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A Tribonacci-Like Sequence of Composite Numbers

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Abstract

We find a new Tribonacci-like sequence of positive integers $\langle x_0, x_1, x_2, \ldots \rangle$ given by $x_n = x_{n-1} + x_{n-2} + x_{n-3}$, $n \geq 3$, and $\gcd(x_0, x_1, x_2) = 1$ that contains no prime numbers. We show that the sequence with initial values $x_0 = 151646890045$, $x_1 = 836564809606$, $x_2 = 942785024683$ is the current record in terms of the number of digits.

1 Introduction

Šiurys [10] found initial values

$$x_0 = 99202581681909167232$$
$$x_1 = 67600144946390082339$$
$$x_2 = 139344212815127987596,$$

satisfying $\gcd(x_0, x_1, x_2) = 1$, such that the Tribonacci-like sequence given by

$$x_n = x_{n-1} + x_{n-2} + x_{n-3} \text{ for } n \geq 3 \tag{1}$$

contains no prime numbers. Similar problems were considered for Fibonacci-like sequences given by $x_n = x_{n-1} + x_{n-2}$ for $n \geq 2$ (Graham [2]; Knuth [5]; Wilf [13]; Nicol [7]; Vsemirnov [12]; Ismailescu and Son [3]), sequences given by $a_n = k2^n + 1$ (Sierpiński [8]; Jaeschke [4]), binary linear recurrent sequences (Dubickas, Novikas, and Šiurys [1]; Somer [11]) and some linear recurrent sequences of higher orders (Šiurys [9]).

The main result of this note is as follows.

2 The main results

Theorem 1. *Let* $\langle x_0, x_1, x_2, \ldots \rangle$ *be defined by* (1) *and* $\gcd(x_0, x_1, x_2) = 1$ *with the following initial values:*

$$x_0 = 151646890045, \quad x_1 = 836564809606, \quad x_2 = 942785024683.$$

Then $\langle x_0, x_1, x_2, \ldots \rangle$ *contains no prime numbers.*

Remark 2. If we allow non-positive values, we can find a slightly smaller (in absolute value) initial triple, namely

$$x_0 = 730344594529, \quad x_1 = -45426674968, \quad x_2 = 151646890045.$$

3 Proof of Theorem 1

In this section we complete the proof of Theorem 1.

Proof of Theorem 1. First, recall Šiurys' idea [10]. Consider the additional sequences $(s_n)_{n=0}^{\infty}$ and $(t_n)_{n=0}^{\infty}$ defined by the same relation (1) with $(s_0, s_1, s_2) = (0, 1, 0)$ and $(t_0, t_1, t_2) = (0, 0, 1)$.

Lemma 3 ([10])**.** *Let* p *be a prime. Suppose that for some integer* $m \geq 2$ *we have* $s_m t_{2m} - s_{2m} t_m \equiv 0 \pmod{p}$*. Then there exist* a, $b \in \mathbb{Z}$ *such that at least one of* a, b *is not divisible by* p *and* $s_{km} a + t_{km} b \equiv 0 \pmod{p}$ *for* $k = 0, 1, 2, \ldots$*.*

The next step is to find a set of pairs (p_i, m_i) satisfying Lemma 3 such that every integer belongs to at least one of the arithmetic progressions

$$A_i = m_i k + r_i, k \in \mathbb{Z}, i = 1, 2, \ldots, 11. \tag{2}$$

In this paper, the following values of p_i and m_i are used: (see Table 1).

Šiurys [10] used $p = 79$ with $m = 40$ instead of $p = 239$.

By Lemma 3, for every pair (p_i, m_i) we can choose $(a_i, b_i) \in \mathbb{Z}^2$ so that at least one of a_i, b_i is not divisible by p_i and

$\mathbf{i}$	$\mathbf{p_i}$	$\mathbf{m_i}$	$\mathbf{\lvert s_{m_i} t_{2m_i} - s_{2m_i} t_{m_i} \rvert}$
1	2	2	$\mathbf{2}$
2	29	5	$\mathbf{29}$
3	17	6	$2 \cdot \mathbf{17}$
4	7	8	$2^6 \cdot \mathbf{7}$
5	11	10	$2 \cdot \mathbf{11} \cdot 29$
6	107	12	$2^3 \cdot 17 \cdot \mathbf{107}$
7	8819	15	$29 \cdot \mathbf{8819}$
8	19	20	$2^3 \cdot 11 \cdot \mathbf{19} \cdot 29 \cdot \mathbf{239}$
9	239	20	$2^3 \cdot 11 \cdot \mathbf{19} \cdot 29 \cdot \mathbf{239}$
10	1151	24	$2^6 \cdot 7 \cdot 17 \cdot 107 \cdot \mathbf{1151}$
11	1621	30	$2 \cdot 11 \cdot 17 \cdot 29 \cdot \mathbf{1621} \cdot 8819$

Table 1: p_i and m_i.

$$s_{km_i} a_i + t_{km_i} b_i \equiv 0 \pmod{p_i} \text{ for } k = 0, 1, 2, \ldots.$$

Next, we construct a sequence $(x_n)_{n=0}^{\infty}$ satisfying

$$x_n \equiv s_{m_i - r_i + n} a_i + t_{m_i - r_i + n} b_i \pmod{p_i}, \qquad i = 1, 2, \ldots, 11, \qquad \text{for } n = 0, 1, 2, \ldots.$$

The initial values satisfy

$$\begin{aligned} &x_0 \equiv s_{m_i - r_i} a_i + t_{m_i - r_i} b_i \pmod{p_i}, \qquad x_1 \equiv s_{m_i - r_i + 1} a_i + t_{m_i - r_i + 1} b_i \pmod{p_i}, \\ &x_2 \equiv s_{m_i - r_i + 2} a_i + t_{m_i - r_i + 2} b_i \pmod{p_i}, \qquad \text{for } i = 1, 2, \ldots, 11. \end{aligned}$$

We can find initial terms (x_0, x_1, x_2) by the Chinese reminder theorem.

In the method described above there is some freedom in the choice of a_i and b_i (up to a common factor). Šiurys [10] used all a_i equal to 1.

We show how to optimize the choice of a_i and b_i. Let $P = \prod_{i=1}^{11} p_i$.

Let us consider the system:

$$\begin{cases} x'_0 \equiv D x_0 \pmod{P} \\ x'_1 \equiv D x_1 \pmod{P} \\ x'_2 \equiv D x_2 \pmod{P} \end{cases}$$

subject to the constraint

$$\gcd(D, P) = 1. \tag{3}$$

The new triple (x'_0, x'_1, x'_2) also satisfies the above properties, i.e., each term of the sequence (1) with starting values (x'_0, x'_1, x'_2) is divisible by at least one of $p_1, \ldots, p_{11}$.

For a moment, let us forget about condition (3). Then the problem can be formulated as follows: find the minimum vector of the form

$$D(x_0, x_1, x_2) + U_1(P, 0, 0) + U_2(0, P, 0) + U_3(0, 0, P),$$

i.e., the vector in the lattice generated by the vectors (x_0, x_1, x_2), $(P, 0, 0)$, $(0, P, 0)$, $(0, 0, P)$. The smallest vector can be found by the LLL-algorithm [6].

For any admissible covering (2) with the above m_i's we build the initial values (x_0, x_1, x_2) for the sequence (1) by using Šiurys' method. Then using the LLL-algorithm we find the smallest lattice basis. Coordinates (x'_0, x'_1, x'_2) for each of the new three basis vectors will suit us, if the condition (3) is satisfied and (x'_0, x'_1, x'_2) are of the same sign (if all of them are negative, then replace (x'_0, x'_1, x'_2) by $(-x'_0, -x'_1, -x'_2)$). Thus, searching through all possible coverings (the total amount is 23040) we find sets $(p_i, m_i, r_i, a_i, b_i)$. Those listed in Table 2 give rise to the smallest initial triple $x_0 = 151646890045$, $x_1 = 836564809606$, $x_2 = 942785024683$, as stated in Theorem 1.

i	1	2	3	4	5	6	7	8	9	10	11
$\mathbf{m_i}$	2	5	6	8	10	12	15	20	20	24	30
$\mathbf{p_i}$	2	29	17	7	11	107	8819	19	239	1151	1621
$\mathbf{r_i}$	1	0	4	0	8	8	6	2	14	12	26
$\mathbf{a_i}$	1	8	16	3	7	70	3246	12	202	1077	180
$\mathbf{b_i}$	0	23	13	1	2	17	8805	8	103	964	291

Table 2: p_i, m_i, r_i, a_i, b_i.

If we allow non-positive terms in the sequence, the same method gives sets $(p'_i, m'_i, r'_i, a'_i, b'_i)$, which give the sequence mentioned in the Remark 2: $x_0 = 730344594529$, $x_1 = -45426674968$, $x_2 = 151646890045$ (see Table 3).

i	1	2	3	4	5	6	7	8	9	10	11
$\mathbf{m_i}$	2	5	6	8	10	12	15	20	20	24	30
$\mathbf{p_i}$	2	29	17	7	11	107	8819	19	239	1151	1621
$\mathbf{r_i}$	1	2	0	2	0	10	8	4	16	14	28
$\mathbf{a_i}$	1	8	7	3	7	70	3246	12	202	1077	180
$\mathbf{b_i}$	0	23	11	1	2	17	8805	8	103	964	291

Table 3: $p'_i, m'_i, r'_i, a'_i, b'_i$.

It is worth noting that in the sequence mentioned in Remark 2 $x_2 = 151646890045$, $x_3 = 836564809606$, $x_4 = 942785024683$, so this means that the sequence in Theorem 1 is a shift of the sequence in Remark 2.

□

Both sequences can be extended to the left. It can be shown that in both cases mentioned above these extended sequences also contain no primes. Since the sequences modulo P are periodic with period $\operatorname{lcm}(m_1, \ldots, m_{11}) = 120$, it is enough to check that $x_j \neq k$ modulo P, $-8819 \leq k \leq 8819 = \max(p_i)$, $j = 0, \ldots, 119$.

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