mathcal{P}_1$ is the value of $f_m(n)$.*

Proof. We have $f_m(1) = 1$, since only the empty word has length 0. Also, $f_m(2) = m + a$ since a word of length 1 may consist of any letter of the alphabet. Assume that $n > 2$. Consider a word of length $n+1$. At the front of such a word, we insert a letter different from the first letter of the word. In this way, we obtain all the words of length $n+2$ beginning with two different letters. The remaining words must begin with two identical letters. Since there are $mf_m(n)$ such words, the statement is true. □

Remark 5. For $a = 2$, the continued fraction $[f_0(1), f_0(2), f_0(3), \dots]$ equals $\sqrt{2}$. The sequence $f_1(1), f_1(2), \dots, f_1(n)$ is the numerator of the nth convergent of $\sqrt{2}$. Also, the value of $f_1(n)$ is the number of ternary words of length $n-1$ avoiding 00 and 11.

Since $f_m(1) = 1$, we may apply (F) to obtain the following result.

Proposition 6. *The number of words of length $n-1$ over $\{0, 1, \dots, a-1, \dots, m+a-1\}$ in which $k-1$ letters equal $m+a-1$, and which satisfy $\mathcal{P}_1$ equals the value of $c_m(n,k)$.*

We next derive an explicit formula for $c_1(n,k)$.

Proposition 7. *We have*

$$c_1(n,n) = 1,$$
$$c_1(n,k) = \sum_{i=0}^{k-1} a^{k-i}(a-1)^{n-2k+i}\binom{k}{i}\binom{n-k-1}{k-i-1}, (k < n).$$

Proof. From (1), we first obtain $c_1(n,n) = 1$. Assume that $k < n$. Since at most $k-1$ of $i_t, (t = 1, 2, \dots, k)$ may equal 1, then

$$\begin{aligned} c_1(n,k) &= \sum_{i=0}^{k-1}\binom{k}{i}\sum_{j_1+j_2+\cdots+j_{k-i}=n-i} f_0(j_1)f_0(j_2)\cdots f_0(j_{k-i}) \\ &= \sum_{i=0}^{k-1}\binom{k}{i}a^{k-i}(a-1)^{n-2k+i}\sum_{j_1+j_2+\cdots+j_{k-i}=n-i} 1, \end{aligned}$$

where the last sum is taken over $j_t \geq 2$. Then we have

$$c_1(n,k) = \sum_{i=0}^{k-1} a^{k-i}(a-1)^{n-2k+i}\binom{k}{i}\binom{n-k-1}{k-i-1}. \tag{2}$$

□

Remark 8. In the preceding formula, terms in which $i < 2k - n$ equal zero.

We use (E) to derive an explicit formula for $c_m(n,k)$. Extracting the term for $i = n$ yields

$$c_m(n,k) = (m-1)^{n-k}\binom{n-1}{k-1} + \sum_{i=k}^{n-1}(m-1)^{i-k}\binom{i-1}{k-1}c_1(n,i).$$

Using (2), we obtain

$$c_m(n,k) = (m-1)^{n-k}\binom{n-1}{k-1} + \sum_{i=k}^{n-1}\sum_{j=0}^{i-1}(m-1)^{i-k}a^{i-j}(a-1)^{n-2i+j}\binom{i}{j}\binom{i-1}{k-1}\binom{n-i-1}{i-j-1}.$$

An explicit formula for $f_m(n)$ can easily be obtained from (C).

The following arrays in Sloane [5] are related to this type: A154929, A113413, A054458, A116412.

3 Type 2

Let a be a positive integer. Define f_0 by

$$f_0(1) = 1, f_0(2) = 0,$$
$$f_0(n+2) = af_0(n), (n \geq 1).$$

Proposition 9. *For $a > 0$, the value of $f_0(n)$ is the number of words of length $n-1$ over the alphabet $\{0, 1, \ldots, a-1\}$ satisfying $\mathcal{P}_2$.*

Proof. Let $d(n)$ denote the number of words of length n which we wish to count. Firstly, $d(0) = 1$ since only the empty word has length 0. Next, $d(1) = 0$ as there are no runs of length 1. Assume that $n > 2$. A word of length n must begin with two identical letters. Hence, there are $ad(n-2)$ such words. We conclude that the following recurrence holds:

$$d(0) = 1, d(1) = 0, d(n) = ad(n-2), (n \geq 2), \tag{3}$$

which yields $d(n-1) = f_0(n), (n \geq 1)$. □

From (3), we easily obtain the following explicit formula for f_0.

$$f_0(n) = \begin{cases} 0, & \text{if } n = 2t; \\ a^t, & \text{if } n = 2t+1. \end{cases} \tag{4}$$

Using (B), we obtain the following result.

Corollary 10. *For $m \geq 0$, we have*

$$f_m(1) = 1, f_m(2) = m,$$
$$f_m(n+2) = mf_m(n+1) + af_m(n), (n \geq 1).$$

Proposition 11. *The value of $f_m(n)$ is the number of words of length $n-1$ over $\{0, 1, \ldots, a-1, \ldots, a+m-1\}$ which satisfy $\mathcal{P}_2$.*

Proof. We let $d(n)$ denote the number of words of length $n-1$. It is clear that $d(0) = 1$ and $d(1) = m$. A word of length $n+1$ may begin with a letter from $\{a, a+1, \ldots, a+m-1\}$. There are $md(n)$ such words. If a word begins with a letter from $\{0, 1, \ldots, a-1\}$, the second letter must be the same. Hence, there are $ad(n-1)$ such words. We conclude that next $d(n) = f_m(n+1)$. □

Some well-known classes of numbers satisfy the recurrence from Corollary 10. We give the appropriate combinatorial meaning for the Fibonacci, the Pell, and the Jacobhstal numbers.

1. The case $a = 1, m = 1$ is related to the Fibonacci numbers. The number of binary words of length $n-1$ in which 0 avoids a run of odd length is F_n.

2. The case $a = 1, m = 2$ is related to the Pell numbers P_n (A000129). The number of ternary words of length $n-1$ in which 0 avoids runs of odd length is P_n.

3. The case $a = 2, m = 1$ is related to the Jacobsthal numbers J_n (A001045). The number of ternary words of length $n-1$ in which 0 and 1 avoid runs of odd length is J_n.

From the combinatorial interpretation, we easily derive an explicit formula for $f_m(n)$.

Proposition 12. *We have*

$$f_m(n) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} m^{n-2j-1} a^j \binom{n-1-j}{j}. \tag{5}$$

Proof. A word of length $n-1$ can contain $2j$ letters from $\{0, 1, \dots, a-1\}$, so that each letter appears in a run of even length, where $0 \le j \le \lfloor \frac{n-1}{2} \rfloor$. The remaining $n-1-2j$ places are filled by letters from $\{a, a+1, \dots, a+m-1\}$ arbitrarily. For a fixed j, we have $a^j \cdot \binom{n-j-1}{j}$ subwords from $\{0, 1, \dots, a-1\}$ and m^{n-1-2j} subwords from $\{a, a+1, \dots, a+m-1\}$. Summing over j, we obtain (5). □

As a consequence, we obtain the following explicit formulas for the Fibonacci, the Pell, and the Jacobsthal numbers:

$$F_n = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-j-1}{j}, \quad P_n = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} 2^{n-2j-1} \binom{n-j-1}{j},$$

$$J_n = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} 2^j \binom{n-j-1}{j}.$$

From (F), we obtain the following result.

Proposition 13. *The value of $c_m(n, k)$ is the number of words of length $n-1$ over the alphabet $\{0, 1, \dots, a-1, \dots, a+m-1\}$ in which the letter $a+m-1$ appears $k-1$ times and which satisfy $\mathcal{P}_2$.*

We now derive an explicit formula for $c_1(n, k)$.

Proposition 14. *The following formula holds:*

$$c_1(n, k) = \begin{cases} a^{\frac{n-k}{2}} \binom{\frac{n+k}{2}-1}{k-1}, & \text{if } n-k \text{ is even;} \\ 0, & \text{if } n-k \text{ is odd.} \end{cases}$$

Proof. According to (4), each term in formula (1) equals zero if some i_t is even. Hence, (1) becomes

$$\begin{aligned} c_1(n,k) &= \sum_{2j_1+1+2j_2+1+\cdots+2j_k+1=n} a^{j_1}\cdot a^{j_2}\cdots a^{j_k} \\ &= a^{\frac{n-k}{2}} \sum_{s_1+s_2+\cdots+s_k=\frac{n+k}{2}} 1 = a^{\frac{n-k}{2}} \binom{\frac{n+k}{2}-1}{k-1}. \end{aligned}$$

Note that the last sum is over positive integers $s_1, s_2, \ldots, s_k$. □

As a consequence of (D), we obtain the following explicit formulas for the Fibonacci and the Jacobsthal numbers:

$$F_{2n} = \sum_{k=1}^{n} \binom{n+k-1}{n-k},\ F_{2n-1} = \sum_{k=1}^{n} \binom{n+k-2}{n-k},$$
$$J_{2n} = \sum_{k=1}^{n} 2^{n-k} \binom{n+k-1}{n-k},\ J_{2n-1} = \sum_{k=1}^{n} 2^{n-k} \binom{n+k-2}{n-k}.$$

Now, we derive an explicit formula for $c_2(2n,k)$. Using (E), we obtain

$$\begin{aligned} c_2(2n,k) &= \sum_{i=k}^{2n} \binom{i-1}{k-1} c_1(2n,i) = \sum_{j=\lceil \frac{k}{2} \rceil}^{n} \binom{2j-1}{k-1} c_1(2n,2j) \\ &= \sum_{j=\lceil \frac{k}{2} \rceil}^{n} a^{n-j} \binom{2j-1}{k-1} \binom{n+j-1}{n-j}. \end{aligned}$$

Furthermore,

$$\begin{aligned} c_2(2n-1,k) &= \sum_{i=k}^{2n-1} \binom{i-1}{k-1} c_1(2n,i) = \sum_{j=\lceil \frac{k+1}{2} \rceil}^{n} \binom{2j-2}{k-1} c_1(2n-1,2j-1) \\ &= \sum_{j=\lceil \frac{k+1}{2} \rceil}^{n} a^{n-j} \binom{2j-2}{k-1} \binom{n+j-2}{n-j}. \end{aligned}$$

In particular, for $a=1$, we obtain the following formulas for the Pell numbers:

$$P_{2n} = \sum_{k=1}^{2n} \sum_{j=\lceil \frac{k}{2} \rceil}^{n} \binom{2j-1}{k-1} \binom{n+j-1}{n-j},\ P_{2n-1} = \sum_{k=1}^{2n-1} \sum_{j=\lceil \frac{k+1}{2} \rceil}^{n} \binom{2j-2}{k-1} \binom{n+j-2}{n-j}.$$

Remark 15. Using (E), we easily obtain an explicit formula for $c_m(n,k)$.

The following arrays in [5] are related to this type: A000129, A001045, A168561, A037027, A054456, A132964, A073370.

4 Type 3

Let $a > b > 0$ be integers. We define f_0 by

$$f_0(1) = 1, f_0(2) = a, f_0(n+2) = af_0(n+1) - bf_0(n), (n \geq 1).$$

Proposition 16. *The value of $f_0(n)$ is the number of words of length $n-1$ over $\{0, 1, \ldots, a\}$ satisfying $\mathcal{P}_3$.*

Proof. We let $d(n)$ denote the number of words of length $n-1$. Since only the empty word has length 0, we have $d(0) = 1$. Since there are no restrictions on words of length 1, we have $d(1) = a$. Assume that $n > 1$. We have $a \cdot d(n-1)$ words beginning with an arbitrary letter. From this number, we must subtract the number of words which begin with subwords $0i, (i = 1, 2, \ldots, b)$. Hence, $d(n)$ satisfies the same recurrence as $f_0(n)$ does. $\square$

Example 17. 1. If $a = 2, b = 1$, we have

$$f_0(1) = 1, f_0(2) = 2, f_0(n+2) = 2f_0(n+1) - f_0(n), (n \geq 1),$$

which yields $f_0(n) = n$. Hence, n is the number of binary words of length $n-1$ avoiding 01, for obvious reasons.

2. If $a = 3, b = 1$, we have the well-known recurrence for the Fibonacci numbers F_{2n}:

$$f_0(1) = 1, f_0(2) = 3, f_0(n+2) = 3f_0(n+1) - f_0(n), (n \geq 1).$$

Thus, we obtain the following combinatorial interpretation of the bisection of the Fibonacci numbers.

Corollary 18. *The number of ternary words of length $n-1$ avoiding 01 is F_{2n}.*

We now consider the case when $a = b + 1$.

Corollary 19. *If $b > 1$ and $a = b + 1$, then*

$$f_0(n) = \frac{b^n - 1}{b - 1}.$$

Proof. We denote $\frac{b^n-1}{b-1}$ by $g_0(n)$. We have $g_0(1) = 1, g_0(2) = 1 + b = a$. Furthermore,

$$(b+1) \cdot g_0(n+1) - b \cdot g_0(n) = (b+1) \cdot \frac{b^{n+1} - 1}{b - 1} - b \cdot \frac{b^n - 1}{b - 1} = \frac{b^{n+2} - 1}{b - 1}.$$

By induction, we conclude that $g_0 = f_0$. $\square$

In particular, for $a = 3, b = 2$, we have $f_0(n) = 2^n - 1$, which yields the following result.

Corollary 20. *The Mersenne number* $2^n - 1$ *is the number of ternary words of length* $n-1$ *avoiding* 01 *and* 02*.*

Using (B), we obtain

$$f_m(1) = 1, f_m(2) = m + a, f_m(n+2) = (a+m)f_m(n+1) - bf_m(n), (n \geq 1).$$

This means that f_m counts the same sort of words as f_0, with $m+a$ instead of a.

Using (F) and (D), we obtain the following combinatorial interpretations of $c_m(n,k)$ and $f_m(n)$.

Corollary 21.

1. *The value of* $c_m(n,k)$ *is the number of words of length* $n-1$ *over* $\{0,1,\ldots,b-1,b\ldots,m+a-1\}$ *having* $k-1$ *letters equal* $m+a-1$ *which satisfy* $\mathcal{P}_3$*.*
2. *The value of* $f_m(n)$ *is the number of words of length* $n-1$ *over the alphabet* $\{0,1,\ldots,b-1,b\ldots,m+a-1\}$ *which satisfy* $\mathcal{P}_3$*.*

Next, we derive an explicit formula for $c_1(n,k)$. A generating function for the sequence $f_0(1), f_0(2), \ldots$ is $\frac{1}{bx^2-ax+1}$. According to [4, Equation (1)], we have

$$\frac{x^k}{(bx^2-ax+1)^k} = \sum_{n=k}^{\infty} c_1(n,k)x^k.$$

The numbers $\alpha = \frac{a+\sqrt{a^2-4b}}{2b}$ and $\beta = \frac{a-\sqrt{a^2-4b}}{2b}$ are the solutions of the equation $bx^2-ax+1 = 0$.

Proposition 22. *The following equality holds:*

$$c_1(n,k) = \frac{1}{b^k}\sum_{j=0}^{n-k} \frac{1}{\alpha^{j+k}\beta^{n-j}}\binom{n-j-1}{k-1}\binom{k+j-1}{k-1}.$$

Proof. We expand $\frac{x^k}{b^k(\alpha-x)^k(\beta-x)^k}$ into powers of x. Since

$$\frac{1}{(\gamma-x)^k} = \sum_{i=0}^{\infty}\binom{k+i-1}{k-1}\frac{x^i}{\gamma^{i+k}},$$

we easily obtain

$$\frac{x^k}{b^k(\alpha-x)^k(\beta-x)^k} = \sum_{i=0}^{\infty}\left[\sum_{j=0}^{i}\frac{1}{b^k\alpha^{j+k}\beta^{i-j+k}}\binom{k+j-1}{k-1}\binom{k+i-j-1}{k-1}\right]x^{i+k},$$

and the statement follows by replacing i with $n-k$. □

In the case $a = b + 1$, we have $\alpha = 1$ and $\beta = \frac{1}{b}$. Therefore, the following formula holds:

$$c_1(n,k) = \sum_{i=0}^{n-k} b^{n-k-i} \binom{n-i-1}{k-1} \binom{k+i-1}{k-1}. \tag{6}$$

Using (1), we obtain the following identity:

Identity 23.

$$\sum_{i_1+i_2+\cdots+i_k=n} \left[\prod_{t=1}^{k} (b^{i_t} - 1) \right] = \sum_{i=0}^{n-k} b^{n-k-i} \binom{n-i-1}{k-1} \binom{k+i-1}{k-1},$$

where $i_t, (t = 1, 2, \ldots, k)$ *are positive integers.*

Remark 24. Using (D) and (E), we obtain explicit formulas for $f_m(n)$ and $c_m(n,k)$.

The following arrays in [5] are related to this type: A078812, A125662, A207823, A207824, A110441, A116414.

5 Type 4

We solve the problem for binary words first.

Proposition 25. *Let* $f_0(n)$ *be the number of binary words of length* $n-1$ *satisfying* $\mathcal{P}_4$*. Then,*

$$\begin{aligned} f_0(1) &= 1, f_0(2) = 0, \\ f_0(n+2) &= f_0(n+1) + f_0(n), (n > 1), \\ f_0(n) &= F_{n-2}, (n > 1). \end{aligned}$$

Proof. We have $f_0(1) = 1$, since only the empty word has length 0. Next, $f_0(2) = 0$, since no words of length 1 satisfy $\mathcal{P}_4$. Also, $f_0(3) = 1$, since 10 is the only word of length 2 satisfying $\mathcal{P}_4$. Next, $f_0(4) = 1$, since 100 is the only word of length 3 which satisfies $\mathcal{P}_4$. Assume that $n > 1$. Then,

$$f_0(n+4) = f_0(n+2) + f_0(n+1) + \cdots,$$

since the word of length greater than 3 must begin with a subword of the form $10\ldots0$. Analogously, we obtain

$$f_0(n+5) = f_0(n+3) + f_0(n+2) + \cdots.$$

Comparing these two equalities, we get

$$f_0(n+5) = f_0(n+4) + f_0(n+3).$$

The explicit formula follows from the preceding recurrence. □

Since $f_0(1) = 1$, and so $f_m(1) = 1$, using (D) and (F), we obtain the following combinatorial interpretations of f_m and $c_m(n, k)$.

Corollary 26. *1. The value of $c_m(n, k)$ is the number of words of length $n - 1$ over $\{0, 1, \ldots, m + 1\}$ having $k - 1$ letters equal $m + 1$ and satisfying $\mathcal{P}_4$.*

2. The value of $f_m(n)$ is the number of words of length $n-1$ over the alphabet $\{0, 1, \ldots, m+1\}$ which satisfy $\mathcal{P}_4$.

We next derive an explicit formula for $c_1(n, k)$. It is known that $c_1(n, k)$ is the coefficient of x^n in the expansion of $\left(\sum_{i=1}^{\infty} F_{i-2}x^i\right)^k$ into powers of x. We consider the following auxiliary initial function:

$$\overline{f}_0(1) = 0, \overline{f}_0(n) = 1, (n > 1).$$

From [2, Proposition 23], we obtain $\overline{f}_1(n) = F_{n-1}$. It is proved in [3, Proposition 13] that

$$\bar{c}_1(n, k) = \binom{n-k-1}{k-1}, \left(k = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right),$$

and $\bar{c}_1(n, k) = 0$, otherwise.

Using (E) implies

$$\bar{c}_2(n, k) = \sum_{i=k}^{\lfloor \frac{n}{2} \rfloor} \binom{i-1}{k-1} \binom{n-i-1}{i-1}.$$

Hence,

$$\left(\sum_{i=1}^{\infty} F_{i-1}x^i\right)^k = \sum_{n=k}^{\infty} \bar{c}_2(n, k)x^n. \tag{7}$$

Let X denote $\sum_{i=1}^{\infty} F_{i-1}x^i$. We expand the expression Y^k, where $Y = \sum_{i=1}^{\infty} F_{i-2}x^i$. Since $F_{-1} = 1$, we have $Y = x(1 + X)$, which yields

$$Y^k = x^k \left(1 + \sum_{i=1}^{k} \binom{k}{i} X^i\right)^k = \sum_{n=k}^{\infty} c_1(n, k)x^n.$$

Using the binomial theorem and (7) yields

$$Y^k = x^k + \sum_{i=1}^{k} \sum_{j=i}^{\infty} \binom{k}{i} \bar{c}_2(j, i)x^{j+k}.$$

For $j + k = n$, the coefficient of x^n on the right-hand side is $\sum_{i=1}^{k} \binom{k}{i} \bar{c}_2(n - k, i)$.

Proposition 27. *We have*

$$c_1(n,n) = 1,$$
$$c_1(n,k) = \sum_{t=1}^{k} \sum_{j=t}^{\lfloor \frac{n-k}{2} \rfloor} \binom{k}{t} \binom{j-1}{t-1} \binom{n-k-j-1}{j-1}, (n > k).$$

Using (B), we easily obtain

$$f_m(1) = 1, f_m(2) = m,$$
$$f_m(n+2) = (m+1)f_m(n+1) - (m-1)f_m(n).$$

We examine two particular cases. In the case $m = 1$, we obtain

$$f_1(1) = 1, f_1(2) = 1,$$
$$f_1(n+2) = 2f_1(n+1), (n > 1),$$

which implies

$$f_1(1) = f_1(2) = 1, f_1(n) = 2^{n-2}, (n > 2).$$

Thus we obtain the following property of powers of 2.

Corollary 28. *For $n \geq 2$, the number 2^{n-2} is the number of ternary words of length $n-1$ which satisfy $\mathcal{P}_4$.*

As a consequence, the following Euler-type identity holds.

Identity 29. *For $n > 2$, the number of binary words of length $n-2$ equals the number of ternary words of length $n-1$, in which 0 and 1 appear only in a run of the form $1i$, where i is the run of zeros of length $i \geq 1$.*

From Propositions 27 and (D), we obtain the following identity for the Mersenne numbers.

Identity 30.

$$2^{n-2} - 1 = \sum_{k=1}^{n} \sum_{i=1}^{k} \sum_{j=i}^{\lfloor \frac{n-k}{2} \rfloor} \binom{k}{i} \binom{j-1}{i-1} \binom{n-k-j-1}{j-1}, \ (n > 2).$$

We now consider the case $m = 2$. We have

$$f_2(1) = 1, f_2(2) = 2,$$
$$f_2(n+2) = 3f_2(n+1) - f_2(n),$$

which is the recurrence for the Fibonacci numbers F_{2n-1}.

Corollary 31. *The number F_{2n-1} is the number of quaternary words of length $n-1$ which satisfy $\mathcal{P}_4$.*

Calculating values for $c_2(n,k)$, we obtain a peculiar expression for F_{2n-1}.

Identity 32.

$$F_{2n-1} = \sum_{k=1}^{n}\sum_{i=k}^{n}\sum_{t=0}^{i}\sum_{j=t}^{\lfloor \frac{n-i}{2} \rfloor} \binom{i-1}{k-1}\binom{i}{t}\binom{j-1}{t-1}\binom{n-i-j-1}{j-1}.$$

Remark 33. Using (E) and (D), we obtain the explicit formulas for $c_m(n,k)$ and $f_m(n)$.

The following arrays in [5] are related to this type: A105422, A105306, A062110, A188137.

6 Type 5

Again, we consider binary words first.

Proposition 34. *The following recurrence holds:*

$$f_0(1) = 1, f_0(2) = 0, f_0(3) = 1,$$
$$f_0(n+3) = f_0(n+1) + f_0(n), (n \geq 1).$$

We have $f_0(n) = p_{n+2}$, where p_n is the nth Padovan number (A000931).

Proof. It is easy to see that the initial conditions are satisfied. A word of length $n+2$ begins with either two zeros or three ones and the recurrence follows.

Since we have a recurrence for the Padovan numbers, the second statement is true. □

This means that the Padovan number p_{n+2} is the number of binary words of length $n-1$ in which 0 appears in runs of even length, while 1 appears in runs of lengths divisible by 3. This is equivalent to the fact that the Padovan numbers count the compositions into parts 2 and 3 (see the comment in A000931).

Corollary 35. *1. The function f_m satisfies*

$$f_m(1) = 1, f_m(2) = m, f_m(3) = m^2 + 1,$$
$$f_m(n+3) = m f_m(n+2) + f_m(n+1) + f_m(n), (n > 1).$$

2. *The value $c_m(n,k)$ is the number of words of length $n-1$ over $\{0, 1, \ldots, m+1\}$ having $k-1$ letters equal to $m+1$, and satisfying $\mathcal{P}_5$.*

3. *The value $f_m(n)$ is the number of words of length $n-1$ over $\{0, 1, \ldots, m+1\}$ which satisfy $\mathcal{P}_5$.*

Proof. Claim 1 follows from (A) easily. Claims 2 and 3 follow from (F) and (D).

We add a short combinatorial proof of 2. Equality $f_m(1) = 1$ means that the empty word satisfies $\mathcal{P}_5$. Furthermore, $f_m(2) = m$ means that a word of length 1 may consist of any letter except 0 and 1. Next, $f_m(3) = m^2 + 1$ means that a word of length 2 may consist of pairs from $\{2, 3, \ldots, m+1\}$, which are m^2 in number, plus the word 00. Finally, a word of length $n > 2$ may begin with any letter from $\{2, 3, \ldots, m+1\}$, or with 00, or with 111. $\square$

The case $m = 1$ in Corollary 35 is the recurrence for Tribonacci numbers.

Corollary 36. *The sequence* $1, 1, 2, 4, 7, \ldots$ *of the Tribonacci numbers is the invert transform of the sequence* $1, 0, 1, 1, 1, 2, \ldots$ *of the Padovan numbers. Also, the Tribonacci numbers count ternary words satisfying* $\mathcal{P}_5$.

Finally, we calculate $c_1(n, k)$. We define the arithmetic function $\overline{f}_0$ such that $\overline{f}_0(2) = \overline{f}_0(3) = 1$, and $\overline{f}_0(n) = 0$ otherwise. It is proved in [3, Propositon 5] that $\overline{c}_1(n, k) = \binom{k}{n-2k}$, and

$$\overline{f}_1(1) = 0, \overline{f}_1(2) = 1, \overline{f}_1(3) = 1,$$
$$\overline{f}_1(n+3) = \overline{f}_0(n+1) + \overline{f}_0(n).$$

This implies that $\overline{f}_1(n) = f_0(n-1), (n > 1)$. The sequence $f_0(1), f_0(2), \ldots$ is thus obtained by inserting 1 at the beginning of the sequence $\overline{f}_1(1), \overline{f}_1(2), \ldots$.

Using (E), we obtain

$$\overline{c}_2(n, k) = \sum_{i=k}^{n} \binom{i-1}{k-1} \cdot \binom{i}{n - 2 \cdot i},$$

which implies

$$\left(\sum_{i=1}^{\infty} \overline{f}_1(i) x^i \right)^k = \sum_{n=k}^{\infty} \overline{c}_2(n, k) x^n. \tag{8}$$

To obtain an explicit formula for $c_1(n, k)$, we need to expand the expression X given by $X = \left(\sum_{i=1}^{\infty} f_0(i) x^i\right)^k$ into powers of x. We have

$$X = \left(x + \sum_{i=2}^{\infty} f_0(i) x^i \right)^k = (x + xY)^k,$$

where $Y = \sum_{i=1}^{\infty} \overline{f}_1(i) x^i$. Hence,

$$X = x^k \sum_{i=0}^{k} \binom{k}{i} Y^i.$$

Applying (8) implies

$$X = \sum_{i=0}^{k} \binom{k}{i} \sum_{j=i}^{\infty} \overline{c}_2(j,i) x^{j+k}.$$

Taking $n = j + k$, we get the following result.

Proposition 37. *The following formulas hold:*

$$c_1(n,n) = 1,$$
$$c_1(n,k) = \sum_{i=0}^{k} \sum_{j=i}^{n-k} \binom{k}{i} \binom{j-1}{i-1} \binom{j}{n-k-2j}, (k < n).$$

In particular, we have the following identity for the Tribonacci numbers.

Identity 38.

$$T_n = 1 + \sum_{k=1}^{n-1} \sum_{i=0}^{k} \sum_{j=i}^{n-k} \binom{k}{i} \binom{j-1}{i-1} \binom{j}{n-k-2j}.$$

Remark 39. Using (E) and (D), we obtain explicit formulas for $c_m(n,k)$ and $f_m(n)$.

The following arrays in Sloane [5] are related to this type: A104578, A104580.

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