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# The Number of Relatively Prime Subsets of a Finite Union of Sets of Consecutive Integers

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### Abstract

Let $A$ be a finite union of disjoint sets of consecutive integers and let $n$ be a positive integer. We give a formula for the number of relatively prime subsets (resp.,

relatively prime subsets of cardinality $k$) of $A$, which generalizes results of Nathanson, El Bachraoui and others. We give as well similar formulas for the number of subsets with gcd coprime to $n$.

# 1 Introduction

A nonempty set $S$ of integers is said to be *relatively prime* if $\gcd(S) = 1$, where $\gcd(S)$ denotes the greatest common divisor of the elements of $S$. Nathanson [10] defines $f(n)$ to be the number of relatively prime subsets of $\{1, 2, \ldots, n\}$ and, for $k \geq 1$, $f_k(n)$ to be the number of relatively prime subsets of $\{1, 2, \ldots, n\}$ of cardinality $k$. By analogy with Euler's phi function $\phi(n)$ that counts the number of positive integers $a$ in the set $\{1, 2, \ldots, n\}$ such that $\gcd(a, n) = 1$, Nathanson [10] defines $\Phi(n)$ to be the number of nonempty subsets $S$ of the set $\{1, 2, \ldots, n\}$ such that $\gcd(S)$ is relatively prime to $n$ and, for integer $k \geq 1$, $\Phi_k(n)$ to be the number of subsets $S$ of the set $\{1, 2, \ldots, n\}$ such that $\gcd(S)$ is relatively prime to $n$ and $|S| = k$. He obtains explicit formulas for these four functions and deduces asymptotic estimates.

For simplicity, we use a more general notation than Nathanson [10]. For a nonempty set of integers $S$, we define

- $f(S) = |\{H \subseteq S : \gcd(H) = 1, H \neq \emptyset\}|$ as the number of nonempty relatively prime subsets of $S$;
- $f_k(S) = |\{H \subseteq S : \gcd(H) = 1, |H| = k\}|$ as the number of relatively prime subsets of $S$ of cardinality $k$;
- $\Phi(S, n) = |\{H \subseteq S : \gcd(H \cup \{n\}) = 1, H \neq \emptyset\}|$ as the number of nonempty subsets of $S$ with gcd relatively prime to integer $n$;
- $\Phi_k(S, n) = |\{H \subseteq S : \gcd(H \cup \{n\}) = 1, |H| = k\}|$ as the number of subsets of $S$ of cardinality $k$ and with gcd relatively prime to integer $n$.

Further, we define $[a, b]_{\mathbb{Z}} = [a, b] \cap \mathbb{Z} = \{a, a + 1, \ldots, b\}$ for integers $a < b$ as the set of consecutive integers from $a$ to $b$, inclusive.

El Bachraoui [4] and Nathanson and Orosz [11] generalize the results of Nathanson [10] to subsets of $[\ell, m]_{\mathbb{Z}}$ for integers $0 \leq \ell < m$, and prove Theorem 1.

**Theorem 1.** *For non-negative integers $\ell < m$ and $k \geq 1$, using the notation $f(\ell, m) = f([\ell, m]_{\mathbb{Z}})$ and $f_k(\ell, m) = f_k([\ell, m]_{\mathbb{Z}})$ of El Bachraoui [4] and Nathanson and Orosz [11] we have*

$$f(\ell, m) = \sum_{d=1}^{m} \mu(d) \left( 2^{\lfloor \frac{m}{d} \rfloor - \lfloor \frac{\ell}{d} \rfloor} - 1 \right) \tag{1}$$

*and*

$$f_k(\ell, m) = \sum_{d=1}^{m} \mu(d) \binom{\lfloor m/d \rfloor - \lfloor \ell/d \rfloor}{k}, \tag{2}$$

*where $\mu$ is the Möbius function.*

For brevity, define the arithmetic sequence $\mathcal{A}_n^{(a,b)} = \{a, a+b, \ldots, a+(n-1)b\}$ for positive integers $n$, $a$, and $b$. Ayad and Kihel [1] generalize Theorem 1 to obtain Theorem 2.

**Theorem 2.** *For all positive integers $n$, $a$, and $b$, with $\gcd(a,b) = 1$, using the notation $f^{(a,b)}(n) = f\left(\mathcal{A}_n^{(a,b)}\right)$ and $f_k^{(a,b)}(n) = f_k\left(\mathcal{A}_n^{(a,b)}\right)$ of Ayad and Kihel [1], we have*

$$f^{(a,b)}(n) = \sum_{\substack{d=1 \\ \gcd(b,d)=1}}^{a+(n-1)b} \mu(d)\left(2^{\left\lfloor \frac{n}{d} \right\rfloor + \varepsilon_d} - 1\right) \tag{3}$$

*and*

$$f_k^{(a,b)}(n) = \sum_{\substack{d=1 \\ \gcd(b,d)=1}}^{a+(n-1)b} \mu(d)\binom{\left\lfloor \frac{n}{d} \right\rfloor + \varepsilon_d}{k}, \tag{4}$$

*where*

$$\varepsilon_d = \begin{cases} 0, & \text{if } d \mid n; \\ 1, & \text{if } d \nmid n \text{ and } (-ab^{-1}) \bmod d \in \left\{\left\lfloor \frac{n}{d} \right\rfloor d, \ldots, n-1\right\}; \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

El Bachraoui [6] extends Theorem 1 to the union of two sets of consecutive integers, to obtain Theorem 3.

**Theorem 3.** *For nonnegative integers $\ell_1 < m_1 < \ell_2 < m_2$ and for $k \geq 1$,*

$$f\left([\ell_1, m_1]_{\mathbb{Z}} \cup [\ell_2, m_2]_{\mathbb{Z}}\right) = \sum_{d=1}^{m_2} \mu(d)\left(2^{\left\lfloor \frac{m_1}{d} \right\rfloor + \left\lfloor \frac{m_2}{d} \right\rfloor - \left\lfloor \frac{\ell_1 - 1}{d} \right\rfloor - \left\lfloor \frac{\ell_2 - 1}{d} \right\rfloor} - 1\right) \tag{6}$$

*and*

$$f_k\left([\ell_1, m_1]_{\mathbb{Z}} \cup [\ell_2, m_2]_{\mathbb{Z}}\right) = \sum_{d=1}^{m_2} \mu(d)\binom{\left\lfloor \frac{m_1}{d} \right\rfloor + \left\lfloor \frac{m_2}{d} \right\rfloor - \left\lfloor \frac{\ell_1 - 1}{d} \right\rfloor - \left\lfloor \frac{\ell_2 - 1}{d} \right\rfloor}{k}. \tag{7}$$

We now switch our attention to analogous results for functions $\Phi$ and $\Phi_k$. For the consecutive integers case, El Bachraoui [4] and Nathanson and Orosz [11] prove Theorem 4.

**Theorem 4.** *For non-negative integers $\ell < m$ and $k \geq 1$, using the notation $\Phi(\ell, m) = \Phi([\ell, m]_{\mathbb{Z}}, m)$ and $\Phi_k(\ell, m) = \Phi_k([\ell, m]_{\mathbb{Z}}, m)$ of El Bachraoui [4] and Nathanson and Orosz [11] we have*

$$\Phi(\ell, m) = \sum_{d \mid m} \mu(d)\, 2^{\left(\frac{m}{d} - \left\lfloor \frac{\ell}{d} \right\rfloor\right)} \tag{8}$$

*and*

$$\Phi_k(\ell, m) = \sum_{d \mid m} \mu(d)\binom{\frac{m}{d} - \left\lfloor \frac{\ell}{d} \right\rfloor}{k}. \tag{9}$$

Ayad and Kihel [1] generalize Theorem 4 to obtain Theorem 5.

**Theorem 5.** *For nonnegative integers $a$, $b$, and $n$, with $\gcd(a,b)=1$, using the notation $\Phi^{(a,b)}(n)=\Phi\left(\mathcal{A}_n^{(a,b)},n\right)$ and $\Phi_k^{(a,b)}(n)=\Phi_k\left(\mathcal{A}_n^{(a,b)},n\right)$ of Ayad and Kihel [1] we have*

$$\Phi^{(a,b)}(n)=\sum_{\substack{d|n\\ \gcd(b,d)=1}}\mu(d)\left(2^{\frac{n}{d}}-1\right) \tag{10}$$

*and*

$$\Phi_k^{(a,b)}(n)=\sum_{\substack{d|n\\ \gcd(b,d)=1}}\mu(d)\binom{\frac{n}{d}}{k}. \tag{11}$$

El Bachraoui and Salim [9] extend Theorem 4 to the union of two sets of consecutive integers, to obtain Theorem 6.

**Theorem 6.** *For nonnegative integers $\ell_1<m_1<\ell_2<m_2$ and for $k\geq 1$,*

$$\Phi\left([\ell_1,m_1]_{\mathbb{Z}}\cup[\ell_2,m_2]_{\mathbb{Z}},n\right)=\sum_{d|n}\mu(d)\,2^{\left\lfloor\frac{m_1}{d}\right\rfloor+\left\lfloor\frac{m_2}{d}\right\rfloor-\left\lfloor\frac{\ell_1-1}{d}\right\rfloor-\left\lfloor\frac{\ell_2-1}{d}\right\rfloor} \tag{12}$$

*and*

$$\Phi_k\left([\ell_1,m_1]_{\mathbb{Z}}\cup[\ell_2,m_2]_{\mathbb{Z}},n\right)=\sum_{d|n}\mu(d)\binom{\left\lfloor\frac{m_1}{d}\right\rfloor+\left\lfloor\frac{m_2}{d}\right\rfloor-\left\lfloor\frac{\ell_1-1}{d}\right\rfloor-\left\lfloor\frac{\ell_2-1}{d}\right\rfloor}{k}. \tag{13}$$

In Section 2, we extend Theorems 3 and 6 to the union of any finite number of disjoint sets of consecutive integers. The approach we take is simple and much different from the approach of El Bachraoui [6] and El Bachraoui and Salim [9] for the union of two sets. Several authors [2, 3, 12, 7, 13, 14, 5] discuss other properties and generalizations.

## 2 Finite union of disjoint sets of consecutive integers

For positive integers $\ell_i\leq m_i$ for $i=1,\ldots,r$, denote $A^{(i)}=[\ell_i,m_i]_{\mathbb{Z}}$ for brevity and assume $A^{(i)}\cap A^{(j)}=\emptyset$ for $i\neq j$. Consider the union

$$A=\bigcup_{i=1}^{r}A^{(i)}. \tag{14}$$

El Bachraoui [6] derives equations for $f(A)$ and $f_k(A)$ for $r=2$, as in Theorem 3. We extend this to any $r\in\mathbb{N}$ in Theorem 8, but first we need Lemma 7. Also, throughout this section, for a set of integers $S$ we denote $\mathcal{P}(S)=\{H\subseteq S : H\neq\emptyset\}$ and $\mathcal{P}_k(S)=\{H\subseteq S : |H|=k\}$.

**Lemma 7.** *Let $A_d = \{x \in A : d \mid x\}$ be all the multiples of $d$ found in $A$, where $A$ is defined in equation (14). Then,*

$$|A_d| = \sum_{i=1}^{r} \left( \left\lfloor \frac{m_i}{d} \right\rfloor - \left\lfloor \frac{\ell_i - 1}{d} \right\rfloor \right).$$

*Proof.* For $i = 1, \ldots, r$, let $A_d^{(i)} = \left\{x \in A^{(i)} : d \mid x\right\}$, $M_d^{(i)} = \{x \in [0, m_i]_{\mathbb{Z}} : d \mid x\}$, and $L_d^{(i)} = \{x \in [0, \ell_i - 1]_{\mathbb{Z}} : d \mid x\}$. Clearly, we have $\left|A_d^{(i)}\right| = \left|M_d^{(i)}\right| - \left|L_d^{(i)}\right|$. But, we simply have $\left|M_d^{(i)}\right| = \left\lfloor \frac{m_i}{d} \right\rfloor$ and $\left|L_d^{(i)}\right| = \left\lfloor \frac{\ell_i - 1}{d} \right\rfloor$. So,

$$\left|A_d^{(i)}\right| = \left\lfloor \frac{m_i}{d} \right\rfloor - \left\lfloor \frac{\ell_i - 1}{d} \right\rfloor.$$

Now, since $A_d = \bigcup\limits_{i=1}^{r} A_d^{(i)}$ and since $A_d^{(i)} \cap A_d^{(j)} = \emptyset$ for $i \neq j$, we have $|A_d| = \sum\limits_{i=1}^{r} \left|A_d^{(i)}\right|$ which completes the proof. □

**Theorem 8.** *For $A$ defined in equation (14), we have*

$$f(A) = \sum_{d=1}^{\max\{m_1,\ldots,m_r\}} \mu(d) \left( 2^{\sum_{i=1}^{r} \left( \left\lfloor \frac{m_i}{d} \right\rfloor - \left\lfloor \frac{\ell_i - 1}{d} \right\rfloor \right)} - 1 \right); \tag{15}$$

$$f_k(A) = \sum_{d=1}^{\max\{m_1,\ldots,m_r\}} \mu(d) \binom{\sum_{i=1}^{r} \left( \left\lfloor \frac{m_i}{d} \right\rfloor - \left\lfloor \frac{\ell_i - 1}{d} \right\rfloor \right)}{k}. \tag{16}$$

*Proof.* We begin by proving equation (15). From the total amount of nonempty subsets of $A$, remove those subsets that are *not* relatively prime:

$$f(A) = |\mathcal{P}(A)| - \left| \bigcup_{p \text{ prime}} \mathcal{P}(A_p) \right|.$$

Using inclusion-exclusion and the same principle as in the proof of Ayad and Kihel [1, Theorem 5], we obtain

$$f(A) = \sum_{d=1}^{\max\{m_1,\ldots,m_r\}} \mu(d) \left(2^{|A_d|} - 1\right).$$

Applying Lemma (7), we obtain equation (15).

To prove equation (16), from the total amount of subsets of $A$ with cardinality $k$, remove those subsets that are *not* relatively prime:

$$f_k(A) = |\mathcal{P}_k(A)| - \left| \bigcup_{p \text{ prime}} \mathcal{P}_k(A_p) \right|.$$

Using inclusion-exclusion and the same principle as in the proof of Ayad and Kihel [1, Theorem 5], we obtain

$$f_k(A) = \sum_{d=1}^{\max\{m_1,\ldots,m_r\}} \mu(d) \binom{|A_d|}{k}.$$

Applying Lemma (7), we obtain equation (16). □

Similarly, we now extend Theorem 6.

**Theorem 9.** *Define $A$ as in equation (14). Then for any integer $k \geq 1$,*

$$\Phi(A,n) = \sum_{d|n} \mu(d) \left( 2^{\sum_{i=1}^{r} \left( \left\lfloor \frac{m_i}{d} \right\rfloor - \left\lfloor \frac{\ell_i - 1}{d} \right\rfloor \right)} - 1 \right); \tag{17}$$

$$\Phi_k(A,n) = \sum_{d|n} \mu(d) \binom{\sum_{i=1}^{r} \left( \left\lfloor \frac{m_i}{d} \right\rfloor - \left\lfloor \frac{\ell_i - 1}{d} \right\rfloor \right)}{k}. \tag{18}$$

*Proof.* We begin by proving equation (17). Notice that

$$\Phi(A,n) = |\mathcal{P}(A)| - \left| \bigcup_{\substack{p \text{ prime} \\ p|n}} \mathcal{P}(A_p) \right|.$$

As in the proof of Theorem 8, we have

$$\Phi(A,n) = \sum_{d|n} \mu(d) \left( 2^{|A_d|} - 1 \right).$$

Applying Lemma 7 proves equation (17).

To prove equation (18), notice that

$$\Phi_k(A,n) = |\mathcal{P}_k(A)| - \left| \bigcup_{\substack{p \text{ prime} \\ p|n}} \mathcal{P}_k(A_p) \right|.$$

As in the proof of Theorem 8, we have

$$\Phi_k(A,n) = \sum_{d|n} \mu(d) \binom{|A_d|}{k}.$$

Applying Lemma 7 proves equation (18). □

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