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Sums of Digits and the Distribution of Generalized Thue-Morse Sequences

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Abstract

In this paper we study the distribution of the infinite word $\mathrm{t}_{q,n} := (s_q(k) \bmod n)_{k=0}^{\infty}$, which we call the generalized Thue-Morse sequence. Here $s_q(k)$ is the digit sum of k in base q. We give an explicit formulation of the exact minimal value of M such that every M consecutive terms in $t_{q,n}$ cover the residue system of n, i.e., $\{0, 1, \ldots, n-1\}$. Also, we prove some stronger related results.

1 Introduction and main results

For $k \in \mathbb{N}$ and $q \in \{2, 3, \ldots\}$, let $s_q(k)$ denote the sum of the digits of k when expressed in base q. By convention, we use $k = (k_1^{d_1} k_2^{d_2} \cdots k_l^{d_l})_q$ to denote

$$k = (\underbrace{k_1 k_1 \cdots k_1}_{d_1} \underbrace{k_2 k_2 \cdots k_2}_{d_2} \cdots \underbrace{k_l k_l \cdots k_l}_{d_l})_q,$$

where we first have d_1 k_1s followed by d_2 k_2s, and so on up until d_l k_ls, and omit every $d_i = 1$. For example, the binary expansion of 6 is 110, so $6 = (1^2 0)_2$ and $s_2(6) = 1 + 1 + 0 = 2$.

The sum of digits is an interesting object in number theory. In recent years, there have been some new results about the distribution of the digit sum sequence $(s_q(k))_{k=1}^{\infty}$. Morgenbesser, Shallit, and Stoll [1] considered the classical Thue-Morse sequence

$$(s_2(k) \bmod 2)_{k=1}^{\infty}.$$

They proved that the least number k satisfying $s_2(d \cdot k) \equiv 1 \pmod 2$ is at most $d+4$, for every fixed positive integer d. For the general infinite word $\mathrm{t}_{q,n} := (s_q(k) \bmod n)_{k=0}^{\infty}$, Allouche and Shallit [2] showed that the sequence $\mathrm{t}_{q,n}$ over the alphabet $\{0, 1, \ldots, n-1\}$ is overlap-free if and only if $n \geq q$.

In this paper we study the distribution of the generalized Thue-Morse sequence $\mathrm{t}_{q,n}$. We give the exact minimal positive integer $M_{q,n}$ (see Definition 3) such that every $M_{q,n}$ consecutive terms in $\mathrm{t}_{q,n}$ contain j for every $j \in \{0, 1, \ldots, n-1\}$. First we give some basic examples to help readers understand.

Example 1 ($M_{10,7} = 13$)**.** Every 13 consecutive positive integers have an element whose digit sum is divisible by 7. But it is false for 12 consecutive positive integers because the sums of digits of the numbers $994, 995, \ldots, 999$, $1000, 1001, \ldots, 1005$, are all not divisible by 7.

Example 2 ($M_{10,11} = 39$)**.** Every 39 consecutive positive integers have an element whose digit sum is divisible by 11, but it is false for 38 consecutive positive integers. In fact, it is easy to check that the sums of digits of the numbers $999981, 999982, \ldots, 999999, 1000000, 1000001, \ldots, 1000018$, are all not divisible by 11.

The general results are given in Corollary 13.

Next, we introduce the positive integers $M_{q,n}$, where $n \in \mathbb{Z}^+$ and $q \in \{2, 3, \ldots\}$.

Definition 3. Let $q, n \in \mathbb{Z}^+$ with $q \geq 2$ and $n = k \cdot (q-1) + l$, where $l \in \{0, 1, \ldots, q-2\}$ and $k \in \mathbb{N}$. For convenience, let

$$r = \begin{cases} \gcd(q-1, l) \cdot \left\lfloor \frac{l-1}{\gcd(q-1,l)} \right\rfloor, & \text{if } 1 \leq l \leq q-2; \\ 0, & \text{if } l = 0. \end{cases} \tag{1}$$

Now the number $M_{q,n}$ is defined to be $(l+r+1) \cdot q^k - 1$, or equivalently $M_{q,n} = ((l+r)(q-1)^k)_q$.

Theorem 4 and Theorem 5 are the main results of this paper.

Theorem 4. *The number $M_{q,n}$ is the least value of M, where every M consecutive terms in $\mathrm{t}_{q,n}$ contain* 0*.*

In other words, every $M_{q,n}$ consecutive positive numbers contain a number whose digit sum is divisible by n. And there exists a sequence of $M_{q,n} - 1$ consecutive positive numbers *such that it contains no integer whose digit sum is divisible by* n*. The least value of the first term of such sequence is*

$$\begin{cases} (1)_q, & \text{if} \quad l-1 < \gcd(q-1,l); \\ ((q-1)^x(q-1-r)0^{k-1}1)_q, & \text{if} \quad l-1 \geq \gcd(q-1,l), \end{cases} \tag{2}$$

where r *is given in Eq.* (1)*, and* x *is the minimal nonnegative integer solution of the congruence equation*

$$(q-1)\cdot(x+1) - r \equiv 0 \pmod{(q-1)\cdot k + l}. \tag{3}$$

Theorem 5 is a strengthened form of Theorem 4.

Theorem 5. *For every* $j \in \{0,1,\ldots,n-1\}$*, the minimum value of* M*, where every* M *consecutive terms in* $\mathrm{t}_{q,n}$ *contain* j*, is also equal to* $M_{q,n}$*. In other words, every* $M_{q,n}$ *consecutive positive integers contain an integer* d *with* $s_q(d) \equiv j \pmod n$*. And there exists a sequence of* $M_{q,n}-1$ *consecutive positive integers containing no integer* d *with* $s_q(d) \equiv j \pmod n$*. The first term of such sequence can be chosen as* $(1^j0(q-1)^x(q-1-r)0^{k-1}1)_q$*, where* x *is a nonnegative integer solution of the congruence equation* $(q-1)\cdot(x+1) - r \equiv 0 \pmod n$*.*

Remark 6. The facts below are true about $M_{q,n}$.

1. For the following set of sequences

   $\mathbb{A}_q(n) = \{(m+i)_{i=0}^{k} | m,\ldots,m+k$ are not divisible by $n,$ where $m,k \in \mathbb{Z}^+\},$

   we have $\max\limits_{S\in\mathbb{A}_q(n)} \text{length}\,(S) = M_{q,n}-1$. Here, length (S) represents the number of terms in sequence S.

2. $M_{q,n}$ is the least value of M, where every M consecutive terms in $\mathrm{t}_{q,n}$ cover the residue system of n, i.e., $\{0,1,\ldots,n-1\}$.

2 Proofs

In this section, we prove our main results. Before that, some lemmas as follows are needed.

Lemma 7. *For positive integers* a, b *and* m*, the equation* $ax \equiv b \pmod m$ *has a positive integer solutions if and only if* $\gcd(a,m)|b$*.*

Lemma 8. *For fixed* $h \in \{0,\ldots,q-2\}$ *and* $t \in \mathbb{N}$*, the following statements are true.*

(1) Consider a sequence of consecutive integers starting with zero. If no integer in this sequence has digit sum over $s := (q-1)\cdot t + h$*, then the length of the sequence is not longer than* $((h+1)(q-1)^t)_q$ *that has sum of digits* $s+1$*.*

(2) *Consider a sequence of consecutive nonnegative integers ending with* $((q-1)^k)_q$. *If every integer in this sequence has digit sum at least* $(q-1)\cdot k - s + 1$, *then the length of the sequence is not longer than* $(h(q-1)^t)_q$.

Proof. (1) If $t \geq 1$, then $((h+1)(q-1)^t)_q - 1 = ((h+1)(q-1)^{t-1}(q-2))_q$ has sum of digits

$$(h+1) + (t-1)\cdot(q-1) + (q-2) = h + t\cdot(q-1) = s.$$

If $t = 0$, then $((h+1)(q-1)^0)_q - 1 = h + 1 - 1 = h = s$ still has sum of digits s. Since the terms of the sequence are consecutive, the length of such sequence is not longer than $((h+1)(q-1)^t)_q - 1 + 1 = ((h+1)(q-1)^t)_q$.

(2) Let $((q-1)^k)_q$ minus $((q-1)^k)_q, \ldots, (1)_q, (0)_q$ respectively. Then the sequence in (2) starts with 0, and no integer in this sequence has digit sum over $s-1$. Combining with $s - 1 = (q-1)\cdot t + h - 1$ and the conclusion of (1), we can easily derive that the length of the sequence satisfying the condition is $(h(q-1)^t)_q$. □

Let $a_k(n)$ be the coefficient of q^k of the representation of n in base q (i.e., $n = \sum_{k=0}^{\infty} a_k(n)q^k$, where $a_k(n) \in \{0, 1, \ldots, q-1\}$) and let $v_q(n) = \max\{k \in \mathbb{N} : q^k | n\}$. We have the following result.

Lemma 9. *For every positive integer* A, *we have* $s_q(A) = s_q(A+1) + (q-1)\cdot x - 1$, *where* x *is the number of consecutive* $(q-1)$ *in the tail of* A, *or equivalently,* $x = v_q(A+1)$.

Hereinafter, we use $[a, b]$ (resp., $[a, b)$, (a, b) and $(a, b]$) to denote the set of integers in the interval $[a, b]$ (resp., $[a, b)$, (a, b) and $(a, b]$), where a, b are integers.

Lemma 10. *Below we make some relevant properties.*

(1) *For every integer* X, $s_q(X) + 1 \geq s_q(X+1)$.

(2) *Let* $A, \ldots, B$ *be consecutive positive integers satisfying* $s_q(A) = \min_{X\in[A,B]} s_q(X)$. *Then* $\{s_q(A), \ldots, s_q(B)\} = [s_q(A), \max_{X\in[A,B]} s_q(X)]$.

(3) *Let* A, A', B *and* B' *be positive integers such that* $B - A = B' - A'$, $s_q(A) = \min_{X\in[A,B]} s_q(X) = s_q(A') = \min_{X'\in[A',B']} s_q(X')$ *and* $s_q(A+C) \geq s_q(A'+C)$ *for every* $C \in [0, B-A]$. *Then* $\{s_q(A), \ldots, s_q(B)\} \supset \{s_q(A'), \ldots, s_q(B')\}$.

Proof. (1) is easy to verify by Lemma 9 and (3) can be deduced from (2). So it suffices to prove (2).

Suppose the contrary, that there exists an integer $N \in (s_q(A), \max_{X\in[A,B]} s_q(X))$ such that $s_q(X) \neq N$ for every $X \in [A, B]$.

Let $C = \min\{D \in [A, B]\,|\,s_q(D) > N\}$. Then by the definitions of C and N, we have $s_q(C) > N$ and $s_q(C-1) < N$. However, according to (1), $s_q(C-1) + 1 \geq s_q(C) > N$, that is $s_q(C-1) \geq N$. This leads to a contradiction. □

Lemma 11. *Let $q, n \in \mathbb{Z}^+$ with $q \geq 2$ and $n = k \cdot (q-1) + l$, where $l \in \{0, 1, \ldots, q-2\}$ and $k \in \mathbb{N}$. Every $(l(q-1)^k)_q$ consecutive terms of $\mathrm{t}_{q,n}$ with indexes in $[(A0^{k+1})_q, ((A+1)0^{k+1})_q)$ for some $A \in \mathbb{N}$, cover $\{0, 1, \ldots, n-1\}$, where A is always written in base q.*

Proof. Consider a sequence of $(l(q-1)^k)_q$ consecutive positive integers. We divide this proof into two cases.

Case 1: $l \geq 1$. In fact, the first $q^k = (10^k)_q$ terms of such sequence must contain a number N satisfying $a_i(N) = 0, i \in \{1, 2, \ldots, k\}$ and $a_{k+1}(N) \in \{0, 1, \ldots, q-l\}$. Therefore, the numbers $N, N+1, \ldots, N+(q-1), N+(1(q-1))_q, \ldots, N+((q-1)(q-1))_q, N+(1(q-1)(q-1))_q, \ldots, N+((q-1)(q-1)(q-1))_q, \ldots, N+((q-1)^k)_q, N+(1(q-1)^k)_q, \ldots, N+((l-1)(q-1)^k)_q$ all belong to such sequence, and they all belong to the interval $[(A0^{k+1})_q, ((A+1)0^{k+1})_q)$ as well. Their digit sums are respectively $s_q(N), s_q(N)+1, \ldots, s_q(N)+(q-1), s_q(N)+q, \ldots, s_q(N)+k \cdot (q-1), \ldots, s_q(N)+k \cdot (q-1)+l-1$, which cover all the residue classes modulo $k \cdot (q-1)+l$.

Case 2: $l = 0$. If the $((q-1)^k)_q$ consecutive positive integers lie in $[(Ab0^k)_q, (Ab(q-1)^k)_q]$ for some $b \in \{0, 1, \ldots, q-1\}$, then they must be $(Ab0^k)_q, (Ab0^{k-1}1)_q, \ldots, (Ab(q-1)^{k-1}(q-2))_q$ or $(Ab0^{k-1}1)_q, (Ab0^{k-1}2)_q, \ldots, (Ab(q-1)^k)_q$, and their digit sums are $s(A)+b, s(A)+b+1, \ldots, s(A)+b+k \cdot (q-1)-1$ or $s(A)+b+1, s(A)+b+2, \ldots, s(A)+b+k \cdot (q-1)$, which cover the residue classes modulo $k \cdot (q-1)$.

If the $((q-1)^k)_q$ consecutive positive integers are not in $[(Ab0^k)_q, (Ab(q-1)^k)_q]$ for every $b \in \{0, 1, \ldots, q-1\}$, then there exists an integer $b \in [0, q-2]$ such that $(Ab(q-1)^k)_q$ and $(A(b+1)0^k)_q$ are contained in these consecutive positive integers. Hence, there exists an integer X with the form of k digits[1] such that these consecutive positive integers can be written as $(AbX)_q, \ldots, (Ab(q-1)^k)_q, (A(b+1)0^k)_q, \ldots, (A(b+1)(X-2))_q$. Note that $s(A(b+1)Y) \geq s(Ab(Y+1))$ for every Y with the form of k digits. Thus, by Lemma 10 (3) it is easy to verify that the digit sums of $(A(b+1)0^k)_q, \ldots, (A(b+1)(X-2))_q$ cover the digit sums of $(Ab0^{k-1}1)_q, \ldots, (Ab(X-1))_q$. Therefore, the digit sums of $(AbX)_q, \ldots, (Ab(q-1)^k)_q, (A(b+1)0^k)_q, \ldots, (A(b+1)(X-2))_q$ cover the digit sums of $(Ab0^{k-1}1)_q, (Ab0^{k-1}2)_q, \ldots, (Ab(q-1)^k)_q$, and consequently they cover the residue classes modulo $k \cdot (q-1)$. □

We are now ready to prove our main results.

Proof of Theorem 4. We divide our proof into three steps.

Step 1. In this step we prove every $M_{q,n}$ consecutive positive integers contain a number whose digit sum is divisible by n.

Suppose the contrary, that there exists a sequence possessing $M_{q,n}$ consecutive positive integers which contains no number whose digit sum is divisible by n.

By Lemma 11 and the fact that $(l(q-1)^k)_q \leq M_{q,n}$ from the definition of $M_{q,n}$, the sequence is not contained in $[(A0^{k+1})_q, ((A+1)0^{k+1})_q)$ for every $A \in \mathbb{N}$. Furthermore, it is obvious that there exists an $A \in \mathbb{N}$ such that the sequence can be written in two parts

$$\underbrace{(As_1 \cdots s_{k+1})_q, \ldots, (A(q-1)^{k+1})_q}_{\text{the first part}}, \underbrace{((A+1)0^{k+1})_q, \ldots, ((A+1)e_1 \cdots e_{k+1})_q}_{\text{the second part}} \qquad (4)$$

[1] In fact, this means that X should be taken from $\{(0^{k-1}1)_q, (0^{k-1}2)_q, \ldots, ((q-1)^k)_q\}$.

where $s_i, e_i \in \{0, 1, \ldots, q-1\}$ satisfy

$$(A(q-1)^{k+1})_q - (As_1 \cdots s_{k+1})_q < (l(q-1)^k)_q - 1,$$

$$((A+1)e_1 \cdots e_{k+1})_q - ((A+1)0^{k+1})_q < (l(q-1)^k)_q - 1$$

and

$$((A+1)e_1 \cdots e_{k+1})_q - (As_1 \cdots s_{k+1})_q = M_{q,n} - 1.$$

Notice that

$$s_q((A(q-1)^{k+1})_q) = s_q(A) + (q-1) \cdot (k+1)$$

and

$$s_q(((A+1)0^{k+1})_q) = s_q(A+1).$$

Suppose

$$s_q(A) + (q-1) \cdot (k+1) \equiv \alpha \pmod{(q-1) \cdot k + l} \tag{5}$$

and

$$s_q(A+1) \equiv \beta \pmod{(q-1) \cdot k + l}, \tag{6}$$

where $\alpha, \beta \in \{0, 1, 2, \ldots, (q-1) \cdot k + l - 1\}$. Since every number F in the sequence shown in (4) satisfies $s_q(F) \not\equiv 0 \pmod{n}$, we indeed have

$$\alpha, \beta \in \{1, 2, \ldots, (q-1) \cdot k + l - 1\}. \tag{7}$$

Then the digit sums of the second part of the sequence shown in (4) are contained in the following $n - \beta$ numbers:

$$s_q(A+1), \ldots, s_q(A+1) + (q-1) \cdot k + l - 1 - \beta. \tag{8}$$

The digit sums of the first part of the sequence shown in (4) are contained in the following α numbers:

$$s_q(A) + (q-1) \cdot (k+1) - \alpha + 1, \ldots, s_q(A) + (q-1) \cdot (k+1). \tag{9}$$

By Lemma 9, we have $s_q(A) = s_q(A+1) + (q-1) \cdot x - 1$, where x is the number of consecutive $(q-1)$ in the tail of A, i.e., $x := v_q(A+1)$. Hence, we deduce from (5) that

$$s_q(A+1) + (q-1) \cdot x - 1 + (q-1) \cdot (k+1) \equiv \alpha \pmod{(q-1) \cdot k + l}.$$

Combining with (6), we have

$$\beta - \alpha + (q-1) \cdot (x + k + 1) - 1 \equiv 0 \pmod{(q-1) \cdot k + l}. \tag{10}$$

According to Lemma 7, (10) has a positive integer solution if and only if $\gcd(q-1, (q-1) \cdot k + l) | \alpha + 1 - \beta$, namely

$$\gcd(q-1, l) | \alpha + 1 - \beta. \tag{11}$$

A quick inspection of (7) reveals $\alpha+1-\beta \leq (q-1)\cdot k+l-1$. Note that $(q-1)\cdot k+r$ is the largest integer which is divisible by $\gcd(q-1,l)$ and not larger than $(q-1)\cdot k+l-1$. Thus, combining with (7) and (11), it is easy to see that $\alpha+1-\beta$ is not larger than $(q-1)\cdot k+r$. Next, in order to get a contradiction, we will estimate the lengths of the two parts of the sequence shown in (4).

(I). We will apply Lemma 8 (1) to the second part of the sequence. Combining with Lemma 11, we know that the number of the consecutive integers behind $(A+1)0^{k+1}$ is less than $(l(q-1)^k)_q$, and then these integers all have the form $((A+1)X)_q$, where X possesses at most $k+1$ digits. Thus, from $s_q(((A+1)X)_q) = s_q(A+1)+s_q(X)$ and (8) we know

$$\begin{aligned} s_q(X) &= s_q(((A+1)X)_q) - s_q(A+1) \\ &\leq s_q(A+1) + (q-1)\cdot k + l - 1 - \beta - s_q(A+1) \\ &= (q-1)\cdot k + l - 1 - \beta \qquad (12) \\ &:= (q-1)\cdot m + \beta_1, \qquad (13) \end{aligned}$$

where $m \leq k$ and $\beta_1 \in \{0,1,\ldots,q-2\}$. Note that the least value of $s_q(X)$ is 0. Therefore, by Lemma 8 (1), we can obtain that the length of the second part of the sequence is at most $((\beta_1+1)(q-1)^m)_q$.

(II). We will apply Lemma 8 (2) to the first part of the sequence. Combining with Lemma 11, we know that the number of the consecutive integers before $(A(q-1)^{k+1})_q$ is less than $(l(q-1)^k)_q$, and then obtain these integers all have the form $(AY)_q$, where Y possesses at most $k+1$ digits. Therefore, from $s_q((AY)_q) = s_q(A)+s_q(Y)$ and (9), we have

$$\begin{aligned} s_q(Y) &= s_q((AY)_q) - s_q(A) \\ &\geq s_q(A) + (q-1)\cdot(k+1) - \alpha + 1 - s_q(A) \\ &= (q-1)\cdot(k+1) - \alpha + 1. \end{aligned}$$

Let $\alpha = (q-1)\cdot t + h$, where $t \leq k$ and $h \in \{0,1,\ldots,q-2\}$. Since Y ends up with $(q-1)^{k+1}$, by Lemma 8 (2), the length of the first part of the sequence must be at most $h(q-1)^t$.

Summing up the above, we obtain the length of this sequence is at most

$$((\beta_1+1)(q-1)^m)_q + (h(q-1)^t)_q.$$

Since $(h(q-1)^t)_q < ((q-1)^k)_q$ for $t \leq k-1$, we should let $t=k$ to make $((\beta_1+1)(q-1)^m)_q+(h(q-1)^t)_q$ as large as possible. And then, we obtain from (II) that $h = \alpha-(q-1)\cdot k$.

Since $((\beta_1+1)(q-1)^m)_q \leq ((q-1)^{k-1})_q < ((q-1)^k)_q$ for $m \leq k-2$, we should take $m \in \{k-1,k\}$ to make $((\beta_1+1)(q-1)^m)_q + (h(q-1)^t)_q$ as large as possible.

If $l = 0$, (12) and (13) imply $m \le k-1$ and thus we should let $m = k-1$. Then, we obtain from (I) that $q-1-1-\beta = \beta_1 \le q-2$. Hence, $((\beta_1+1)(q-1)^m)_q \le ((q-1)^k)_q$. To make $((\beta_1+1)(q-1)^m)_q + (h(q-1)^t)_q$ as large as possible, we should take $\beta_1 = q-2$. And in this case, $\beta = l = 0$.

If $l \ge 1$, (12) and (13) imply $m \le k$ and thus we should let $m = k$. Then, we obtain from (I) that $l-1-\beta = \beta_1 \ge 0$.

Note that in both cases, we have

$$\begin{aligned}((\beta_1+1)(q-1)^m)_q + (h(q-1)^t)_q &= ((l-\beta)(q-1)^k)_q + ((\alpha-(q-1)\cdot k)(q-1)^k)_q \\ &= ((l+\alpha+1-\beta-(q-1)\cdot k)(q-1)^k)_q - 1 \\ &\le ((l+r)(q-1)^k)_q - 1 = M_{q,n} - 1, \end{aligned} \tag{14}$$

which is a contradiction. The proof in Step 1 is completed.

Step 2. In this step, we will prove there exists a sequence of $M_{q,n}-1$ consecutive positive numbers containing no integer whose digit sum is divisible by n.

Note that in the case of $l-1 < \gcd(q-1,l)$, we have $M_{q,n} = (l(q-1)^k)_q$ by Definition 3, and thus the numbers $1, 2, \ldots, (l(q-1)^{k-1}(q-2))_q$ are $M_{q,n}-1$ consecutive positive integers containing no integer whose digit sum is divisible by $n = k\cdot(q-1)+l$. So we only need to explain the case of $l-1 \ge \gcd(q-1,l)$ for detail. Now, we verify the following $M_{q,n}-1 = ((l+r)(q-1)^k)_q - 1$ consecutive positive integers

$$\underbrace{((q-1)^x(q-1-r)0^{k-1}1)_q, \ldots, ((q-1)^{x+1+k})_q}_{\text{the first part}}, \underbrace{(10^{x+1+k})_q, \ldots, (10^x(l-1)(q-1)^{k-1}(q-2))_q}_{\text{the second part}} \tag{15}$$

contain no integer whose digit sum is divisible by n.

The sums of digits of the integers in the first part shown in (15) are contained in $(q-1)\cdot x+q-r, \ldots, (q-1)\cdot(x+1+k)$, which equal to $1, 2, \ldots, (q-1)\cdot k + r$ modulo n according to Eq. (3).

The sums of digits of the integers in the second part shown in (15) are contained in $1, \ldots, (q-1)\cdot k + l - 1$, which equal to $1, 2, \ldots, (q-1)\cdot k + l - 1$ modulo n by Eq. (3).

Step 3. To complete our proof, it remains to show that (15) is the smallest $M_{q,n}-1$ consecutive positive numbers containing no integer whose digit sum is divisible by n.

According to Step 2, it suffices to consider the case of $l-1 \ge \gcd(q-1,l)$.

In this step, the length of the sequence is $M_{q,n}-1$. So in inequality (14), $((\beta_1+1)(q-1)^m)_q + (h(q-1)^t)_q = M_{q,n}-1$, which means

$$\alpha + 1 - \beta = (q-1)\cdot k + r. \tag{16}$$

Combining with (10), we have

$$(q-1)\cdot(x+1) - r \equiv 0 \pmod{(q-1)\cdot k + l}. \tag{17}$$

As the priori proof in Step 1, the smallest number possesses the form $(AY)_q$. By $x = v_q(A+1)$, the minimal possible value of A is $((q-1)^x)_q$. Continuing the proof in Step 1, one can get the number of digits of Y (followed by A) must be $k+1$. To make $(AY)_q$ as small as possible, we let $A = ((q-1)^x)_q$.

Equality (16) implies that $\alpha = (q-1)\cdot k + r + \beta - 1 \geq (q-1)\cdot k + r$. So, we may assume that $\alpha = (q-1)\cdot k + r'$, where $r' \in \{0, 1, \ldots, l-1\}$. Then we can simplify (5) further to

$$s_q(A) + q - 1 - r' \equiv 0 \pmod{(q-1)\cdot k + l}. \tag{18}$$

Together with (17), (18) and $A = ((q-1)^x)_q$, we immediately obtain $r' = r$.

From (II) in Step 1, we can obtain that the length of the first part of the sequence is $r(q-1)^k$. Thus, the first term of such sequence is

$$(A(q-1)^{k+1})_q - (r(q-1)^k)_q + 1 = ((q-1)^x(q-1)^{k+1})_q - (r(q-1)^k)_q + 1 = ((q-1)^x(q-1-r)0^{k-1}1)_q.$$

By Step 2 and the discussions above in Step 3, the first term $((q-1)^x(q-1-r)0^{k-1}1)_q$ is the smallest possible one. □

Proof of Theorem 5. Suppose the contrary, that for some integer j, there exists a sequence of $M_{q,n}$ consecutive positive integers containing no integer d with $s_q(d) \equiv j \pmod{n}$. For simplicity, we use $m_1, m_2, \ldots, m_{M_{q,n}}$ to denote these consecutive integers. Let c be the number of the digits of $m_{M_{q,n}}$, and let $b = \sum_{k=1}^{n-j} q^{k+c}$. Then $s_q(m_i) + n - j = s_q(m_i + b)$, $i \in \{1, 2, \ldots, M_{q,n}\}$.

Note that $s_q(m_i) \neq j \pmod{n}$ means $s_q(m_i+b) \neq 0 \pmod{n}$. Therefore, the new sequence $\{m_k + b\}_{k=1}^{M_{q,n}}$ is consecutive, but it contains no integer $m_i + b$ with $s_q(m_i + b) \equiv 0 \pmod{n}$, which is a contradiction with Theorem 4. □

3 Further conclusions and open problems

From Theorem 4, it is easy to obtain the special results below.

Corollary 12 (Binary system case)**.** $M_{2,n} = 2^n - 1$.

Corollary 13 (Decimalism case)**.** *Let $n = 9\cdot k + l$, $k \in \mathbb{N}$, and $l \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, and then we have $M_{10,n} = (M_{10,l} + 1)\cdot 10^k - 1$. The details are presented in Table 1.*

l	$M_{10,9k+l}$	example on $k=0$	example on $k \in \mathbb{Z}^+$
1	$(19^k)_{10}$	$(1)_{10}$	$(1)_{10}$
2	$(39^k)_{10}$	$(9)_{10}$	$(9^{4k}80^{k-1}1)_{10}$
3	$(39^k)_{10}$	$(1)_{10}$	$(1)_{10}$
4	$(79^k)_{10}$	$(997)_{10}$	$(9^{6k+2}60^{k-1}1)_{10}$
5	$(99^k)_{10}$	$(6)_{10}$	$(9^k50^{k-1}1)_{10}$
6	$(99^k)_{10}$	$(7)_{10}$	$(9^k60^{k-1}1)_{10}$
7	$(139^k)_{10}$	$(994)_{10}$	$(9^{3k+2}30^{k-1}1)_{10}$
8	$(159^k)_{10}$	$(9999993)_{10}$	$(9^{7k+6}20^{k-1}1)_{10}$
9	$(99^k)_{10}$	$(1)_{10}$	$(1)_{10}$

Table 1: This table is a detailed explanation about Corollary 13, in which we symbolically set $(19^0)_{10} = 1$, and the third and fourth columns of this table are the concrete realizations of Eq. (2).

Corollary 13 is a general form of Examples 1 and 2. To illustrate Table 1, we show two examples:

For $l = 8$ and $k = 0$, we obtain from Table 1 that $M_{10,8} = 15$ and the 14 numbers, $9999993, 9999994, \ldots, 10000006$, are the smallest 14 consecutive positive integers whose digit sums are all not divisible by 8.

For $l = 6$ and $k = 1$, Table 1 shows that $M_{10,15} = 99$ and the 98 numbers, $961, 962, \ldots, 1058$, are the smallest 98 consecutive positive integers whose digit sums are all not divisible by 15.

Finally, we give some open problems.

1. How to extend the method in this paper to prove results about $(s_q(ak) \bmod n)_{k=1}^{\infty}$ for given integers $q, n, a \geq 2$?

2. How often do $M_{q,n}$ consecutive terms in $\mathrm{t}_{q,n}$ cover $\{0, 1, \ldots, n-1\}$?

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