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On the Binomial Interpolated Triangles

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Abstract

The binomial interpolated transform of a sequence is a generalization of the well-known binomial transform. We examine a Pascal-like triangle, on which a binomial interpolated transform works between the left and right diagonals, focusing on binary recurrences. We give the sums of the elements in rows and in rising diagonals, and we define two special classes of these arithmetical triangles.

1 Introduction

Let us define the sequence $b = \{b_n\}_{n=0}^{\infty} \in \mathbb{R}^{\infty}$ as the binomial transform of the given sequence $a = \{a_n\}_{n=0}^{\infty} \in \mathbb{R}^{\infty}$ by $b_n = \sum_{i=0}^{n} \binom{n}{i} a_i$. This transformation is invertible with formula $a_n = \sum_{i=0}^{n} \binom{n}{i}(-1)^{n-i} b_i$. Several researchers [1, 2, 3, 5, 7] examined the properties and the generalizations of the binomial transformation. One of its generalizations is the so-called binomial interpolated transform [1] given by

$$b_n = \sum_{i=0}^{n} \binom{n}{i} u^i v^{n-i} a_i \tag{1}$$

for any non-zero $u, v \in \mathbb{R}$. Bhadouria at al. [2] and Falcon and Plaza [3] showed for some (falling and rising) k-binomial transform cases, when $u = 1$, $v = k$; $u = k$, $v = 1$ or $u = v = k$, that there are special infinite (Pascal-like) triangles, whose left diagonal (left leg) contains the terms of a and the right one (right leg) the terms of b.

In our paper, we examine the inner part of this type of triangle in a generalized form focusing exclusively on the binary recurrence sequences a. We determine the sums and alternating sums of rows, rising diagonals and central term sequence and give explicit forms for entries of the triangle. Finally, we define and examine two special classes of our arithmetical triangles.

2 Binomial interpolated triangle

Let an arithmetical triangle (called "derangement triangle" [2]) be defined by terms $a_{n,k}$ (k-th entry in row n — see Figure 1) from the sequence $\{a_{n,0}\}_{n=0}^{\infty}$, where

$$a_{n,k} = ua_{n,k-1} + va_{n-1,k-1} \quad (1 \le k \le n), \tag{2}$$

$u, v \in \mathbb{R}$ and $uv \neq 0$.

We shall show that the right diagonal sequence is a binomial interpolated transform of the left diagonal sequence with parameters u and v. Moreover, we shall prove that the converse also holds, with the parameters $-v/u$ and $1/u$.

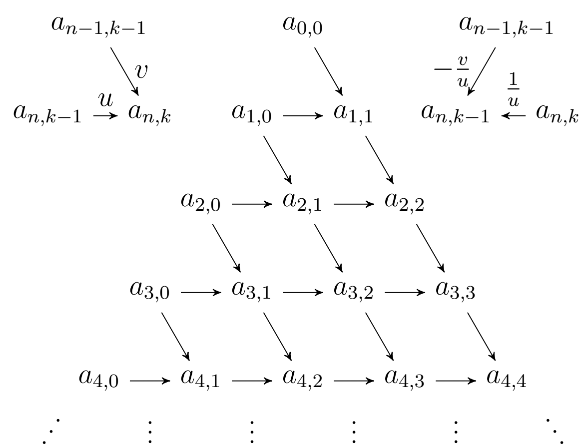

Figure 1: Binomial interpolated triangle

Theorem 1. *For all integer r, k such that $k_0 \le k \le n$, where $k_0 \ge 0$ is a fixed integer, we have*

$$a_{n,k} = \sum_{i=0}^{k-k_0} \binom{k-k_0}{i} u^i v^{k-k_0-i} a_{n-k+k_0+i,k_0}. \tag{3}$$

Proof. We prove it by induction on n and k. If $k = k_0$, then the formula (3) trivially holds for any n. We suppose that (3) is true up to $k_0 \le k-1$. Let $1 \le \overline{k} = k - k_0$. Then

$$
\begin{aligned}
a_{n,k} &= u a_{n,k-1} + v a_{n-1,k-1} \\
&= u \sum_{i=0}^{\overline{k}-1} \binom{\overline{k}-1}{i} u^i v^{\overline{k}-i-1} a_{n-\overline{k}+i+1,k_0} + v \sum_{i=0}^{\overline{k}-1} \binom{\overline{k}-1}{i} u^i v^{\overline{k}-i-1} a_{n-\overline{k}+i,k_0} \\
&= \sum_{i=0}^{\overline{k}-1} \binom{\overline{k}-1}{i} u^{i+1} v^{\overline{k}-i-1} a_{n-\overline{k}+i+1,k_0} + \sum_{i=0}^{\overline{k}-1} \binom{\overline{k}-1}{i} u^i v^{\overline{k}-i} a_{n-\overline{k}+i,k_0} \\
&= \sum_{i=1}^{\overline{k}} \binom{\overline{k}-1}{i-1} u^i v^{\overline{k}-i} a_{n-\overline{k}+i,k_0} + \sum_{i=0}^{\overline{k}-1} \binom{\overline{k}-1}{i} u^i v^{\overline{k}-i} a_{n-\overline{k}+i,k_0} \\
&= u^{\overline{k}} a_{n,k_0} + \sum_{i=1}^{\overline{k}-1} \binom{\overline{k}-1}{i-1} u^i v^{\overline{k}-i} a_{n-\overline{k}+i,k_0} + v^{\overline{k}} a_{n-\overline{k},k_0} + \sum_{i=1}^{\overline{k}-1} \binom{\overline{k}-1}{i} u^i v^{\overline{k}-i} a_{n-\overline{k}+i,k_0} \\
&= v^{\overline{k}} a_{n-\overline{k},k_0} + \sum_{i=1}^{\overline{k}-1} \left(\binom{\overline{k}-1}{i-1} + \binom{\overline{k}-1}{i} \right) u^i v^{\overline{k}-i} a_{n-\overline{k}+i,k_0} + u^{\overline{k}} a_{n,k_0} \\
&= \sum_{i=0}^{\overline{k}} \binom{\overline{k}}{i} u^i v^{\overline{k}-i} a_{n-\overline{k}+i,k_0}.
\end{aligned}
$$

□

Considering the substitutions $k_0 = 0$ and $k = n$, or considering a fixed term a_{n_0,k_0} leads us to the following corollary.

Corollary 2. *The right diagonal sequence is the binomial interpolated transform of the left diagonal sequence, so*

$$
b_n = a_{n,n} = \sum_{i=0}^{n} \binom{n}{i} u^i v^{n-i} a_{i,0}. \tag{4}
$$

Furthermore, let us fix k_0 and n_0, so that $0 \le k_0 \le n_0$. Then the terms $\overline{a}_{i,j} = a_{n,k}$ form a binomial interpolated sub-triangle (see Figure 2), where $i = n - n_0$, $j = k - k_0$ $n_0 \le n$, $k_0 \le k \le n-(n_0-k_0)$. The sub-triangle's right diagonal sequence is the binomial interpolated transform of its left diagonal sequence.

We now express the terms $a_{n,k}$ by the right diagonal sequence $\{a_{n,n}\}_{n=0}^{\infty}$.

Theorem 3. *For any $0 \le k \le n$*

$$
a_n^k = \sum_{i=0}^{n-k} \binom{n-k}{i} \left(\frac{1}{u}\right)^i \left(-\frac{v}{u}\right)^{n-k-i} a_{k+i}^{k+i}. \tag{5}
$$

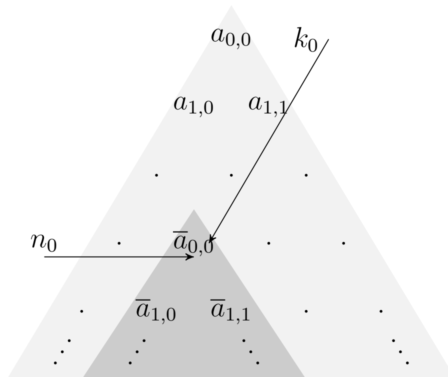

Figure 2: Sub-triangle of the binomial interpolated triangle

Proof. We suppose that the sequence $\{b_n = a_{n,n}\}$ with $\{b_n\}_{n=0}^{\infty} = \{a_{n,n}\}_{n=0}^{\infty}$ is given. Let $U = 1/u$ and $V = -v/u$, then the right equality is $a_{n,k-1} = Ua_{n,k} + Va_{n-1,k-1}$, according to (2). Based on this connection, we can write the entries of the binomial interpolated triangle (from right to left) by $b_{n,r}$, where $b_{n,0} = b_n$, $b_{n,r} = Ub_{n,r-1} + Vb_{n-1,r-1}$ $(1 \leq r \leq n)$. Using relation (3), proved in Theorem 1, we obtain

$$b_{n,r} = \sum_{i=0}^{r} \binom{r}{i} U^i V^{r-i} b_{n-r+i,0}.$$

Since $b_{n,r} = a_{n,n-r}$, considering the substitution $k = n - r$, the thesis follows. □

If $k = 0$, as a direct consequence of Theorem 3, the inverse transformation of the binomial interpolated transform (1) is

$$a_{n,0} = \sum_{i=0}^{n} \binom{n}{i} \left(\frac{1}{u}\right)^i \left(-\frac{v}{u}\right)^{n-i} a_{i,i}. \tag{6}$$

3 Binary binomial interpolated triangle

Let $a_{n,0} = a_n$, where $\{a_n\}_{n=0}^{\infty}$ is a binary recursive sequence defined by

$$a_n = \alpha a_{n-1} + \beta a_{n-2} \qquad (2 \leq n,\ \alpha, \beta \in \mathbb{R},\ \alpha\beta \neq 0), \tag{7}$$

with initial values $a_0, a_1 \in \mathbb{R}$ $(|a_0|+|a_1| \neq 0)$. Then we call the binomial interpolated triangle *binary binomial interpolated triangle* and we let $\mathcal{BIT}(a_0, a_1, \alpha, \beta; u, v)$ (in short $\mathcal{BIT}$) denote it.

Bhadouria et al. [2] gave three special examples, where the triangles are generated by the 4-Lucas sequence. They are $\mathcal{BIT}(2, 4, 4, 1; 1, 1)$, $\mathcal{BIT}(2, 4, 4, 1; 4, 1)$ and $\mathcal{BIT}(2, 4, 4, 1; 4, 4)$.

From now on, we will use two important variables

$$\mathcal{A} = u\alpha + 2v \qquad \text{and} \qquad \mathcal{B} = u^2\beta - uv\alpha - v^2.$$

First of all, we give some recursive formulas for terms $a_{n,k}$. The results of the next technical lemma will be useful during the proofs of the further theorems.

Lemma 4. *The following recurrence relations hold*

$$a_{n,k} = \alpha a_{n-1,k} + \beta a_{n-2,k} \qquad (2 \le k+2 \le n), \tag{8}$$

$$a_n^k = \frac{u\beta}{v} a_{n-1,k} - \frac{\mathcal{B}}{v} a_{n-1,k-1} \qquad (2 \le k+1 \le n), \tag{9}$$

$$a_{n,k} = \frac{u\alpha + v}{u} a_{n-1,k} + \frac{\mathcal{B}}{u} a_{n-2,k-1} \qquad (2 \le k+1 \le n), \tag{10}$$

$$a_{n,k} = (u\alpha + v) a_{n-1,k-1} + u\beta a_{n-2,k-1} \qquad (2 \le k+1 \le n), \tag{11}$$

$$a_{n,k} = \frac{u^2\beta + v^2}{v} a_{n-1,k-1} - \frac{u\mathcal{B}}{v} a_{n-1,k-2} \qquad (2 \le k \le n), \tag{12}$$

$$a_{n,k} = \mathcal{A}\, a_{n-1,k-1} + \mathcal{B}\, a_{n-2,k-2} \qquad (2 \le k \le n), \tag{13}$$

$$a_{n,k} = \frac{2u\beta - v\alpha}{\beta} a_{n,k-1} - \frac{\mathcal{B}}{\beta} a_{n,k-2} \qquad (2 \le k \le n). \tag{14}$$

Proof. First we prove relation (8) by induction. For $k = 0$ it corresponds to definition (7). Let us suppose that this formula is true up to $k-1$. So, if $x = a_{n-3,k-1}$ and $y = a_{n-2,k-1}$, then $a_{n-1,k-1} = \beta x + \alpha y$ and $a_{n,k-1} = \alpha\beta x + (\beta + \alpha^2)y$ hold. Figure 3, which is a suitable part of Figure 1, depicts it. Now $a_{n-2,k} = vx + uy$, $a_{n-1,k} = u\beta x + (u\alpha + v)y$ and $a_{n,k} = (u\alpha\beta + v\beta)x + (u\alpha^2 + v\alpha + u\beta)y$, which gives (8). Moreover, we proved with it, that Figure 3 could be any part of Figure 1.

In order to prove the further equations we can also use the relevant part of Figure 3. For example, in the case (11) $a_{n,k} = (u\alpha + v)\, y + u\beta\, x = u\beta x + (u\alpha + v)y$. □

The relation (13) gives that the right diagonal sequence b (and the i-th diagonals parallel to it) satisfies the binary recurrence relation

$$\forall n \ge 2 \quad b_n = \mathcal{A} b_{n-1} + \mathcal{B} b_{n-2} \tag{15}$$

with initial values $b_0 = a_0$, $b_1 = ua_1 + va_0$ (generally, $b_0 = a_{i,0}$, $b_1 = a_{i+1,1}$, $i \ge 0$). Moreover, the system of equations $\mathcal{A} = 0$, $\mathcal{B} = 0$ gives $v = -u\alpha/2$ and $u^2(\alpha^2 + 4\beta) = 0$. Thus, as the condition $uv \ne 0$ holds, if $v \ne -u\alpha/2$ or $\alpha^2 + 4\beta \ne 0$, then $|\mathcal{A}| + |\mathcal{B}| \ne 0$, otherwise $\mathcal{A} = \mathcal{B} = 0$.

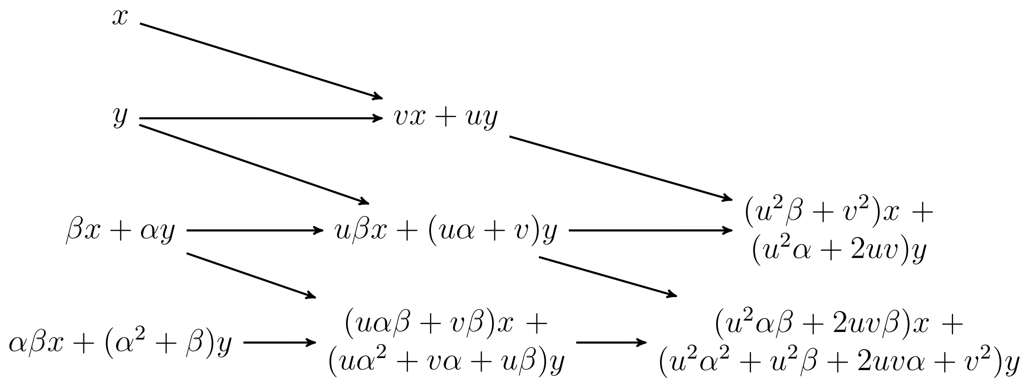

Figure 3: A part of the growing

3.1 Sums and alternating sums of rows

Let $s = \{s_n\}_{n=0}^{\infty}$ be the sequence whose elements are the sums of the values belonging to the n-th row of a binomial interpolated triangle. We have

$$s_n = \sum_{k=0}^{n} a_{n,k} = \sum_{k=0}^{n} \sum_{i=0}^{k} \binom{k}{i} u^i v^{k-i} a_{n-k+i,0}. \tag{16}$$

We obtain a linear recurrence for s, whose coefficients depend only on α, β, $\mathcal{A}$ and $\mathcal{B}$, i.e., the coefficients of the binary recurrences related to sequences a and b.

Theorem 5. *The sequence s satisfies the following fourth order linear homogeneous recurrence*

$$\forall n \geq 4 \quad s_n = (\alpha + \mathcal{A})s_{n-1} + (\beta - \alpha\mathcal{A} + \mathcal{B})s_{n-2} - (\alpha\mathcal{B} + \beta\mathcal{A})s_{n-3} - \beta\mathcal{B}s_{n-4}. \tag{17}$$

Proof. The sum of row n can be given by the previous two ones in form

$$s_n = \alpha(s_{n-1} - a_{n-1,n-1}) + \beta s_{n-2} + a_{n,n-1} + a_{n,n}. \tag{18}$$

Since $ua_{n,n-1} = a_{n,n} - va_{n-1,n-1}$ then from (18) we obtain

$$u\, s_n = u\alpha\, s_{n-1} + u\beta\, s_{n-2} + (u+1)a_{n,n} - (\alpha u + v)a_{n-1,n-1}. \tag{19}$$

Applying (13) and as $\mathcal{A} - \alpha u - v = v$ we have

$$u\, s_n = u\alpha\, s_{n-1} + u\beta\, s_{n-2} + (\mathcal{A}u + v)a_{n-1,n-1} + \mathcal{B}(u+1)a_{n-2,n-2}. \tag{20}$$

Now we transform (19) into

$$u\, s_{n-1} = u\alpha\, s_{n-2} + u\beta\, s_{n-3} + (u+1)a_{n-1,n-1} - (\alpha u + v)a_{n-2,n-2}. \tag{21}$$

Transforming again (19) into form $u\,s_{n-2} = u\alpha\,s_{n-3} + \dots$ and using (13) we gain

$$u\mathcal{B}\,s_{n-2} = u\mathcal{B}(\alpha\,s_{n-3} + \beta\,s_{n-4}) - (\alpha u + v)a_{n-1,n-1} + (\mathcal{B}(u+1) + (\alpha u + v)\mathcal{A})\,a_{n-2,n-2}. \quad (22)$$

Finally, let us express $a_{n-1,n-1}$ and $a_{n-2,n-2}$ from (20) and (21), respectively. We substitute them into (22) and obtain (17). □

Remark 6. It should be noticed that the recurrence (18) is the minimal order recurrence of s when

$$\mathcal{B}(u+1)^2 + (\alpha u + v)(\mathcal{A}u + v) \neq 0.$$

Indeed, in this case the determinant of coefficients of the system (from (20) and (21))

$$\begin{cases} (\mathcal{A}u + v)a_{n-1,n-1} + \mathcal{B}(u+1)a_{n-2,n-2} = u\,s_n - u\alpha\,s_{n-1} - u\beta\,s_{n-2} \\ (u+1)a_{n-1,n-1} - (\alpha u + v)a_{n-2,n-2} = u\,s_{n-1} - u\alpha\,s_{n-2} - u\beta\,s_{n-3} \end{cases}$$

is different from 0, so the system has a unique solution, otherwise when

$$\mathcal{B}(u+1)^2 + (\alpha u + v)(\mathcal{A}u + v) = 0$$

or, equivalently, with a little bit of calculations, when $\mathcal{B}(u+1)^2 + (\alpha - u)(v + u\alpha) = 0$ the minimal order of the linear recurrence of s should be less then four. For example, the solution $u = -1$ and $v = \alpha$ implies that $\mathcal{A} = \alpha$ and $\mathcal{B} = \beta$. Then the relation (20) becomes

$$\forall n \geq 2 \quad s_n = \alpha\,s_{n-1} + \beta\,s_{n-2}. \quad (23)$$

Obviously, summing relations $s_n = \alpha s_{n-1} + \beta s_{n-2}$, $-\alpha s_{n-1} = -\alpha^2 s_{n-2} - \alpha\beta s_{n-3}$ and $-\beta s_{n-2} = -\alpha\beta s_{n-3} - \beta^2 s_{n-4}$ we find that the recurrence (17) also holds for sequence s, but it is not the minimal order one.

Let the sequence $\bar{s} = \{\bar{s}_n\}_{n=0}^{\infty}$ be the alternating sum sequence, where the terms are the values of rows of a binomial interpolated triangle, so that

$$\bar{s}_n = \sum_{k=0}^{n} (-1)^k a_{n,k} = \sum_{k=0}^{n} (-1)^k \sum_{i=0}^{k} \binom{k}{i} u^i v^{k-i} a_{n-k+i,0}.$$

Theorem 7. *The sequence $\bar{s}$ satisfies the following fourth order linear homogeneous recurrence*

$$\forall n \geq 4 \quad \bar{s}_n = (\alpha - \mathcal{A})\bar{s}_{n-1} + (\beta + \alpha\mathcal{A} + \mathcal{B})\bar{s}_{n-2} - (\alpha\mathcal{B} - \beta\mathcal{A})\bar{s}_{n-3} - \beta\mathcal{B}\bar{s}_{n-4}. \quad (24)$$

Proof. Row by row the signs of the entries in the alternating sums do not change in directions parallel to the left diagonal ($\mathrm{sign}((-1)^k a_{n,k}) = \mathrm{sign}((-1)^k a_{n-1,k})$), but parallel to the right diagonal they change ($\mathrm{sign}((-1)^k a_{n,k}) \neq \mathrm{sign}((-1)^{k-1} a_{n-1,k-1})$). Hence we only have to change the sign of $\mathcal{A}$ in the summation relation (17). □

Remark 8. If $u = -1$ and $v = \alpha$, then the relation (24) becomes more simple,

$$\forall n \geq 3 \quad \bar{s}_n = (\alpha^2 + 2\beta)\bar{s}_{n-2} - \beta^2\bar{s}_{n-4}. \quad (25)$$

3.2 Rising diagonal sum sequence

The sequence $d = \{d_n\}_{n=0}^{\infty}$ of sums of elements in rising diagonals has the following definition

$$\forall n \geq 0 \quad d_n = \sum_{k=\lceil \frac{n}{2} \rceil}^{n} a_{k,n-k} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} a_{n-k,k}. \tag{26}$$

When we consider the Pascal arithmetical triangle it is well-known that these sums provide the Fibonacci sequence. We give the recurrence relation for the sequence of sums of elements belonging to rising diagonal (i.e.,"shallow diagonal" as they are defined in Wolfram Math World [8]) of our binomial interpolated triangles.

Theorem 9. *The rising diagonals sums sequence d of a $\mathcal{BIT}$ satisfies the sixth order linear recurrence relation*

$$\forall n \geq 6 \quad D_n = \mathcal{A} D_{n-2} + \mathcal{B} D_{n-4}, \tag{27}$$

where $D_n = -d_n + \alpha d_{n-1} + \beta d_{n-2}$. Moreover, for even n ($n = 2k$, $n \geq 2$), $D_n = -b_k$ also holds.

We point out that with the equation (27) is a concise expression for the following sixth order recurrence relation

$$d_n = \alpha d_{n-1} + (\beta + \mathcal{A}) d_{n-2} - \alpha \mathcal{A} d_{n-3} + (-\beta \mathcal{A} + \mathcal{B}) d_{n-4} - \alpha \mathcal{B} d_{n-5} - \beta \mathcal{B} d_{n-6}. \tag{28}$$

Proof. Without loss of generality, let $n = 2k$. The values of d_{n+2} and d_{n+3} can be given by the previous two ones in forms

$$\begin{aligned} d_{n+2} &= \alpha d_{n+1} + \beta d_n + a_{k+1,k+1}, \\ d_{n+3} &= \alpha \left(d_{n+2} - a_{k+1,k+1} \right) + \beta d_{n+1} + a_{k+2,k+1}, \end{aligned}$$

generally for even and odd indexes we have

$$\begin{aligned} d_{n+2i} &= \alpha d_{n+2i-1} + \beta d_{n+2i-2} + a_{k+i,k+i}, \\ d_{n+2i+1} &= \alpha \left(d_{n+2i} - a_{k+i,k+i} \right) + \beta d_{n+2i-1} + a_{k+i+1,k+i}. \end{aligned}$$

When we consider D_n, with n even, we obtain

$$\forall i \geq 1 \quad a_{k+i,k+i} = d_{n+2i} - \alpha d_{n+2i-1} - \beta d_{n+2i-2} = -D_{n+2i},$$

and according to (13)

$$D_{n+6} - \mathcal{A} D_{n+4} - \mathcal{B} D_{n+2} = -a_{k+3,k+3} + \mathcal{A} a_{k+2,k+2} + \mathcal{B} a_{k+1,k+1} = 0.$$

If we deal with the case D_n, with n odd, then, using relations (10) and (2), we find

$$\begin{aligned}
d_{n+3} &= \alpha d_{n+2} + \beta d_{n+1} + \frac{v}{u} a_{k+1,k+1} + \frac{\mathcal{B}}{u} a_{k,k}, \\
d_{n+5} &= \alpha d_{n+4} + \beta d_{n+3} + \frac{v\mathcal{A} + \mathcal{B}}{u} a_{k+1,k+1} + \frac{v\mathcal{B}}{u} a_{k,k}, \\
d_{n+7} &= \alpha d_{n+6} + \beta d_{n+5} + \frac{v\mathcal{A}^2 + v\mathcal{B} + \mathcal{A}\mathcal{B}}{u} a_{k+1,k+1} + \frac{v\mathcal{A}\mathcal{B} + \mathcal{B}^2}{u} a_{k,k}.
\end{aligned}$$

Thus

$$\begin{aligned}
& u(D_{n+7} - \mathcal{A}D_{n+5} - \mathcal{B}D_{n+3}) = \\
& \quad (-(v\mathcal{A}^2 + v\mathcal{B} + \mathcal{A}\mathcal{B}) + \mathcal{A}(v\mathcal{A} + \mathcal{B}) + v\mathcal{B}) a_{k+1,k+1} \\
& \quad + (-(v\mathcal{A}\mathcal{B} + \mathcal{B}^2) + v\mathcal{A}\mathcal{B} + \mathcal{B}^2) a_{k,k} = 0.
\end{aligned}$$

□

3.3 Central elements

The central elements of the binomial triangles are the terms $a_{2k,k}$ with $k \geq 0$. In this subsection we give the relation between them, moreover it turns out that the before mentioned recurrence relation holds for all the columns (which are parallel to the vertical axis of the triangle). These sequences are defined by $\{a_{2k+\ell,k}\}_{k=k_0}^{\infty}$ with $\ell \in \mathbb{Z}$ and

$$k_0 = \begin{cases} 0, & \text{if } \ell \geq 0; \\ |\ell|, & \text{if } \ell < 0. \end{cases} \tag{29}$$

Moreover, if $\ell = 0$, then it is the sequence of the central elements.

Theorem 10. *All the sequences $\{a_{2k+\ell,k}\}_{k=k_0}^{\infty}$ of the binomial interpolated triangle with $\ell \in \mathbb{Z}$ and k_0 defined by (29) satisfy the same binary homogeneous recurrence relation*

$$c_{k+2} = (\alpha^2 u + \alpha v + 2\beta u) c_{k+1} - \beta \mathcal{B} c_k, \tag{30}$$

where $c_k = a_{2k+\ell,k}$.

Proof. We have to prove that

$$a_{2(k+2)+\ell,k+2} = (\alpha^2 u + \alpha v + 2\beta u) a_{2(k+1)+\ell,k+1} - \beta(u^2\beta - uv\alpha - v^2) a_{2k+\ell,k}. \tag{31}$$

First, supposing that $u\alpha + v \neq 0$ and recalling (11) and (8) we obtain

$$\begin{aligned}
a_{2k+3+\ell,k+1} &= (u\alpha + v) a_{2k+2+\ell,k} + u\beta a_{2k+1+\ell,k} \\
&= (u\alpha + v)\left(\alpha a_{2k+1+\ell,k} + \beta a_{2k+\ell,k}\right) + u\beta a_{2k+1+\ell,k} \\
&= (u\alpha^2 + v\alpha + u\beta) \frac{a_{2k+2+\ell,k+1} - u\beta a_{2k+\ell,k}}{u\alpha + v} + (u\alpha + v)\beta a_{2k+\ell,k},
\end{aligned}$$

furthermore

$$\begin{aligned}
a_{2k+4+\ell,k+2} &= (u\alpha+v)a_{2k+3+\ell,k+1} + u\beta a_{2k+2+\ell,k+1} \\
&= (u\alpha^2+v\alpha+u\beta)\left(a_{2k+2+\ell,k+1} - u\beta a_{2k+\ell,k}\right) + (u\alpha+v)^2\beta a_{2k+\ell,k} \\
&\qquad + u\beta a_{2k+2+\ell,k+1} \\
&= (u\alpha^2+v\alpha+2u\beta)a_{2k+2+\ell,k+1} + \beta(-u^2\beta+uv\alpha+v^2)a_{2k+\ell,k}.
\end{aligned}$$

Second, when $v=-u\alpha$ the equality (30) becomes

$$c_{k+2} = 2u\beta c_{k+1} - u^2\beta^2 c_k,$$

or explicitly

$$a_{2(k+2)+\ell,k+2} = 2u\beta a_{2(k+1)+\ell,k+1} - u^2\beta^2 c a_{2k+\ell,k}. \tag{32}$$

Thus, since the sequence is a geometric progression we have

$$a_{2(k+2)+\ell,k+2} = u\beta a_{2(k+1)+\ell,k+1} = u^2\beta^2 a_{2k+\ell,k},$$

and clearly (32) holds because it corresponds to the identity

$$u\beta a_{2(k+1)+\ell,k+1} = 2u\beta a_{2(k+1)+\ell,k+1} - u\beta a_{2(k+1)+\ell,k+1}.$$

□

3.4 Explicit formula

Theorem 11. *Let $D=\sqrt{\alpha^2+4\beta}$. If $D\neq 0$, then the explicit formula of $a_{n,k}$ is*

$$a_{n,k} = \frac{(vD+\alpha v-2\beta u)a_{n,0}+2\beta a_{n,1}}{2vD}\left(\frac{\beta u - v x_2}{\beta}\right)^k + \frac{(vD-\alpha v+2\beta u)a_{n,0}-2\beta a_{n,1}}{2vD}\left(\frac{\beta u - v x_1}{\beta}\right)^k, \tag{33}$$

where $x_1=(\alpha+D)/2$, $x_2=(\alpha-D)/2$ and

$$a_{n,0} = \frac{(D-\alpha)a_{0,0}+2a_{1,0}}{2D}x_1^n + \frac{(D+\alpha)a_{0,0}-2a_{1,0}}{2D}x_2^n, \tag{34}$$

$$a_{n,1} = \frac{(D-\alpha)a_{0,0}+2a_{1,0}}{2D}x_1^{n-1}(ux_1+v) + \frac{(D+\alpha)a_{0,0}-2a_{1,0}}{2D}x_2^{n-1}(ux_2+v). \tag{35}$$

If $D=0$ and $\mathcal{A}\neq 0$, then the explicit formula is

$$a_{n,k} = \left(a_{n,0} + k\left(\frac{\alpha a_{n,1}}{\mathcal{A}} - a_{n,0}\right)\right)\left(\frac{\mathcal{A}}{\alpha}\right)^k,$$

where

$$a_{n,0} = \left(a_{0,0} + n\frac{2a_{1,0} - \alpha a_{0,0}}{\alpha}\right)\left(\frac{\alpha}{2}\right)^n \quad (36)$$

$$a_{n,1} = ua_{n,0} + v\left(a_{0,0} + (n-1)\frac{2a_{1,0} - \alpha a_{0,0}}{\alpha}\right)\left(\frac{\alpha}{2}\right)^{n-1}. \quad (37)$$

If $D = 0$ and $\mathcal{A} = 0$, then

$$a_{n,0} = \left(a_{0,0} + n\frac{2a_{1,0} - \alpha a_{0,0}}{\alpha}\right)\left(\frac{\alpha}{2}\right)^n \quad (38)$$

$$a_{n,1} = u\frac{2a_{1,0} - \alpha a_{0,0}}{\alpha}\left(\frac{\alpha}{2}\right)^n. \quad (39)$$

and $a_{n,k} = 0$, if $k \geq 2$.

Proof. We suppose that $\alpha^2 + 4\beta \neq 0$. Firstly, we give the explicit form of the elements in case of the main path ($k = 0$). As the recursion is binary with coefficient α and β, then the characteristic equation of (7) is $x^2 - \alpha x - \beta = 0$ with roots $x_1 \neq x_2$. Consequently, for all $n \geq 0$ the term $a_{n,0}$ is a linear combination of the powers x_1^n, x_2^n. In order to determine the coefficients of this linear combination, we need to solve the system of equations $px_1^i + qx_2^i = a_{i,0}$, $i = 0, 1$. From this system we find $p = \frac{a_{1,0} - a_{0,0}x_2}{x_1 - x_2}$ and $q = \frac{a_{1,0}x_1 - a_{1,0}}{x_1 - x_2}$. Thus observing that

$$x_1 = \frac{a + D}{2}, x_2 = \frac{a - D}{2}, x_1 - x_2 = D,$$

where $D = \sqrt{\alpha^2 + 4\beta}$, we easily find equality (34). Moreover $a_{n,1} = ua_{n,0} + va_{n-1,0}$ yields (35).

Secondly, we give similarly the explicit formula of the sequence $\{a_{n,k}\}_{k=0}^n$ with initial conditions $a_{n,0}$ and $a_{n,1}$. Because of (14), its characteristic equation is

$$y^2 - \frac{2\beta u - \alpha v}{\beta}y - \frac{\alpha uv - \beta u^2 + v^2}{\beta} = 0$$

and the roots are

$$y_1 = \frac{2\beta u - \alpha v + vD}{2\beta} = u + \frac{v(D - \alpha)}{2\beta} = \frac{\beta u - vx_2}{\beta} \quad \text{and} \quad y_2 = \frac{\beta u - vx_1}{\beta}.$$

When we consider the case $D^2 = \alpha^2 + 4\beta = 0$ and $\mathcal{A} \neq 0$ (or, equivalently, $\beta = -\alpha^2/4$ and $v = -u\alpha/2$) we use the same method adopted before. Taking in account that the characteristic equation $x^2 - \alpha x - \beta = 0$ has the unique root $x_0 = \alpha/2$, we have to solve the system $(p + qi)x_0^i = a_{i,0}$, $i = 0, 1$, which provides the coefficients p, q for $a_{n,0} = (p + qn)x_0^n$ in equality (36). In this case the characteristic equation of sequence $\{a_n^k\}_{k=0}^n$ also has just one root $y_0 = \mathcal{A}/\alpha \neq 0$ and the equations $(\bar{p} + \bar{q}i)y_0^i = a_n^i$, $i = 0, 1$ and k yield the final formula.

Finally, we examine the case $D^2 = \alpha^2 + 4\beta = 0$ and $\mathcal{A} = 0$. Now the equivalent conditions $\beta = -\alpha^2/4$ and $v = -u\alpha/2$ imply that $\mathcal{B} = 0$ and the equality (37) is simplified into (39). Moreover if $k \geq 2$, then the relation (14) becomes $a_{n,k} = 0\, a_{n,k-1} + 0\, a_{n,k-2} = 0$. □

4 Special types of binary binomial interpolated triangles

The classical Pascal triangle has vertical symmetry and its inner elements satisfy the well-known rule of addition, namely every elements is the sum of the two terms directly above it. In this section we give the classes of our triangles which have the same properties, and we answer the question, "Is Pascal's triangle a binomial interpolated triangle?"

A binomial interpolated triangle is (vertically) symmetrical if

$$a_{n,k} = a_{n,n-k} \quad (0 \leq k \leq n).$$

Theorem 12. *Let $a_{0,0} \neq 0$ and $a_{1,0}$ be given. A binary binomial interpolated triangle is symmetrical if and only if*

$$u = -1, \; v = \alpha = \frac{2a_{1,0}}{a_{0,0}}$$

or

$$\alpha = \frac{2a_{1,0}}{a_{0,0}}, \; \beta = -\frac{\alpha^2}{4}, \; v = -\frac{\alpha(u-1)}{2},$$

where $u \neq 1$ or

$$u = -1, \; v = \alpha = \frac{2a_{1,0}}{a_{0,0}}, \; \beta = -\frac{\alpha^2}{4}.$$

(See [1] for the binomial transform in a special case).

Proof. Indeed clearly the condition $a_{n,k} = a_{n,n-k}$ implies the following system of equations

$$\begin{cases} \alpha = \mathcal{A} = u\alpha + 2v \\ \beta = \mathcal{B} = u^2\beta - uv\alpha - v^2 \\ b_1 = a_{1,0} = ua_{1,0} + va_{0,0} \end{cases} .$$

From the first equation we find $v = -\alpha(u-1)/2$ and substituting in the second equation we obtain, with some calculations,

$$(\alpha^2 + 4\beta)(u+1)(u-1) = 0.$$

Now, we have the three possibilities $\alpha^2 + 4\beta = 0$, $u = -1$, $u = 1$. Obviously the case $u = 1$ implies $v = 0$, a contradiction. The case $u = -1$ gives $v = \alpha$ and $\alpha = 2a_{1,0}/a_{0,0}$. Considering the case $\alpha^2 + 4\beta = 0$ we obtain the second part of the statement of the theorem. Finally, the equalities $\alpha^2 + 4\beta = 0$ and $u = -1$ yield the third last case. □

Corollary 13. *Let $\lambda = a_{1,0}/a_{0,0}$ and $a_{0,0} \neq 0$, $u \neq 1$. Then the form of a symmetrical binomial interpolated triangle can be written by*

$$\mathcal{BIT}(a_0, \lambda a_0, 2\lambda, \beta; -1, 2\lambda), \tag{40}$$

$$\mathcal{BIT}(a_0, \lambda a_0, 2\lambda, -\lambda^2; u, \lambda(1-u)), \tag{41}$$

or

$$\mathcal{BIT}(a_0, \lambda a_0, 2\lambda, -\lambda^2; -1, 2\lambda). \tag{42}$$

Figure 4 shows a symmetrical binomial interpolated triangle, namely $\mathcal{BIT}(2,1,1,1;-1,1)$ generated by Lucas numbers (A000032 in [6]).

$$\begin{array}{ccccccccccccc}
 & & & & & & 2 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 3 & & -2 & & 3 & & & & \\
 & & & 4 & & -1 & & -1 & & 4 & & & \\
 & & 7 & & -3 & & 2 & & -3 & & 7 & & \\
 & 11 & & -4 & & 1 & & 1 & & -4 & & 11 & \\
18 & & -7 & & 3 & & -2 & & 3 & & -7 & & 18
\end{array}$$

Figure 4: Symmetrical binomial interpolated triangle $\mathcal{BIT}(2,1,1,1;-1,1)$

Remark 14. According to (23) the recursion of s in the symmetrical binomial interpolated triangles case is $s_n = 2\lambda s_{n-1} + \beta s_{n-2}$ or $s_n = 2\lambda s_{n-1} - \lambda^2 s_{n-2}$. Moreover, when we consider the triangle (42) for the sequence d we obtain $d_n = \lambda d_{n-1} + \lambda d_{n-2} - \lambda^2 d_{n-3}$.

Theorem 15. *Pascal's triangle is not a binomial interpolated triangle.*

Proof. The "sum of above terms" condition and relation (9) should imply $a\beta/v = -\mathcal{B}/v = 1$ and the vertical symmetry implies $\beta = \mathcal{B}$. Thus we find $u = -1$ and $v = -\beta$. Now the first set of solutions coming from Theorem 12 gives $v = \beta = \alpha = 2$, since in Pascal's triangle we have $a_{0,0} = a_{1,0} = 1$. The recurrence (7) becomes $a_{n,0} = 2a_{n-1,0} - 2a_{n-2,0}$ which is clearly not satisfied in Pascal's triangle by the first elements (all equal to 1) of any row. From the second set of solutions we find $\alpha = 2$, $\beta = -1$, $v = 1 - u$, but, since $u = -1$, from these solutions we must have $v = 2$ and on the other hand since $v = -\beta = -1$ we find a contradiction. □

Now we give the conditions of u and v, so that an inner entry of $\mathcal{BIT}$ could be the sum of values not only left but also directly above it by coefficients u and v. Using (9), we can gain it in two different ways (see Figure 5). Our cases are

$$\textbf{case 1:} \qquad u = \frac{u\beta}{v}, \qquad v = -\frac{\mathcal{B}}{v},$$
$$\textbf{case 2:} \qquad v = \frac{u\beta}{v}, \qquad u = -\frac{\mathcal{B}}{v}.$$

Figure 5: Coefficients of summing downwards

$$a_0$$

$$a_1 \qquad \beta a_0 + \alpha a_1$$

$$\beta a_0 + \alpha a_1 \qquad \begin{matrix} \alpha\beta a_0 + \\ (\alpha^2+\beta)a_1 \end{matrix} \qquad \begin{matrix} (\alpha^2\beta+\beta^2)a_0 + \\ (\alpha^3+2\alpha\beta)a_1 \end{matrix}$$

$$\begin{matrix} \alpha\beta a_0 + \\ (\alpha^2+\beta)a_1 \end{matrix} \qquad \begin{matrix} (\alpha^2\beta+\beta^2)a_0 + \\ (\alpha^3+2\alpha\beta)a_1 \end{matrix} \qquad \begin{matrix} (\alpha^3\beta+2\alpha\beta^2)a_0 + \\ (\alpha^4+3\alpha^2\beta+\beta^2)a_1 \end{matrix} \qquad \begin{matrix} (\alpha^4\beta+3\alpha^2\beta^2+\beta^3)a_0 + \\ (\alpha^5+4\alpha^3\beta+3\alpha\beta^2)a_1 \end{matrix}$$

Figure 6: $\mathcal{BIT}(a_0, a_1, \alpha, \beta; \alpha, \beta)$

4.1 Case 1

Solving the system

$$\begin{cases} u = \frac{u\beta}{v} \\ v = -\frac{\mathcal{B}}{v} \end{cases},$$

we can easily obtain that $u = \alpha$ and $v = \beta$, thus the triangle is

$$\mathcal{BIT}(a_0, a_1, \alpha, \beta; \alpha, \beta). \tag{43}$$

We derive some interesting properties of this triangle. From Figure 6, which shows the first four rows of triangle $\mathcal{BIT}(a_0, a_1, \alpha, \beta; \alpha, \beta)$ (see also Figure 7), we notice that the rows and the left diagonal satisfy the same recurrence. Indeed, using the equalities $u = \alpha$, $v = \beta$, the recurrence relation (14) becomes $a_{n,k} = \alpha a_{n,k-1} + \beta a_{n,k-2}$. Moreover, the terms along the rising diagonals are the same. The next theorem and corollary will provide it precisely.

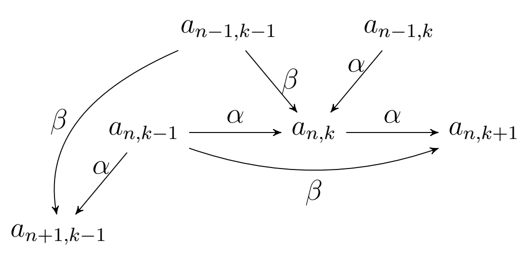

Figure 7: Coefficients of summing in case 1

Theorem 16. *The entries in triangle $\mathcal{BIT}(a_0, a_1, \alpha, \beta; \alpha, \beta)$ can be written ($n \geq 1$) by*

$$a_{n,k} = a_0 \sum_{i=0}^{\lfloor \frac{n+k-2}{2} \rfloor} \binom{n+k-i-2}{i} \alpha^{n+k-2i-2} \beta^{i+1} + a_1 \sum_{i=0}^{\lfloor \frac{n+k-1}{2} \rfloor} \binom{n+k-i-1}{i} \alpha^{n+k-2i-1} \beta^{i}.$$

Proof. We consider the results of Theorem 11. Indeed, using the equalities $u = \alpha$, $v = \beta$ and $x_1 + x_2 = \alpha$, $\alpha x_1 + \beta = x_1^2$, $\alpha x_2 + \beta = x_2^2$, $D + \alpha = 2x_1$, $D - \alpha = -2x_2$ the equations (33)–(35) become

$$a_{n,k} = \frac{-a_{n,0} x_2 + a_{n,1}}{D} x_1^k + \frac{a_{n,0} x_1 - a_{n,1}}{D} x_2^k,$$

where $D = \sqrt{\alpha^2 + 4\beta} = x_1 - x_2 \neq 0$ and

$$\begin{aligned} a_{n,0} &= \frac{-a_0 x_2 + a_1}{D} x_1^n + \frac{a_0 x_1 - a_1}{D} x_2^n, \\ a_{n,1} &= \frac{-a_0 x_2 + a_1}{D} x_1^{n+1} + \frac{a_0 x_1 - a_1}{D} x_2^{n+1}. \end{aligned}$$

A little calculation shows that

$$a_{n,k} = a_0 \beta \left(\frac{x_1^{n+k-1} - x_2^{n+k-1}}{x_1 - x_2} \right) + a_1 \left(\frac{x_1^{n+k} - x_2^{n+k}}{x_1 - x_2} \right). \tag{44}$$

Using the Girard-Waring formula [4]

$$\frac{X^{N+1} - Y^{N+1}}{X - Y} = \sum_{i=0}^{\lfloor \frac{N}{2} \rfloor} (-1)^i \binom{N-i}{i} (X+Y)^{N-2i} (XY)^i,$$

where, in our case, $X = x_1$, $Y = x_2$, $X + Y = \alpha$, $XY = -\beta$ and $N = n + k - 2$ or $N = n + k - 1$, the thesis follows. The Girard-Waring formula also holds in the case $x_1 = x_2$, i.e., $D = 0$, taking the limit $x_1 \to x_2$ on both members. We mention that $D = 0$ implies $\mathcal{A} \neq 0$. □

Corollary 17. *In case of the triangle $\mathcal{BIT}(a_0, a_1, \alpha, \beta; \alpha, \beta)$ the rising diagonal sequence $\{a_{n-k,k}\}_{k=0}^{\lfloor \frac{n}{2} \rfloor}$ is a constant sequence for any k.*

Proof. We apply the formula proved in Theorem 16 to $a_{n-k,k}$ and obtain an expression that does not depend on the index k. □

Figure 8 shows $\mathcal{BIT}(a_0, a_1, 1, 1; 1, 1)$ as an example for case 1. Using the result of Theorem 16, in the case $\alpha = 1$ and $\beta = 1$, the binomial coefficients are the coefficients of the elements in the rising diagonals. Thus $a_{n,k} = a_0 f_{n+k-1} + a_1 f_{n+k}$, where f_n is the n^{th} Fibonacci number (A000045), so that, $f_0 = 0, f_1 = f_{-1} = 1$. (We also find from the above relation (44) the connection with Fibonacci numbers.) The special Fibonacci binomial interpolated triangle is $\mathcal{BIT}(1, 1, 1, 1; 1, 1)$ (A199512). Falcon and Plaza [3] provided an other example in Table 4 for this case, namely $\mathcal{BIT}(0, 1, 3, 1; 3, 1)$, which is generated by the 3-Fibonacci sequence (A006190).

$$
\begin{array}{ccccccccc}
 & & & & a_0 & & & & \\
 & & & a_1 & & p+a_1 & & & \\
 & & a_0+a_1 & & a_0+2a_1 & & 2a_0+3a_1 & & \\
 & a_0+2a_1 & & 2a_0+3a_1 & & 3a_0+5a_1 & & 5a_0+8a_1 & \\
2a_0+3a_1 & & 3a_0+5a_1 & & 5a_0+8a_1 & & 8a_0+13a_1 & & 13a_0+21a_1
\end{array}
$$

Figure 8: $\mathcal{BIT}(a_0, a_1, 1, 1; 1, 1)$

4.2 Case 2

From the equations system we obtain

$$
\begin{aligned}
v_1 &= \frac{\alpha - 1 + \sqrt{(\alpha-1)^2 + 4\beta}}{2}, \qquad u_1 = \frac{v_1^2}{\beta}, \\
v_2 &= \frac{\alpha - 1 - \sqrt{(\alpha-1)^2 + 4\beta}}{2}, \qquad u_2 = \frac{v_2^2}{\beta}.
\end{aligned}
$$

If $(\alpha-1)^2 + 4\beta = 0$, i.e., $\beta = -(1/4)(\alpha-1)^2$, then, replacing the corresponding values for u and v, after simplification we have

$$
\mathcal{BIT}\left(a_0, a_1, \alpha, -\frac{(\alpha-1)^2}{4}; -1, \frac{\alpha-1}{2}\right). \tag{45}
$$

If $(\alpha-1)^2 + 4\beta \neq 0$, then the triangles are

$$
\mathcal{BIT}(a_0, a_1, \alpha, \beta; u_1, v_1), \tag{46}
$$

$$
\mathcal{BIT}(a_0, a_1, \alpha, \beta; u_2, v_2). \tag{47}
$$

Finally, we compare the triangles (43)–(47) with the symmetrical ones (40)–(42). In all the cases for the symmetrical binary interpolated binomial triangle with some calculations we gain the following corollary.

Corollary 18. *The symmetrical binary interpolated binomial triangles whose an inner entry could be the sum of values not only left but also directly above it by coefficients u and v are*

$$
\mathcal{BIT}(a_0, -\frac{a_0}{2}, -1, -1; -1, -1),
$$

which has only terms a_0 and $\pm a_0/2$ and

$$
\mathcal{BIT}(a_0, a_0, 2, -1; 2, -1),
$$

which has only terms a_0.

Remark 19. When both parameters $a_{0,0}$, $a_{1,0}$ are zero we find the trivial triangle exclusively composed of null entries.
The triangle with all entries equal to 1 correspond to more binomial interpolated triangles, for example, $\mathcal{BIT}(1,1,2,-1;-1,2)$ or $\mathcal{BIT}(1,1,2,-1;-1,2)$.
The triangle whose left diagonal and rows are the sequences of natural numbers (A000027) is $\mathcal{BIT}(1,2,2,-1;2,-1)$ (A094727).

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