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Nim Fractals

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Abstract

We enumerate P-positions in the game of Nim in two different ways. In one series of sequences we enumerate them by the maximum number of counters in a pile. In another series of sequences we enumerate them by the total number of counters.

We show that the game of Nim can be viewed as a cellular automaton, where the total number of counters divided by 2 can be considered as a generation in which P-positions are born. We prove that the three-pile Nim sequence enumerated by the total number of counters is a famous toothpick sequence based on the Ulam-Warburton cellular automaton. We introduce 10 new sequences.

1 Introduction

The study of the game of Nim is fundamental to the field of combinatorial game theory. Nim is known as an *impartial combinatorial game*, a game in which each player has the same moves available at each point in the game and has a complete amount of information about the game and the potential moves. In addition, there is no randomness in the game (such as rolling dice).

Originally introduced by Charles Bouton in 1901 [4], Nim has played a role in many combinatorial games. The relationship between Nim and other impartial combinatorial games can be described with the Sprague-Grundy Theorem [5, 10], which states that all impartial games are equivalent to a Nim heap.

Although the game of Nim has been studied extensively [1, 3, 6, 8], in this paper, we invent new sequences related to enumeration of the P-positions of Nim.

In Section 2 we introduce the game of Nim as well as Bouton's general formula for P-positions. In Section 3 we define the sequences we want to count and provide examples for the games with one and two piles. One set of sequences is indexed by the maximum number of counters in a P-position and the other by the total number of counters. All of these sequences exhibit fractal-like behavior.

We continue with calculating formulae for the sequences indexed by the maximum number of counters in Section 4. We calculate the three-piles case in Section 4.1 and the four-piles case in Section 4.2. It can be noted that the calculation method is different for an odd and an even number of piles. But these sections provide enough background for a general formula in Section 4.3.

Then we turn our attention to the sequences indexed by the total number of counters in Section 5. We start with calculating the three-piles case in Section 5.1. We were motivated by the fractal-like patterns in this sequence to discover that this sequence describes an evolution of a particular cellular automaton. We explain in Section 5.2 how Nim can be viewed as an automaton. We extend the definitions to allow any impartial combinatorial game to be viewed as an automaton in Section 5.3. In Section 5.4 we define the Ulam-Warburton automaton, three branches of which correspond to Nim with three piles. We proceed to enumerating four piles in Section 5.5 and arbitrarily many piles in Section 5.6.

2 The game of Nim

In Nim, there are k piles of counters, with p_i counters in each pile. Two players alternate turns by taking some or all of the counters in a single pile. The player who takes the last counter (or equivalently, makes the last move) wins. We may denote the state or position of a game with an ordered tuple $P = (p_1, p_2, \ldots, p_k)$.

We begin by introducing some general definitions in game theory. Assuming that both players use an optimal strategy, there are two types of positions in a game such as Nim:

Definition 1. A *P-position* is a position in which the previous player will win (the one who just moved). An *N-position* is a position in which the next player will win (the one about to move).

We denote the set of P-positions as $\mathcal{P}$, and the set of N-positions as $\mathcal{N}$. Thus, any move from a P-position must be an N-position, and conversely, every N-position has at least one move that results in a P-position. This motivates the following theorem [1]:

Theorem 2. *Suppose that the positions of a finite impartial game can be partitioned into disjoint sets A and B with the properties:*

1. *Every move of a position in A is to a position in B.*
2. *Every position in B has at least one move to a position in A.*
3. *The terminal positions are in A.*

Then $A = \mathcal{P}$ and $B = \mathcal{N}$.

By definition, the Nim position $(0, 0, \ldots, 0)$ will be a (terminal) P-position. Note that the general winning strategy is to move to a P-position if possible.

To explicitly give a formula for P-positions, we need the following definition:

Definition 3. The *nim-sum* of two non-negative integers x, y is their bit-wise XOR: $x \oplus y$. Suppose that $x = (b_j \ldots b_2 b_1)_2$ and $y = (c_j \ldots c_2 c_1)_2$ in binary with leading zeroes as necessary, where j is the minimum number of digits sufficient for the binary representations of x and y. Then the nim-sum of x and y is $(d_j \ldots d_2 d_1)_2$, where $d_i = b_i + c_i \pmod 2$ for $1 \leq i \leq j$.

The nim-sum is clearly associative and commutative, and 0 is the identity element. Further, $x \oplus y = x \oplus z$ implies $y = z$. We can extend the concept of nim-sum to a position $(p_1, p_2, \ldots, p_k)$: it is simply $p_1 \oplus p_2 \oplus \cdots \oplus p_k$.

The following theorem [4] describes the set of P-positions in Nim.

Theorem 4 (Bouton, 1901)**.** *$\mathcal{P}$ is the set of positions in the game of Nim with nim-sum 0, and $\mathcal{N}$ is the complement.*

From here, we can show that the last pile in a P-position is a function of the previous $k - 1$ piles.

Corollary 5. *A position $(p_1, \ldots, p_{k-1}, p_k)$ is a P-position if and only if $p_k = p_1 \oplus \cdots \oplus p_{k-1}$.*

3 Nim sequences

We would like to enumerate P-positions in the game of Nim. We assume that the number of piles, k, is fixed, which means that piles of zero are allowed. There are two natural ways to enumerate these P-positions.

In the first set of sequences, we want to count the number of P-positions where the number of counters in each pile is bounded by some number n. We call these sequences *indexed-by-maximum*.

- $a_k(n)$ is the number of P-positions in the game of Nim with k piles such that each pile has no more than n counters.
- $d_k(n)$ is the number of P-positions in the game of Nim with k piles such that the largest pile has exactly n counters.

Note that $d_k(n)$ is the sequence of first differences of $a_k(n)$, and $a_k(n)$ is the sequence of partial sums of $d_k(n)$.

Another natural way to enumerate P-positions is bounding the total number of counters in all the piles. Note that the total number of counters in a P-position is even. We call these sequences *indexed-by-total*.

- $A_k(n)$ is the number of P-positions in the game of Nim with k piles such that the total number of counters is no more than $2n$.
- $D_k(n)$ is the number of P-positions in the game of Nim with k piles such that the total number of counters is exactly $2n$.

Once again, note that $D_k(n)$ is the sequence of first differences of $A_k(n)$, and $A_k(n)$ is the sequence of partial sums of $D_k(n)$.

These sequences exhibit fractal-like behavior. The source of this behavior is the self-similarity in the set of P-positions. Consider the positions (p_1, p_2, p_3), $(2^k + p_1, 2^k + p_2, p_3)$, $(2^k + p_1, p_2, 2^k + p_3)$, and $(p_1, 2^k + p_2, 2^k + p_3)$, where $\max(p_1, p_2, p_3) < 2^k$. If one of these is a P-position then the other three are also P-positions.

In our proofs, we loosely use the term "pile" to refer to the number of counters in the pile.

3.1 Relationship between sequences

Let $\#(P)$ be the total number of counters in a P-position $P = (p_1, p_2, \ldots, p_k)$; that is, $\#(P) = \sum p_i$. We will refer to this as the *total sum* of the P-position. Further, let $\max(P)$ denote the largest pile of P; that is, $\max(P) = \max\{p_i\}$.

The sequences $a_k(n)$ and $A_k(n)$ can bound each other due to the following lemma:

Lemma 6. $2\max(P) \leq \#(P) \leq k\max(P)$.

Proof. The upper bound is obvious. To prove the lower bound, consider the place values of the ones in the binary representation of $\max(P)$. By Theorem 4, the nim-sum is 0, and the only way for this to occur is if the other piles collectively have ones in each of those place values. The lower bound then follows immediately since the sum of the numbers other than $\max(P)$ is at least $\max(P)$. □

Corollary 7. $a_k(\lfloor 2n/k \rfloor) \leq A_k(n) \leq a_k(n)$.

Proof. The sequence $A_k(n)$ enumerates P-positions with no more than $2n$ counters. These positions cannot have more than n counters in any pile by Lemma 6, so they all are included in the enumeration corresponding to $a_k(n)$. Further, every position with a maximum pile of no more than $2n/k$ must have a total sum of less than $2n$. Thus, P-positions that are counted by $a_k(\lfloor 2n/k \rfloor)$ are all included in the count of $A_k(n)$. □

In the sections below we provide recursive formulae for sequences a, d, A, and D. For indices of the form $2^m - 1$ the formula for a_k is particularly simple.

Lemma 8. $a_k(2^m - 1) = 2^{m(k-1)}$.

Proof. There are 2^m choices for each of the first $k-1$ piles, for a total of $2^{m(k-1)}$ choices. By Corollary 5, the last pile is uniquely determined, and since no pile is greater than $2^m - 1$, which is the largest number that has m digits in binary, the last pile will also not be greater than $2^m - 1$. □

Together with Corollary 7, we can prove the following bound:

Corollary 9.

$$2^{(k-1)\lfloor \log_2(\lfloor 2n/k \rfloor + 1) \rfloor} \leq a_k(\lfloor 2n/k \rfloor) \leq A_k(n) \leq a_k(n) \leq 2^{(k-1)\lceil \log_2(n+1) \rceil}.$$

3.2 One or two piles

If there is only one pile, there is only one P-position: (0). Thus, $a_1(n) = A_1(n) = 1$ for all n, which is sequence A000012 in the OEIS [7]. Correspondingly, $d_1(0) = D_1(0) = 1$ and $d_1(n) = D_1(n) = 0$ for $n \geq 1$, which is sequence A000007.

The P-positions for the game with two piles are described by the following lemma:

Lemma 10. *A position P is a P-position if and only if $P = (x, x)$ for a non-negative integer x.*

This means that $a_2(n) = A_2(n) = n + 1$, which is sequence A000027 with an initial offset of 0. In addition, $d_2(n) = D_2(n) = 1$, which is sequence A000012.

From here, the formulae for the sequences become cumbersome to express only in terms of n, so we define $n = 2^b - 1 + c$, where $b = \lfloor \log_2 n \rfloor$, and $1 \leq c = n + 1 - 2^{\lfloor \log_2 n \rfloor} \leq 2^b$. In other words, b is the number of digits in the binary representation of n, and $c - 1$ is n without its first digit.

4 Indexed-by-maximum sequences

4.1 Three piles

Consider the set of P-positions in Nim with three piles. We want to find a formula for the number of such P-positions P with $\max(P) = n$.

Theorem 11. *For $n > 0$, $d_3(n) = 6c - 3$.*

Proof. There are three P-positions that are permutations of $(n, n, 0)$. All other P-positions have exactly one pile with n counters. However, in order for the nim-sum to be 0, one of the other piles must be at least 2^b, and then the last pile is uniquely defined. There are 3 choices for which pile has n counters, 2 choices for the pile that is at least 2^b, and $c - 1$ choices for the number of counters in this pile. This results in a total of $3 + 6(c-1) = 6c - 3$ P-positions. □

In other words, $d_3(n) = 6(n + 1 - 2^{\lfloor \log_2 n \rfloor}) - 3$. This is sequence A241717 in the OEIS [7]: 1, 3, 3, 9, 3, 9, 15, 21, 3, 9, 15, 21, 27, 33, 39, 45, 3, 9, ...

If we arrange the numbers into a triangle as follows, the fractal-like behavior of the sequence can be seen:

3,
3, 9,
3, 9, 15, 21,
3, 9, 15, 21, 27, 33, 39, 45,
3, 9, 15, 21, 27, 33, 39, 45, 51, 57, 63, 69, 75, 81, 87, 93,
⋮

The length of each line is a power of 2, and each line converges to A016945—the sequence $6n + 3$.

Theorem 12. *For $n > 0$, $a_3(n) = 2^{2b} + 3c^2$.*

Proof. This statement is true for $n = 2^b - 1$ by Lemma 8. Now we prove the statement for all $n = 2^b - 1 + c$. There are 2^{2b} P-positions such that all piles are less than 2^b. If one of the piles is at least 2^b, then exactly one other pile must also be at least 2^b. The leftover pile is uniquely defined and is less than 2^b. There are 3 ways to designate the two piles greater than or equal to 2^b, and c different choices for each of those two piles, so the total is $2^{2b} + 3c^2$ P-positions, as desired. □

In other words, $a_3(n) = 2^{2\lfloor \log_2 n \rfloor} + 3(n + 1 - 2^{\lfloor \log_2 n \rfloor})^2$. This is sequence A236305 in the OEIS [7]: 1, 4, 7, 16, 19, 28, 43, 64, 67, 76, 91, 112, ... This sequence also displays fractal-like behavior, much like the previous sequence.

4.2 Four piles

When there are four piles, we cannot apply the argument used in Theorem 12 because there is a possibility that all four piles have more than $2^b - 1$ counters, and we need to make sure that each of them does not exceed n. However, a slight modification of our argument shows that we can find a recursive formula:

Theorem 13. *For $n > 0$, $a_4(n) = 2^{3b} + 6c^2 2^b + a_4(c-1)$.*

Proof. Suppose that all of the piles are not greater than $2^b - 1$. Similar to the argument in Theorem 12, the first three piles that are not less than 2^b uniquely define a P-position where all the piles are not less than 2^b. There are 2^{3b} such positions.

In addition to that, we can have either 2 or 4 piles that are greater than or equal to 2^b. If there are 2 such piles, we can choose them in 6 different ways, and each of those piles can be any of c possible numbers. We can then choose another pile in 2^b ways, and the last pile will thus be fixed and less than 2^b. This accounts for the total of $6 \cdot 2^b c^2$ ways. If there are 4 piles that are greater than $2^b - 1$, we can remove 2^b counters from each pile without changing the nim-sum, thus reducing this situation to one when all piles are no greater than $c - 1$, which can be done in $a_4(c-1)$ ways. □

The sequence $a_4(n)$ is sequence A241522: 1, 8, 21, 64, 89, 168, 301, 512, 561, 712, ...

Corollary 14. *For $n > 0$, $d_4(n) = (12c - 6)2^b + d_4(c - 1)$.*

Proof. We can either use a similar argument as before or the fact that this sequence is the first difference sequence of the sequence above. □

The sequence $d_4(n)$ is sequence A241718: 1, 7, 13, 43, 25, 79, 133, 211, 49, 151, 253, ...

4.3 Many piles

We will now prove a more general formula for $a_k(n)$ based on the parity of k.

Theorem 15. *If k is odd, $a_k(n) = \frac{(2^b + c)^k + (2^b - c)^k}{2^{b+1}}$, for $n > 0$.*

Proof. Suppose that $2i$ of the piles are at least 2^b. There are $\binom{k}{2i}$ ways to choose which piles these are, and there are c choices for each of these $2i$ piles. Of the remaining $k - 2i$ piles, there are 2^b choices for the first $k - 2i - 1$ piles. The last pile will be uniquely determined by Lemma 5, and its size will not exceed 2^b. Hence, we get a total of $\binom{k}{2i} 2^{b(k-2i-1)} c^{2i}$ P-positions.

Since $2i$ can range from 0 to $k - 1$, we get the following formula:

$$a_k(n) = \binom{k}{0} 2^{b(k-1)} c^0 + \binom{k}{2} 2^{b(k-3)} c^2 + \cdots + \binom{k}{k-1} 2^0 c^{k-1}.$$

We multiply both sides of this equation by 2^b:

$$2^b a_k(n) = \binom{k}{0} 2^{bk} c^0 + \binom{k}{2} 2^{b(k-2)} c^2 + \cdots + \binom{k}{k-1} 2^b c^{k-1}.$$

Since

$$(2^b \pm c)^k = \binom{k}{0} 2^{bk} c^0 \pm \binom{k}{1} 2^{b(k-1)} c^1 + \binom{k}{2} 2^{b(k-2)} c^2 \pm \cdots,$$

we have that

$$2^b a_k(n) = \frac{(2^b + c)^k + (2^b - c)^k}{2}$$

and

$$a_k(n) = \frac{(2^b + c)^k + (2^b - c)^k}{2^{b+1}},$$

as desired. □

Note that if $k = 3$, we get $a_3(n) = 2^{2b} + 3c^2$ as expected. If $k = 5$, we get $a_5(n) = 2^{4b} + 10 \cdot 2^{2b} c^2 + 5c^4$. This is sequence A241523: 1, 16, 61, 256, 421, 976, 2101, 4096, 4741, ...

We can calculate $d_k(n)$ for odd k in a similar manner, or by subtracting consecutive terms:

Theorem 16. *If k is odd,*

$$d_k(n) = \frac{(2^b + c)^k + (2^b - c)^k - (2^b + c - 1)^k - (2^b - c + 1)^k}{2^{b+1}}.$$

For example, if $k = 5$, $d_5(n) = 10 \cdot 2^{2b}(2c - 1) + 20c^3 - 30c^2 + 20c - 5$. This sequence is sequence A241731: 1, 15, 45, 195, 165, 555, 1125, 1995, 645, ...

Theorem 17. *If k is even, $a_k(n) = \dfrac{(2^b + c)^k + (2^b - c)^k - 2c^k}{2^{b+1}} + a_k(c - 1)$, for $n > 0$.*

Proof. We can use the same argument as above for all cases except for when all of the piles are at least 2^b. So if we do not consider this case, there are $\dfrac{(2^b + c)^k + (2^b - c)^k - 2c^k}{2^{b+1}}$ such P-positions. Suppose, now, that all of the piles are at least 2^b. Note that if we subtract 2^b from each of these piles, the nim-sum will not be changed, and now each pile is no more than $n - 2^b = c - 1$, so there are $a_k(c - 1)$ such P-positions. So our formula is $a_k(n) = \dfrac{(2^b + c)^k + (2^b - c)^k - 2c^k}{2^{b+1}} + a_k(c - 1)$. □

We can calculate $d_k(n)$ for even k in a similar manner:

Theorem 18. *If k is even, $d_k(n)$ equals*

$$\frac{(2^b + c)^k + (2^b - c)^k - (2^b + c - 1)^k - (2^b - c + 1)^k - 2c^k + 2(c - 1)^k}{2^{b+1}} + d_k(c - 1).$$

5 Indexed-by-total

We will now fix the total number of counters as $2n$. Let $s_2(n)$ denote the *binary weight* of n; that is, the number of ones in the binary expansion of n.

5.1 Three piles

Theorem 19. $D_3(n) = 3^{s_2(n)}$.

Proof. Represent each pile as a sum of distinct powers of 2. Each power of two, 2^i, can be present in exactly two piles, or not present at all. That means if we sum all the piles we get that n is the sum of powers of two that are present in exactly two piles. For each power of two that is present in the binary representation of n we can chose in 3 ways in which piles they occur, for a total of $3^{s_2(n)}$ ways. □

This sequence is sequence A048883: 1, 3, 3, 9, 3, 9, 9, 27, 3, 9, 9, 27, 9, 27, 27, 81, 3, 9, ... Indexing starts as follows: $D_3(0) = 1$, and $D_3(1) = 3$. This sequence satisfies the following recursion: $D_3(2n) = D_3(n)$ and $D_3(2n + 1) = 3D_3(n)$.

We can see the fractal-like behavior of this sequence by considering a recursive definition: Start with the multiset $S_0 = \{1\}$. Then form a new multiset S_{i+1} by concatenating the elements of S_i with the three times the elements of S_i, and repeat *ad infinitum*. For example, $S_2 = \{1, 3, 3, 9\}$. We form S_3 by concatenating the elements of the set $\{3, 9, 9, 27\}$.

The sequence $A_3(n)$: partial sums of $D_3(n)$ is also present in the database. It is sequence A130665: 1, 4, 7, 16, 19, 28, 37, 64, 67, 76, 85, 112, 121, 148, 175, ... The sequence satisfies the recursion: $A_3(2n) = 3A_3(n - 1) + A_3(n)$ and $A_3(2n + 1) = 4A_3(n)$.

We calculated this sequence and discovered that it is in the database as the sequence describing the number of cells in three branches of the *Ulam-Warburton cellular automaton* (see Ulam [12], Singmaster [9], Stanley and Chapman [11], Wolfram [13]). It is amazing how the On-Line Encyclopedia of Integer Sequences makes it possible to connect different areas of mathematics.

5.2 Evolution of Nim

A natural question that arises is that if P-positions in Nim can be enumerated by cells in an automaton, can we find a bijection between P-positions of Nim and cells in the automaton? We provide the construction in this subsection.

Call a P-position P_1 a *parent* of a P-position P_2 if $\#(P_1) + 2 = \#(P_2)$ and P_1 differs from P_2 in exactly two piles with the same index, by one counter in each. Correspondingly, if P_1 is a parent of P_2, we call P_2 a *child* of P_1. The following lemma connects the parent-child relationship to the game.

Lemma 20. *A parent P_1 can be achieved in a game from P-position P_2.*

Proof. Suppose piles i and j have one fewer counter in P_1 than in P_2. Then in the first move a player takes one counter from the i-th pile. In the next move the next player takes one counter from the j-th pile. □

The zero position: $(0, 0, \ldots, 0)$ does not have a parent. But any other P-position has a parent that is described by the following lemma:

Lemma 21. *Any non-zero P-position has a parent. Each parent can be achieved by subtracting 1 from piles i and j, where p_i and p_j are non-empty piles with the same number of zeros at the end of their binary representations.*

Proof. The nim-sum of p_i and p_j should not change: $p_i \oplus p_j = (p_i - 1) \oplus (p_j - 1)$. This is only possible if p_i and p_j have the same number of zeros at the end of their binary representations since they have to regroup in the same number of places. □

Corollary 22. *If there are 3 piles, each non-zero P-position has exactly one parent.*

Proof. Consider the rightmost 1 in the binary representations of p_1, p_2 and p_3. This 1 must appear in exactly two of the representations, and so these numbers have the same number of zeroes at the end of their binary representation. □

Similarly, we can also describe a child.

Lemma 23. *Each child can be achieved by adding 1 to piles i and j, where p_i and p_j has the same number of ones at the end of their binary representation.*

Proof. The nim-sum of p_i and p_j should not change: $p_i \oplus p_j = (p_i + 1) \oplus (p_j + 1)$. This is only possible if p_i and p_j have the same number of ones at the end of their binary representations since they have to regroup in the same number of places. □

If we play the game with 3 piles each P-position has exactly 0, 1, or 3 children.

This way we get a cellular automaton. We start with a zero position and call it alive. At each step the children of the living positions are born. Children that are born at step n are called n-generation and $D_k(n)$ enumerates them. Similarly, $A_k(n)$ enumerates all the cells that are alive by the time n.

After we wrote this paper, we discovered that there is a well-known notion of a game position being *born on day n* [1, 8]. If a P-position is of generation n, then it is born on day n.

5.3 Evolution of an impartial combinatorial game

Note that we can describe an evolution of any impartial combinatorial game using the following definition. We assume that the players behave optimally. That is, if they can move to a P-position they will do so.

A P-position P_1 is a *parent* of P_2 if there exists an optimal game of maximal length in which P_1 is achieved from P_2 in exactly two moves.

If the longest game starting with P_1 takes $2n$ moves, then n is the generation number of P_1. For Nim this definition coincides with the previous one because the longest game starting from a P-position with $2n$ counters cannot take more than $2n$ moves.

There is a standard algorithm for finding P-positions. Start with the terminal P-positions and assume they were found at step 0. Then proceed by induction. Denote the set of P-positions found at steps up to i as $\mathcal{P}_i$. Denote the positions that are one move away from $\mathcal{P}_i$ as $\mathcal{N}_i$. Then the P-positions that do not belong to $\mathcal{P}_i$ and all moves from which belong to $\mathcal{N}_i$ are the P-positions from $\mathcal{P}_{i+1} \setminus \mathcal{P}_i$. Note that $\mathcal{P}_i \subset \mathcal{P}_{i+1}$ and $\mathcal{N}_i \subset \mathcal{N}_{i+1}$.

Lemma 24. *P-positions found at step i are born in generation i.*

Proof. All optimal moves from $\mathcal{N}_i$ lead to $\mathcal{P}_i$. All moves from $\mathcal{P}_i$ lead to $\mathcal{N}_{i-1}$. Thus, if there is an optimal game where the P-position P_1 is reached after P-position P_2, then P_1 was found at an earlier step. That means an optimal game starting with $P_1 \in \mathcal{P}_i$ can not take more than $2i$ steps.

Now suppose $P_1 \in \mathcal{P}_i \setminus \mathcal{P}_{i-1}$. That means there exists a move from P_1 to $\mathcal{N}_{i-1} \setminus \mathcal{N}_{i-2}$. Similarly, there exists a move from $\mathcal{N}_{i-1}$ to $\mathcal{P}_{i-1} \setminus \mathcal{P}_{i-2}$. That means there exists an optimal game from P_1 that takes $2i$ moves, so P_1 is born in generation i. □

5.4 Ulam-Warburton cellular automaton

Now we will describe an automaton that produces the same sequences as P-positions in the game of Nim with three piles.

Consider points on an infinite square grid on the plane. Start with the point $(0, 0)$ and forbid any growth in the south branch. That is, points with coordinates (x, y), where $y < 0$ and $y \leq -|x|$ are not allowed to be born. At each moment a child is born if it has exactly one alive neighbor horizontally or vertically. Note that this corresponds to three branches of the Ulam-Warburton automaton when any direction is allowed. Figure 1 shows 6 generations of the automaton. We can clearly see the fractal formed by this automaton. The dots represent cells, and the dots are connected if they form a parent-child pair. The starting cell is in the bottom center.

The description of points born in generation n is well-known [2, 9, 11, 12, 13]. Suppose $n = \sum_{j=1}^{i} 2^{r_j}$ for distinct integers $r_1 > r_2 > \cdots > r_i \geq 0$. Then the points that are born in generation n have coordinates $\Sigma_{j=1}^{i} 2^{r_j} \mathbf{v}_j$, where $\mathbf{v}_j \in \{(0,1), (0,-1), (1,0), (-1,0)\}$ and $\mathbf{v}_j \neq -\mathbf{v}_{j-1}$ for $j > 1$.

We can describe three branches of this automaton in the following manner: Start in any of three directions (N, E, W) and move 2^{r_1} steps, then either continue forward or turn 90 degrees and move 2^{r_2} steps, and so on.

Theorem 25. *The evolution graph of the game of Nim with three piles is the same as three branches of the evolution graph of the Ulam-Warburton automaton.*

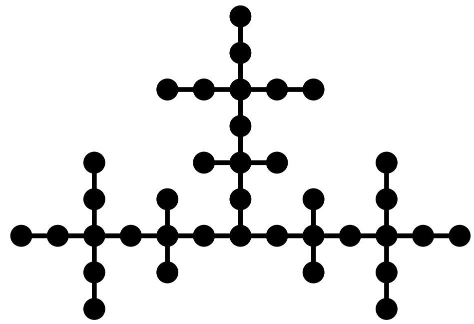

Figure 1: Ulam-Warburton automaton without the South branch after 6 generations

Proof. Consider a P-position in the game of Nim with three piles: (p_1, p_2, p_3). If this P-position was born on step n, it means $p_1 \oplus p_2 \oplus p_3 = 0$ and $p_1 + p_2 + p_3 = 2n$. Suppose that we decompose each p_i into distinct powers of 2. Then the powers of 2 that appear will be the r_j. Further, each r_j is present in exactly two out of three piles. Now let us describe the ancestors of this P-position. Start with the zero position, then chose pile i_1 and i_2 in which the power r_1 is present. Add 1 to both piles, and continue adding 2^{r_1} times. Then move to the next power and so on.

Analogously, with respect to the graph of the cellular automaton, chose a legal direction for each pair of piles. Pick a direction corresponding to the largest power of 2 in n and make 2^{r_1} steps forward in this direction. Take the next power of 2. If it corresponds to the same two piles continue forward, otherwise turn 90 degrees either left or right depending on the new pair and move 2^{r_2} steps.

Now we want to make an explicit bijection between cells in the automaton and P-positions. To start, we identify the P-position $(0, 1, 1)$ with the point $(-1, 0)$ and West direction, the P-position $(1, 0, 1)$ with the point $(0, 1)$ and East direction, and the P-position $(1, 1, 0)$ with the point $(1, 0)$ and North direction. Now we define turns:

- Left turn: changing direction from $(0, 1, 1)$ to $(1, 1, 0)$, from $(1, 1, 0)$ to $(1, 0, 1)$, and from $(1, 0, 1)$ to $(0, 1, 1)$
- Right turn: changing direction from $(0, 1, 1)$ to $(1, 0, 1)$, from $(1, 0, 1)$ to $(1, 1, 0)$, and from $(1, 1, 0)$ to $(0, 1, 1)$.

Each cell in the automaton (correspondingly, P-position) has exactly one parent. The cell (P-position) is uniquely described by the path from the starting point (terminal position). We showed the bijection between the paths which establishes the bijection between the cells and the P-positions. □

For example, consider the P-position $(14, 11, 5)$, which can be decomposed into powers of 2: $(8 + 4 + 2, 8 + 2 + 1, 4 + 1)$. This means that the evolution happens in the following way. Start with the P-position $(0, 0, 0)$, then 8 generations are born in the direction $(1, 1, 0)$ until the P-position $(8, 8, 0)$ is reached. After that 4 generations are born in the direction $(1, 0, 1)$ until the P-position $(12, 8, 4)$ is reached. After that 2 generations are born in the direction $(1, 1, 0)$ reaching $(14, 10, 4)$, then the child is born in the direction $(0, 1, 1)$ reaching the final destination $(14, 11, 5)$. This corresponds to the following walk on the automaton: 8 steps to the right until the coordinates $(8, 0)$, turn right, 4 more steps reaching $(8, -4)$, turn left and make 2 more steps reaching $(10, -4)$, then turn left again and make one step to get to $(10, -3)$.

It is more natural to place the Nim evolution in 3D, but such a graph is more difficult to draw and to visualize; see Figure 2.

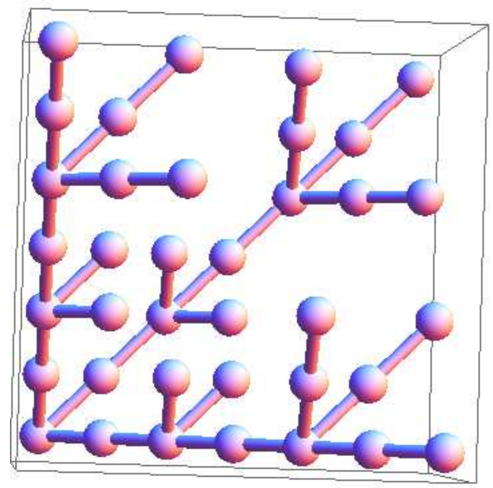

Figure 2: Nim evolution after 6 generations in 3D

5.5 Four piles

Let us move to four piles. Each P-position can be represented as four non-negative integers. The number of possible parents is 1, 2, or 6. We describe the possibilities by considering the binary representation of the four numbers:

- 1: if there is exactly one pair of binary numbers with the same number of zeros at the end,
- 2: if the binary numbers can be paired so that the number of zeros at the end is the same within each pair and different for different pairs,
- 6: all four binary numbers have the same number of zeros at the end.

Similarly, the number of possible children is 1, 2, or 6. Table 1 shows examples of P-positions with different numbers of parents and children.

	1 child	2 children	6 children
1 parent	(0,1,2,3)	(0,0,1,1)	(0,0,2,2)
2 parents	(0,1,4,5)	(1,1,2,2)	(2,2,4,4)
6 parents	(1,3,5,7)	(1,1,3,3)	(1,1,1,1)

Table 1: P-positions with different numbers of parents/children

Suppose the total number of counters is $2n$. We computed the sequence $D_4(n)$, which is sequence A237711: 1, 6, 7, 36, 13, 42, 43, 216, 49, 78, 55, 252, 85, ... Sequence $D_4(n)$ can be described recursively:

Lemma 26. *$D_4(0) = 1$, $D_4(1) = 6$, $D_4(2n+1) = 6D_4(n)$ for $n \geq 0$, and $D_4(2n+2) = D_4(n+1) + D_4(n)$ for $n \geq 0$.*

Proof. As we discussed before, the total number of counters in a P-position is even. Moreover, given a position, each 1 in the binary representation of a pile has a matching 1 in the binary representation of another pile. Thus, the ones not in unit positions do not affect the total number of counters modulo 4. That means if $\#(P) = 4n+2$, there are exactly two odd piles, and, if $\#(P) = 4n+4$, the total number of odd piles is either 0 or 4.

Suppose $\#(P) = 4n+2$, so there are two odd piles. Consider the following operation: subtract one counter from both odd piles and then divide all piles by two. This operation is a map from P-positions with total sum $4n+2$ to all P-positions with total sum $2n$. This map is a surjection, and each pre-image has 6 elements because by doubling a P-position with total sum $2n$ and adding two counters in any pair of 4 piles, we can get from any P-position with sum $2n$ to 6 P-positions with total sum $4n+2$. Therefore, $D_4(2n+1) = 6D_4(n)$.

Similarly, suppose $\#(P) = 4n+4$, so there are 0 or 4 odd piles. We can create a bijection between all P-positions with total sum $2n+2$ and P-positions with all even piles by doubling the piles in the former P-position. Doubling all piles and adding one counter to each of the piles is a bijection between all P-positions with total sum $2n$ and P-positions with all odd piles. Therefore, $D_4(2n+2) = D_4(n+1) + D_4(n)$. □

The corresponding sequence of partial sums $A_4(n)$ is sequence A237686: 1, 7, 14, 50, 63, 105, 148, 364, 413, 491, 546, 798, 883, 1141, ... This sequence can also be described recursively:

Lemma 27. *$A_4(0) = 1$, $A_4(1) = 7$, $A_4(2n+1) = 7A_4(n) + A_4(n-1)$ for $n \geq 1$, and $A_4(2n+2) = 7A_4(n) + A_4(n+1)$ for $n \geq 1$.*

Proof. We use induction to prove this statement. We can see that the base cases hold via direct computation. Now assume that the recurrence relation holds for $k \leq 2n$. By definition, $A_4(2n+1) = A_4(2n) + D_4(2n+1)$. Using the inductive hypothesis, $A_4(2n+1) = 7A_4(n-1) + A_4(n) + 6D_4(n) = A_4(n-1) + 6(A_4(n-1) + D_4(n)) + A_4(n) = 7A_4(n) + A_4(n-1)$.

Similarly, by definition: $A_4(2n+2) = A_4(2n+1) + D_4(2n+2)$. Using the previous result, $A_4(2n+2) = 7A_4(n) + A_4(n-1) + D_4(n) + D_4(n+1) = 7A_4(n) + A_4(n+1)$, which completes the induction. □

5.6 Many piles

We now calculate these sequences for any number of piles. First, we compute the initial terms of D_k.

Lemma 28. $D_k(0) = 1,\ D_k(1) = \binom{k}{2}$.

Proof. It is easy to see that $D_k(0) = 1$ because this is just the position $(0, \ldots, 0)$. In the case of $D_k(1)$, we can choose two piles to have one counter each in $\binom{k}{2}$ ways, and this is the only way for this position to have a nim-sum of zero. □

The following theorem provides a recursive formula for $D_k(n)$.

Theorem 29. *Assuming $D_k(j) = 0$ for negative j:*

$$D_k(2n+1) = \binom{k}{2} D_k(n) + \binom{k}{6} D_k(n-1) + \binom{k}{10} D_k(n-2) + \ldots,$$

$$D_k(2n+2) = \binom{k}{0} D_k(n+1) + \binom{k}{4} D_k(n) + \binom{k}{8} D_k(n-1) + \ldots,$$

Proof. We exploit the fact that each P-position with only even piles and total sum $2m$ can be realized by doubling each pile in a corresponding P-position with total sum m, which means that there is a bijection between all P-positions and all P-positions with only even piles.

If the total number of counters is $4n + 2$, then the number of odd piles could be $4i + 2$, where $4i + 2 \leq k$. If there are $4i + 2$ odd piles, then we can choose which piles are odd in $\binom{k}{4i+2}$ ways. Then we can remove one counter from every odd pile and divide each pile by 2. Using the bijection above, there will be $\binom{k}{4i+2} D_k(n-i)$ such P-positions for each choice of i. The case $4n + 4$ is similar. □

For example, if $k = 5$, then, $D_5(0) = 1$, $D_5(1) = 10$, $D_5(2n + 1) = 10D_5(n)$, and $D_5(2n + 2) = D_5(n + 1) + 5D_5(n)$. This is sequence A238759: 1, 10, 15, 100, 65, 150, 175, 1000, 565, ...

Similarly, we can prove a recursive formula for $A_k(n)$.

Theorem 30.

$$A_k(2n+1) = \left(\binom{k}{2} + \binom{k}{0}\right) A_k(n) + \left(\binom{k}{6} + \binom{k}{4}\right) A_k(n-1) + \left(\binom{k}{10} + \binom{k}{8}\right) A_k(n-2) + \cdots,$$

$$A_k(2n+2) = \binom{k}{0} A_k(n+1) + \left(\binom{k}{2} + \binom{k}{4}\right) A_k(n) + \left(\binom{k}{6} + \binom{k}{8}\right) A_k(n-1) + \cdots,$$

Proof. We can show this by using the partial sums of D_k. □

For example, if $k = 5$, then, $A_5(0) = 1$, $A_5(1) = 11$, $A_5(2n + 1) = 11A_5(n) + 5A_5(n - 1)$, and $A_5(2n + 2) = A_5(n + 1) + 15A_5(n)$. This is sequence A238147: 1, 11, 26, 126, 191, 341, 516, 1516, 2081, ...

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References

[1] M. H. Albert, R. J. Nowakowski, and D. Wolfe, *Lessons in Play*, A K Peters, 2007.

[2] D. Applegate, O. E. Pol, and N. J. A. Sloane, The toothpick sequence and other sequences from cellular automata, http://arxiv.org/pdf/1004.3036v2.pdf, 2010.

[3] E. R. Berlekamp, J. H. Conway, and R. K. Guy, *Winning Ways for your Mathematical Plays*, vol. 1, second edition, A K Peters, 2001.

[4] C. L. Bouton. Nim, a game with a complete mathematical theory. *Ann. of Math.*, **3** (1901–1902), 35–39.

[5] P. M. Grundy, Mathematics and games, *Eureka* **2** (1939), 6–8; reprinted **27** (1964), 9–11.

[6] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford University Press, 1979.

[7] *The On-Line Encyclopedia of Integer Sequences*, published electronically at http://oeis.org, 2014.

[8] A. Siegel. *Combinatorial Game Theory*, American Mathematical Society, 2013.

[9] D. Singmaster, On the cellular automaton of Ulam and Warburton, *M500 Magazine of The Open University*, **195** (2003), 2–7.

[10] R. P. Sprague, Über mathematische Kampfspiele [On mathematical war games], *Tôhoku Math. Journal* **41** (1935–1936), 438–444.

[11] R. P. Stanley (proposer) and R. J. Chapman (solver), A tree in the integer lattice, Problem 10360, *Amer. Math. Monthly*, **101** (1994), 76; **105** (1998), 769–771.

[12] S. M. Ulam, On some mathematical problems connected with patterns of growth of figures, *Mathematical Problems in Biological Sciences*, **14** (1962), 215–224.

[13] S. Wolfram, *A New Kind of Science*, Wolfram Media, 2002.

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