Aperiodic Compositions and Classical Integer Sequences

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Abstract
In this paper we define the notion of singular composition of a positive integer. We provide a characterization of these compositions, together with methods for decomposing and also extending a singular composition in terms of other singular compositions. Consecutive extensions of particular compositions determine sequences of integers which coincide with classical integer sequences involving Fibonacci and Lucas numbers.

1 Introduction
Let $k, n$ be integers where $1 \leq k \leq n$, and let $\alpha = (a_1, a_2, \ldots, a_k)$ denote a composition of $n$ into $k$ parts [3]. We call $\alpha$ $(h, i)$-singular if

$$ (a_1, a_2, \ldots, a_i + a_{i+1}, \ldots, a_k) = (a_{1+h}, a_{2+h}, \ldots, a_{i+h} + a_{i+1+h}, \ldots, a_{k+h}), $$

(1)

where $1 \leq h \leq k - 1$, $1 \leq i \leq k$ and the indices are modulo $k$. Note that shifting a $(h, i)$-singular composition of one position to the right, we obtain a $(h, i+1)$-singular composition. Consequently, the choice of a single index $i$ is sufficient for identifying such compositions.

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Thus we fix $i = 1$ and we call the composition $\alpha = (a_1, a_2, \ldots, a_k)$ $h$-singular if
\[
(a_1 + a_2, a_3, \ldots, a_k) = (a_{1+h}, a_{2+h}, a_{3+h}, \ldots, a_{k+h}).
\]

A $k$-composition of $n$ is singular when it is $h$-singular for a suitable value of $1 \leq h \leq k-1$.

**Example 1.** The 5-composition $(1, 2, 2, 1, 2)$ of $n = 8$ is 2-singular.

Kramer [2] used singular compositions in order to define the middle levels partition graph of $n$.

The concatenation of the compositions $\alpha = (a_1, a_2, \ldots, a_k)$ and $\beta = (b_1, b_2, \ldots, b_h)$ of the positive integers $n$ and $m$ respectively is the composition $\alpha \beta = (a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_h)$ of $n + m$. We let $\alpha^i$ denote the concatenation of $\alpha$ with itself $i$ times. A composition $\alpha$ is periodic if $\alpha = \pi^j$, where $1 < j \leq k$ and $\pi$ is a suitable composition.

Fibonacci and Lucas numbers will appear in some of our results. Recall that the Fibonacci sequence $(F_n)_{n \geq 0}$ is defined by setting $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$. The Lucas sequence $(L_n)_{n \geq 0}$ is defined by setting $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$, for $n \geq 2$.

The paper is outlined as follows. In Section 2 we determine a characterization of aperiodic singular compositions which allows us to obtain a method for constructing such compositions (Theorem 11). In Section 3 we study decompositions (Theorem 14) and also extensions (Theorem 18) of a singular composition in terms of other singular compositions. In Section 4 we prove that consecutive extensions of particular compositions determine sequences of integers which coincide with classical sequences involving Fibonacci and Lucas numbers. We conclude the paper by posing a more general definition of singular composition together with an open problem.

## 2 A characterization of singular compositions

Let $\alpha$ be an $h$-singular $k$-composition of $n$; from (2) it follows that $(a_3, \ldots, a_k) = (a_{3+h}, \ldots, a_{k+h})$.

This equality determines the function $f_h : \{3, 4, \ldots, k\} \to \{3+h, 4+h, \ldots, k+h\}$ on the indices of the elements of previous sequences such that
\[
f_h(i) = i + h,
\]
where $3 \leq i \leq k$ and the integers are modulo $k$. We may represent $f_h$ in two-line notation
\[
\begin{pmatrix}
3 & 4 & \cdots & k \\
3+h & 4+h & \cdots & k+h
\end{pmatrix}.
\]

Note that the second line is obtained by shifting of $h$ positions to the left the elements of the sequence $(1, 2, 3, 4, \ldots, k)$ and ignoring the first two elements $1+h$ and $2+h$. A consequence is that the elements 1 and 2, which do not belong to the first line, belong to the second one,
except for \( h = 1 \) and \( h = k - 1 \). Indeed for \( h = 1 \) the second line contains 1, but not 2; for \( h = k - 1 \) the second line contains 2, but not 1. In a similar way, the elements \( 1 + h \) and \( 2 + h \) do not belong to the second line while they belong to the first one, except for \( h = 1 \) and \( h = k - 1 \).

**Proposition 2.** An \( h \)-singular \( k \)-composition of \( n \), where \( h \) and \( k \) are not coprime, is periodic.

**Proof.** Let \( \gcd(k, h) = t > 1 \), where \( h = th', k = tk' \) and \( \gcd(k', h') = 1 \). Note that the sets \( H_i = \{i, i+h, \ldots, i+(k'-1)h\} \), \( 1 \leq i \leq t \), determine a partition of the set \([k]\). Then, for an \( h \)-singular \( k \)-composition \( \alpha = (a_1, a_2, \ldots, a_k) \), the elements of the sets \( \{a_i, a_i+h, \ldots, a_i+(k'-1)h\} \), \( 1 \leq i \leq t \), coincide and \( \alpha \) turns out to be the concatenation \( (a_1, a_2, \ldots, a_t)^{k'} \).

Throughout the paper we consider only aperiodic compositions.

Beggas et al. [1] proved that a particular bijection, called widened permutation, between two \( n \)-sets having \( n - 1 \) elements in common has a decomposition into a linear order and a possible permutation. In this case we have a similar function in which the two sets have \( n - 2 \) elements in common, but for \( h = 1 \) and \( h = k - 1 \).

**Lemma 3.** Let \( h, k \) be coprime integers, where \( 1 \leq h \leq k - 1 \). The function \( f_h \) does not contain cycles.

**Proof.** By way of contradiction we assume there is a cycle

\[
C = (d, d + h, \ldots, d + (r - 1)h),
\]

where \( 1 \leq d \leq k \) and \( d + rh \equiv d \pmod{k} \). This means that \( rh \equiv 0 \pmod{k} \) and therefore \( k \) divides \( rh \). Then, because \( \gcd(k, h) = 1 \), \( k \) divides \( r \). The unique possibility is \( r = k \); so the cycle contains all the elements. But this implies the impossible condition that also every line of \((3)\) contains all the elements. \( \square \)

**Theorem 4.** Let \( h, k \) be coprime integers, where \( 1 \leq h \leq k - 1 \). The function \( f_h \) is decomposed into the linear orders:

1. \( E_h = (1 + h, 1 + 2h, \ldots, 1 + rh) \) (4)
   \[
   \text{and} \quad F_h = (2 + h, 2 + 2h, \ldots, 2 + sh),
   \]
   \( \text{where } r = h^{-1}, s = (k - 1)h^{-1} \text{ in } \mathbb{Z}_k, \text{ for } h \neq 1, k - 1; \)
2. \( E_1 = (2) \) and \( F_1 = (3, 4, \ldots, k, 1) \), for \( h = 1 \);
3. \( E_{k-1} = (k, k - 1, \ldots, 2) \) and \( F_{k-1} = (1) \), for \( h = k - 1 \).
Proof. Let $h \neq 1, k-1$. Starting from $1+h$ we obtain the sequence $(1+h, 1+2h, \ldots, 1+rh = a)$, where $a$ is one of the two elements which are in the second but not in the first line. So we have either $a = 1$ or $a = 2$. If $a = 1$, we obtain the impossible relation $rh \equiv 0 \pmod{k}$.

If $a = 2$, we obtain $rh \equiv 1 \pmod{k}$, which is satisfied for $r = h^{-1}$ in $\mathbb{Z}_k$. Now starting from $2+h$ we obtain the sequence $(2+h, 2+2h, \ldots, 2+sh = b)$, where either $b = 1$ or $b = 2$. The unique possibility is $b = 1$, which holds for $s = (k-1)h^{-1}$ in $\mathbb{Z}_k$. By Lemma 3 the function does not contain cycles; therefore it is decomposed into the previous linear orders.

Now let $h = 1$. The function $f_1$ is decomposed into $F_1 = (3, 4, \ldots, k, 1)$ and $E_1 = (2)$. In the case $h = k - 1$, $f_{k-1}$ is decomposed into $E_{k-1} = (k, k-1, \ldots, 2)$ and $F_{k-1} = (1)$. This completes the proof of the theorem.

In the following we let $E_h$ and $F_h$ also denote the sets of the elements of the assigned linear orders.

**Corollary 5.** For every $1 \leq h \leq k-1$ such that $\gcd(k, h) = 1$, $E_h \cup F_h = [k]$ and, for $k > 2$, $|E_h| \neq |F_h|$.

**Proof.** If $k-h^{-1} = h^{-1}$, then $k = 2h^{-1}$ and $kh = 2$ in $\mathbb{Z}_k$. This implies that $2 \equiv 0 \pmod{k}$, a contradiction for $k > 2$.

**Lemma 6.** Let $h_1, h_2$ be two integers such that $1 \leq h_1 < h_2 \leq k-1$ and $\gcd(k, h_1) = \gcd(k, h_2) = 1$. Then $E_{h_1} \neq E_{h_2}$ and $F_{h_1} \neq F_{h_2}$.

**Proof.** The cardinalities of $E_{h_1}$ and $E_{h_2}$ coincide with $h_1^{-1}$ and $h_2^{-1}$ in $\mathbb{Z}_k$ respectively. Because $h_1 < h_2$, their inverses are distinct; then also the sets $E_{h_1}$ and $E_{h_2}$ are distinct. The same argument applies for $F_{h_1}$ and $F_{h_2}$.

The following result is straightforward.

**Corollary 7.** If $k$ is a prime integer, then all the sets $E_h$ (respectively $F_h$), $1 \leq h \leq k-1$, are distinct.

Note that when $k$ and $h$ are coprime, then also $k$ and $k-h$ are coprime. In the following result we establish a relation between $E_{k-h}$ (respectively $F_{k-h}$) and $F_h$ (respectively $E_h$).

**Proposition 8.** For every $1 \leq h \leq \left\lfloor \frac{k}{2} \right\rfloor$ such that $\gcd(k, h) = 1$, $E_{k-h} = (F_h \setminus \{1\}) \cup \{2\}$ and $F_{k-h} = (E_h \setminus \{2\}) \cup \{1\}$.

**Proof.** The result is easy to prove for $h = 1$. Let $h' = k-h$. Since $\gcd(k, h) = 1$, then $\gcd(k, h') = 1$, $E_{h'} = \{1+h', 1+2h', \ldots, 1+(r'-1)h', 2\}$ and $F_{h'} = \{2+h', 2+2h', \ldots, 2+(s'-1)h', 1\}$, where $r' = (h')^{-1}$ and $s' = k-(h')^{-1}$ in $\mathbb{Z}_k$.

Let $s$ denote $k-h^{-1}$ in $\mathbb{Z}_k$; it follows that

\[1+k-h \equiv 2 + (s-1)h \pmod{k}.
\]

Then $1+2(k-h) \equiv 2+(s-2)h$ and so on until $1+(s-1)(k-h) \equiv 2+h$ and $1+s(k-h) \equiv 2 \pmod{k}$. Thus $E_{k-h} = \{2+(s-1)h, 2+(s-2)h, \ldots, 2+h, 2\} = (F_h \setminus \{1\}) \cup \{2\}$.

Moreover, $2+k-h \equiv 1+(r-1)h \pmod{k}$, where $r = h^{-1}$ in $\mathbb{Z}_k$; thus $F_{k-h} = \{1+(r-1)h, 1+(r-2)h, \ldots, 1+h, 1\} = (E_h \setminus \{2\}) \cup \{1\}$. 

\[\square\]

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Corollary 9. If \( \alpha = (a_1, a_2, \ldots, a_k) \) is an aperiodic \( h \)-singular \( k \)-composition of \( n \), then every \( a_i \) is equal to \( a_1 \) or \( a_2 \), \( 1 \leq i \leq k \), as long as \( i \in F_h \) or \( i \in E_h \) respectively. Then \( a_1 \) and \( a_2 \) are distinct, and they satisfy the relation
\[
(k - h^{-1})a_1 + h^{-1}a_2 = n. \tag{6}
\]

Corollary 10. If an aperiodic composition contains more than two distinct elements, then it is not singular.

Previous results allow us to give a characterization of singular compositions, which turns out to be a method for their construction.

Theorem 11. Let \( h, k, n \) be positive integers such that \( 1 \leq h < k \leq n \) and \( \gcd(k, h) = 1 \). An aperiodic \( k \)-composition \( \alpha = (a_1, a_2, \ldots, a_k) \) is \( h \)-singular if and only if \( a_1 \neq a_2 \) and the pair of elements \((a_1, a_2)\) is a solution of the equation
\[
(k - h^{-1})x_1 + h^{-1}x_2 = n, \tag{7}
\]
where \( h^{-1}, k - h^{-1} \in \mathbb{Z}_k \), and each \( a_i \) coincides with \( a_1 \) or \( a_2 \) for \( i \in F_h \) or \( i \in E_h \) respectively.

Proof. If \( \alpha \) is \( h \)-singular, then by Corollary 9 the property holds.

Now let us assume that the pair of distinct integers \((a_1, a_2)\) is solution of the equation (7) and each \( a_i \) coincides with \( a_1 \) or \( a_2 \) for \( i \in F_h \) or \( i \in E_h \) respectively. Hence, for \( h \neq 1, k - 1 \), the composition \( \alpha = (a_1, a_2, \ldots, a_k) \) which has the elements \( a_1 \) and \( a_2 \) in the positions given by (5) and (4) respectively, is \( h \)-singular. Lastly, if \( h = 1, \alpha = (a_1, a_2, a_3, \ldots, a_1) \) is 1-singular, while if \( h = k - 1, \alpha = (a_1, a_2, \ldots, a_2) \) is \((k - 1)\)-singular. \( \square \)

Example 12. The list of \( h \)-singular 9-compositions with \( a_1 = 1 \) and \( a_2 = 2 \) is

1. for \( h = 1, \alpha_1 = (1, 2, 1, 1, 1, 1, 1, 1, 1); \)
2. for \( h = 2, \alpha_2 = (1, 2, 2, 1, 2, 1, 2, 1, 2); \)
3. for \( h = 4, \alpha_4 = (1, 2, 2, 2, 2, 1, 2, 2, 2); \)
4. for \( h = 5, \alpha_5 = (1, 2, 1, 1, 1, 2, 1, 1, 1); \)
5. for \( h = 7, \alpha_7 = (1, 2, 1, 2, 2, 1, 2, 1, 2); \)
6. for \( h = 8, \alpha_8 = (1, 2, 2, 2, 2, 2, 2, 2, 2) \)

where the corresponding integers are \( n_1 = 10, n_2 = 14, n_4 = 16, n_5 = 11, n_7 = 13 \) and \( n_8 = 17 \). Note that the compositions \( \alpha_5, \alpha_7 \) and \( \alpha_8 \) are obtained from \( \alpha_4, \alpha_2 \) and \( \alpha_1 \) respectively, by exchanging 1 with 2 after the first two positions.
Let $\alpha = (a_1, a_2, \ldots, a_k)$ be an $h$-singular composition. By Proposition 8, it follows that by exchanging $a_1$ and $a_2$ after the first two positions, we obtain a $(k-h)$-singular composition. We now prove that by exchanging only the first two elements we obtain again a $(k-h)$-singular composition.

**Proposition 13.** Let $\alpha = (a_1, a_2, \ldots, a_k)$ be an aperiodic $h$-singular composition of $n$, where $1 \leq h \leq k-1$. Then $\alpha^* = (a_2, a_1, a_3, \ldots, a_k)$ is a $(k-h)$-singular composition of $n$, obtained from $\alpha$ by rotation.

**Proof.** Consider the composition $\alpha^* = (a_1^*, a_2^*, \ldots, a_k^*) = (a_2, a_1, a_3, \ldots, a_k)$ of $n$. The set $E^*$ of indices of the elements equal to $a_2^*$ in $\alpha^*$ satisfies $E^* = (F_h \setminus \{1\}) \cup \{2\} = E_{k-h}$ (Proposition 8). The same relation holds for $F^* = F_{k-h}$, where $F^*$ is the set of indices of the elements equal to $a_2^*$ in $\alpha^*$. Then $\alpha^*$ is a $(k-h)$-singular composition of $n$. Note that the composition $\alpha' = (a_1+1, a_2+h, \ldots, a_k, a_1, \ldots, a_h)$ is $(k-h)$-singular and is obtained from $\alpha$ by rotation. Moreover $a_2 = a_{1+h}$ and $a_1 = a_{2+h}$. Since the first two elements of $\alpha^*$ coincide with the first two of $\alpha'$ and both the compositions are $(k-h)$-singular, $E^* = E'$ and $F^* = F'$. Thus $\alpha^* = \alpha'$, and the result follows.

\[\square\]

### 3 Decompositions and extensions

In this section we investigate two decompositions and some extensions of an aperiodic singular composition.

**Theorem 14.** Let $\alpha = (a_1, a_2, \ldots, a_k)$ be an aperiodic $h$-singular $k$-composition of $n$, where $k = hq + r$ and $1 \leq r < h$. Then $\alpha = \lambda \mu \lambda \cdots \lambda$, where $\lambda = (a_1, a_2, \ldots, a_h)$, $\mu$ is the sequence of the last $r$ elements of $\lambda$ and $q$ is the multiplicity of $\lambda$. Moreover $\lambda$ is a $(h-r)$-singular $h$-composition of $a_1 + \cdots + a_h$.

**Proof.** Since $\alpha$ is $h$-singular, the sequences $\beta = (a_1 + a_2, a_3, \ldots, a_k)$ and $\gamma = (a_{1+h} + a_{2+h}, a_{3+h}, \ldots, a_{k+h})$ coincide. In particular this holds for the subsequences $\beta'$ and $\gamma'$ obtained by deleting the first $h-1$ elements of $\beta$ and $\gamma$ respectively. If $1 \leq h \leq \lfloor \frac{k}{2} \rfloor$, by comparing $\beta' = (a_{1+h}, a_{2+h}, \ldots, a_k)$ and $\gamma' = (a_{1+2h}, a_{2+2h}, \ldots, a_k, a_1, \ldots, a_h) = (a_{1+2h}, \ldots, a_k) \lambda$, where $\lambda = (a_1, a_2, \ldots, a_h)$, we obtain that the sequence $(a_{k-(h-1)}, \ldots, a_k)$ formed by the last $h$ elements of $\beta'$ coincides with $\lambda$. Then the sequence of length $h$ in $\gamma'$ which precedes the last subsequence $\lambda$ coincides again with $\lambda$. We continue until we find a subsequence $\mu$ of length less than $h$ in $\beta'$, which is formed by the last $r$ elements of $\lambda$. Thus $\mu = (a_{h-(r-1)}, a_{h-(r-2)}, \ldots, a_h)$. If $\lfloor \frac{k}{2} \rfloor < h \leq k-1$, by comparing $\beta'$ and $\gamma' = \mu$ we obtain $\alpha = \lambda \mu \lambda \cdots \lambda$, where $\lambda$ occurs $q$ times.

Let us assume that $r > 1$. Since $\alpha$ is $h$-singular, the sequence

$$(a_1 + a_2, a_3, \ldots, a_h, a_{h-(r-1)}, a_{h-(r-2)}, \ldots, a_h) \lambda^{q-1}$$

coincides with

$$(a_{h-(r-1)} + a_{h-(r-2)}, a_{h-(r-3)}, \ldots, a_h) \lambda^q.$$
Therefore the sequences of the first $h - 1$ elements coincide

$$(a_1 + a_2, a_3, \ldots, a_h) = (a_{h-(r-1)} + a_{h-(r-2)}, a_{h-(r-3)}, \ldots, a_h, a_1, \ldots, a_{h-r}).$$

Thus the composition $\lambda$ is $(h-r)$-singular. A similar argument applies in the case $r = 1$. □

**Proposition 15.** Let $\alpha = (a_1, a_2, \ldots, a_k)$ be an aperiodic $h$-singular $k$-composition of $n$, where $k = hq + r$ and $1 < r < h$. Then $\alpha = \sigma \lambda \cdots \lambda$, where $\lambda = (a_1, a_2, \ldots, a_h)$, $\sigma = (a_1, a_2, \ldots, a_r)$ and the multiplicity of $\lambda$ is $q$. Moreover $\lambda$ is a $(h-r)$-singular $h$-composition of $a_1 + \cdots + a_h$.

**Proof.** Let $\lambda = (a_1, a_2, \ldots, a_h)$ and $\sigma = (a_1, a_2, \ldots, a_r)$. By applying the same argument used in the proof of Theorem 14 to the subsequences obtained by deleting the first $r - 1$ elements of $\beta$ and $\gamma$, the result follows. □

**Corollary 16.** In the case of $r = 1$, there is not a decomposition $\alpha = \sigma \lambda \lambda \cdots \lambda$.

**Proof.** In the case of $r = 1$, $\sigma$ is reduced to the element $a_1$. This implies the relation $a_1 + a_2 = 2a_1$; then $a_2 = a_1$, a contradiction to the assumption that $\alpha$ is aperiodic. □

**Corollary 17.** If $k = hq + r$ and $1 < r < h$, then $\sigma \lambda = \lambda \mu$.

Now we investigate an operation which can be considered the inverse of the decomposition; namely we want to determine an extension of a singular composition which turns out to be again a singular composition.

**Theorem 18.** Let $\alpha$ be an aperiodic $h$-singular $k$-composition of $n$, and let $\nu$ denote the sequence formed by the last $k - h$ elements of $\alpha$. The $k'$-composition $\beta = \alpha \nu \alpha \cdots \alpha$, where $k' = kq' + k - h$ and $q'$ is the multiplicity of $\alpha$, is $k$-singular.

**Proof.** Let $\alpha = (a_1, a_2, \ldots, a_k)$ be an aperiodic $h$-singular $k$-composition of $n$, where $k > 2$ and $1 \leq h < k - 1$. The composition $\beta = \alpha \nu \alpha \cdots \alpha$, where $\nu$ denotes the sequence formed by the last $k - h$ elements of $\alpha$, is $k$-singular if

$$(a_1 + a_2, \ldots, a_k, a_{1+h}, \ldots, a_k)\alpha^{q'-1} = (a_{1+h} + a_{2+h}, \ldots, a_k)\alpha^{q'}.$$

In order to prove the equality, it is sufficient to show that

$$(a_1 + a_2, a_3, \ldots, a_k, a_{1+h}, \ldots, a_k) = (a_{1+h} + a_{2+h}, \ldots, a_k, a_1, \ldots, a_k). \quad (8)$$

Since $\alpha$ is $h$-singular, $(a_1 + a_2, a_3, \ldots, a_k) = (a_{1+h} + a_{2+h}, \ldots, a_k, a_1, \ldots, a_h)$. Thus the left side of (8) coincides with $(a_{1+h} + a_{2+h}, \ldots, a_k, a_1, \ldots, a_h, a_{1+h}, \ldots, a_k)$ and the result follows. A similar argument applies in the cases $k = 2$ and $h = k - 1$. □
4 Classical integer sequences

Let $\alpha$ be an $h$-singular $k$-composition of $n$. The composition $\beta = \alpha \nu \alpha \ldots \alpha$, where $\nu$ is the sequence formed by the last $k - h$ elements of $\alpha$ and $\alpha$ is repeated $q$ times, is called a $q$-extension of $\alpha$. By consecutive extensions, we determine a sequence of singular compositions and therefore a sequence of integers corresponding to the numbers of their parts.

4.1 Fibonacci sequences

Let us consider the $h_0$-singular $k_0$-composition $\alpha_0 = (a, b)$, with $a \neq b$, $k_0 = 2$ and $h_0 = 1$. The 2-extension of $\alpha_0$ is the $h_1$-singular $k_1$-composition $\alpha_1 = \alpha_0 \nu_0 \alpha_0 = (a, b, b, a, b)$, where $k_1 = k_0 \cdot 2 + 1$, $h_1 = k_0 = 2$ and $\nu_0$ is the composition formed by last $(k_0 - h_0) = 1$ element of $\alpha_0$. The consecutive 2-extension is the $h_2$-singular $k_2$-composition $\alpha_2 = \alpha_1 \nu_1 \alpha_1 = (a, b, b, a, b, a, b, a, b, b, a, b)$, where $k_2 = k_1 \cdot 2 + 3$, $h_2 = k_1$ and $\nu_1$ is the composition formed by last $(k_1 - h_1) = 3$ elements of $\alpha_1$ and so on.

The first values of the sequence of the numbers $(k_n)_{n \geq 0}$ of parts of the 2-extensions of $\alpha_0$ are

$$2, 5, 13, 34, 89, 233, \ldots$$

These numbers appear as the first integers, but the first two, in the sequence A001519 [4], which is obtained from the recursive relation

$$a_n = 3a_{n-1} - a_{n-2}, \quad (9)$$

with the initial conditions $a_0 = 1$, $a_1 = 1$. We prove that the integers $k_n$ satisfy the same recursive relation.

**Lemma 19.** The integers $k_n$ of the parts of the 2-extensions of the 1-singular 2-composition $(a, b)$, with $a \neq b$, satisfy the recursive relation:

$$k_n = 3k_{n-1} - k_{n-2}$$

with the initial conditions $k_0 = 2$, $k_1 = 5$.

**Proof.** Recall that, by Theorem 18,

$$k_n = 2k_{n-1} + k_{n-1} - h_{n-1}.$$

Because $h_{n-1} = k_{n-2}$, the result follows. \qed

The following corollary is straightforward.

**Corollary 20.** The integers $h_n$ associated to the 2-extensions of the 1-singular 2-composition $(a, b)$, with $a \neq b$, satisfy the recursive relation:

$$h_n = 3h_{n-1} - h_{n-2}$$

with the initial conditions $h_0 = 1$, $h_1 = 2$. 

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It is easy to prove that the generating function of the sequence of the integers \( k_n \) is
\[
\frac{2 - x}{1 - 3x + x^2},
\]
and
\[
k_n = \frac{2 + \sqrt{5}}{\sqrt{5}} \left( \frac{3 + \sqrt{5}}{2} \right)^n + \frac{-2 + \sqrt{5}}{\sqrt{5}} \left( \frac{3 - \sqrt{5}}{2} \right)^n.
\]

Proposition 21. The sequence
\[ k_0, k_1 - h_1, k_1, k_2 - h_2, k_2, k_3 - h_3, \ldots \quad (10) \]
coincides with the sequence of Fibonacci numbers \( F_n \), with initial conditions \( F_2 = 2, F_3 = 3 \).

Proof. We have to prove that every element of (10) is the sum of the preceding two elements and the initial conditions coincide. For \( i \geq 1 \), \( k_i = k_i - h_i + k_{i-1} \), because \( h_i = k_{i-1} \). Moreover, for \( i \geq 2 \), \( k_i - h_i = k_{i-1} + k_{i-1} - h_{i-1} \) by Lemma 19. Because \( k_0 = 2, k_1 = 5 \) and \( h_1 = 2 \), the initial conditions are 2 and 3, which coincide with \( F_2 \) and \( F_3 \) of the Fibonacci sequence \( A000045 \).

Another consequence of Proposition 21 is that the elements \( k_i, i \geq 0 \), form a bisection of the Fibonacci sequence; this result turns out to be one of the comments to \( A001519 \).

By repeating the previous procedure for \( q > 2 \), we easily obtain a sequence satisfying the recursive relation
\[ a_n = (q + 1)a_{n-1} - a_{n-2}, \]
with the initial conditions \( a_0 = 2, a_1 = 2q + 1 \).

In the particular case of \( q = 3 \), we obtain the sequence whose first elements are
\[ 2, 7, 26, 97, \ldots \]
which coincides with \( A001075 \), but the first element.

Again, for \( q = 4 \) we obtain a sequence whose first elements are
\[ 2, 9, 43, 206, \ldots \]
which coincides with \( A002310 \), but the first element.

4.2 Lucas sequences

The first values of the sequence of the numbers \( (p_n)_{n \geq 0} \) of parts of the 2-extensions of the 2-singular 3-composition \( (a, b, b) \), with \( a \neq b \), are
\[ 3, 7, 18, 47, 123, \ldots \]
These integers coincide with the first integers, but the first one, of A005248, which is obtained from the recursive relation (9), with the initial conditions \(a_0 = 2, a_1 = 3\).

Using the same procedure of Lemma 19, the numbers \(p_n\) satisfy the same recursive relation with initial conditions \(p_0 = 3\) and \(p_1 = 7\). Moreover the generating function of the sequence of the integers \(p_n\) is

\[
\frac{3 - 2x}{1 - 3x + x^2},
\]

and

\[
p_n = \left(\frac{3 + \sqrt{5}}{2}\right)^{n+1} + \left(\frac{3 - \sqrt{5}}{2}\right)^{n+1}.
\]

**Proposition 22.** The sequence

\[
h_0, p_0 - h_0, p_0, p_1 - h_1, p_1, p_2 - h_2, p_2, p_3 - h_3, \ldots
\]

(11)

coincides with the sequence of Lucas numbers \(L_n\), with initial conditions \(L_0 = 2, L_1 = 1\).

Another consequence of the previous result is that the elements \(p_i, i \geq 0\), form a bisection of the Lucas sequence A000032, as noted in a comment to A005248.

### 4.3 Other integer sequences

We now consider the sequence of the numbers \((t_n)_{n \geq 0}\) of parts of 2-extensions of the 3-singular 4-compositions \((a, b, b, b)\), with \(a \neq b\), that is

\[
4, 9, 23, 60, 157, \ldots
\]

This sequence, which is not contained in [4], satisfies the recursive relation (9), with initial conditions \(t_0 = 4\) and \(t_1 = 9\). The corresponding generating function is

\[
\frac{4 - 3x}{1 - 3x + x^2},
\]

and

\[
t_n = \frac{3 + 2\sqrt{5}}{\sqrt{5}}\left(\frac{3 + \sqrt{5}}{2}\right)^n + \frac{-3 + 2\sqrt{5}}{\sqrt{5}}\left(\frac{3 - \sqrt{5}}{2}\right)^n.
\]

By continuing, we may obtain other integer sequences by \(q\)-extension, with \(q \geq 2\), of the singular composition \((a, b, \ldots, b)\), where \(b\) occurs more than three times.

### 5 Conclusion

The notion of singular composition can be generalized as follows. We call the composition \(\alpha = (a_1, a_2, \ldots, a_k) (h, i, j)\)-singular, if

\[
(a_1, a_2, \ldots, a_{i-1}, a_i + a_j, a_{i+1}, \ldots, a_{j-1}, a_{j+1}, \ldots, a_k) = (a_1 + h, a_2 + h, \ldots, a_{i-1} + h, a_i + h, a_{i+1} + h, \ldots, a_{j-1} + h, a_{j+1} + h, \ldots, a_k + h),
\]

(12)
where $1 \leq h \leq k - 1$, $1 \leq i < j \leq k$ and the indices are modulo $k$.

This definition leads to compositions which cannot be obtained from equation (1). In fact, $(1, 1, 2, 2, 2)$ satisfies $(a_1 + a_3, a_2, a_4, a_5) = (a_1 + h + a_3 + h, a_2 + h, a_4 + h, a_5 + h)$ for $h = 4$, but it does not satisfy any equation (1).

Thus this definition poses the problem to find necessary and sufficient conditions based on which a given aperiodic sequence with two distinct elements satisfies (12).

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References


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