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The Inverse Problem on Subset Sums, II

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Abstract

For a set T of integers, let $P(T)$ be the set of all finite subset sums of T, and let $T(x)$ be the set of all integers of T not exceeding x. Let $B = \{b_1 < b_2 < \cdots\}$ be a sequence of integers and $d_1 = 10$, $d_2 = 3b_1 + 4$, and $d_n = 3b_{n-1} + 2$ $(n \geq 3)$. In this paper, we prove that

(i) if $b_n > d_n$ for all $n \geq 1$, then there exists a sequence of positive integers $A = \{a_1 < a_2 < \cdots\}$ such that, for all $k \geq 2$, $P(A(b_k)) = [0, 2b_k] \setminus \{b_u, 2b_k - b_u : 1 \leq u \leq k\}$;

(ii) if $b_m = d_m$ for some $m \geq 1$ and $b_n > d_n$ for all $n \neq m$, then there is no such sequence A.

We also pose a problem for further research.

1 Introduction

For a sequence of integers $A = \{a_1 < a_2 < \cdots\}$, let

$$P(A) = \left\{ \sum \varepsilon_i a_i : \ a_i \in A, \varepsilon_i = 0 \text{ or } 1; \ \sum \varepsilon_i < \infty \right\}.$$

Here $0 \in P(A)$. Burr [1] asked the following question: which sets S of integers are equal to $P(A)$ for some A? Let $B = \{b_1 < b_2 < \cdots\} = \mathbb{N} \setminus S$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the set of all natural numbers. Burr mentioned that if $b_1 > b_0$ and $b_{n+1} \geq b_n^2$ for all $n \geq 1$,

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then there exists an A such that $P(A) = \mathbb{N} \setminus B$. Hegyvári [4] proved that if $b_1 \geq b_0$ and $b_{n+1} \geq 5b_n$ for all $n \geq 1$, then such A exists. The condition $b_{n+1} \geq 5b_n$ has been improved to $b_{n+1} \geq 3b_n + 5$ by Chen and Fang [2]. Recently, Chen and the author [3] proved that, if $B = \{b_1 < b_2 < \cdots\}$ is a sequence of integers with $b_1 \in \{4, 7, 8\} \cup \{b : \ b \geq 11, \ b \in \mathbb{N}\}$, $b_2 \geq 3b_1 + 5$, $b_3 \geq 3b_2 + 3$ and $b_{n+1} > 3b_n - b_{n-2}$ for all $n \geq 3$, then there exists a sequence of positive integers $A = \{a_1 < a_2 < \cdots\}$ such that $P(A) = \mathbb{N} \setminus B$.

For any set T of integers and any real number x, let $T(x)$ be the set of all integers of T not exceeding x. Let $[a, b] = \{n : n \in \mathbb{N}, \ a \leq n \leq b\}$, and let $x + T = \{x + a : a \in T\}$.

In this paper, we prove the following result.

Theorem 1. *Let $B = \{b_1 < b_2 < \cdots\}$ be a sequence of integers, and let $d_1 = 10$, $d_2 = 3b_1+4$, and $d_n = 3b_{n-1} + 2$ $(n \geq 3)$. Then*

(i) if $b_n > d_n$ for all $n \geq 1$, then there exists a sequence of positive integers $A = \{a_1 < a_2 < \cdots\}$ such that, for all $k \geq 2$,

$$P(A(b_k)) = [0, 2b_k] \setminus \{b_u, 2b_k - b_u : 1 \leq u \leq k\};$$

(ii) if $b_m = d_m$ for some $m \geq 1$ and $b_n > d_n$ for all $n \neq m$, then, for any sequence of positive integers $A = \{a_1 < a_2 < \cdots\}$, there exists an index $k \geq 2$ such that

$$P(A(b_k)) \neq [0, 2b_k] \setminus \{b_u, 2b_k - b_u : 1 \leq u \leq k\}.$$

Remark 2. Theorem 1 gives a segment version of the original problem. The symmetry of the missing set is related to the structure of a subset sum. The analogous result for $P(A(b_k - b_{k-1}))$ was given in [3].

We pose a problem here.

Problem 3. Determine all sequences of integers $B = \{b_1 < b_2 < \cdots\}$ for which there exist two sequences of positive integers $A = \{a_1 < a_2 < \cdots\}$ and $X = \{x_1 < x_2 < \cdots\}$ such that, for all $k \geq 2$,

$$P(A(x_k)) = [0, 2b_k] \setminus \{b_u, 2b_k - b_u : 1 \leq u \leq k\}.$$

2 Proof of Theorem 1

First, we prove Theorem 1 (i). By the proof of [2, Theorem 1], there exists a subset A_2 of $[1, b_2 - b_1] \subset [1, b_2]$ such that $P(A_2) = [0, 2b_2] \setminus \{b_1, b_2, 2b_2 - b_1\}$. Suppose that $k \geq 2$, $A_k \subseteq [1, b_k]$ and

$$P(A_k) = [0, 2b_k] \setminus \{b_u, 2b_k - b_u : 1 \leq u \leq k\}. \tag{1}$$

We deal with the case $k + 1$. If $b_{k+1} \geq 3b_k + 5$, then, by the proof of [2, Theorem 1], we can construct the required A_{k+1}. So we consider the case $3b_k + 3 \leq b_{k+1} \leq 3b_k + 4$. Similar to the arguments in [2] and [3], we have

$$P(A_k \cup \{b_k + 1\}) = [0, 3b_k + 1] \setminus \{b_u, 3b_k - b_u + 1 : 1 \leq u \leq k\},$$

$$P(A_k \cup \{b_k + 1, b_{k+1} - 2b_k - 1\}) = [0, b_{k+1} + b_k] \setminus \{b_u, b_{k+1} + b_k - b_u : 1 \le u \le k\},$$

$$P(A_k \cup \{b_k + 1, b_{k+1} - 2b_k - 1, b_{k+1} - b_k\}) = [0, 2b_{k+1}] \setminus \{b_u, 2b_{k+1} - b_u : 1 \le u \le k + 1\}.$$

Let

$$A_{k+1} = A_k \cup \{b_k + 1, b_{k+1} - 2b_k - 1, b_{k+1} - b_k\}.$$

Thus, we have constructed a sequence of sets $\{A_k\}_{k=1}^{\infty}$ such that $A_1 \subseteq A_2 \subseteq \cdots$, $A_{k+1} \setminus A_k \subseteq (b_k, b_{k+1}]$ $(k \ge 1)$ and (1) holds for all $k \ge 2$. Let $A = \bigcup_{k=1}^{\infty} A_k$. Then, for all $k \ge 2$,

$$P(A(b_k)) = [0, 2b_k] \setminus \{b_u, 2b_k - b_u : 1 \le u \le k\}.$$

Now we prove Theorem 1 (ii). The proof is similar to that of [3, Theorem 2].

Suppose that there exists a sequence $A = \{a_1 < a_2 < \cdots\}$ of positive integers such that

$$P(A(b_s)) = [0, 2b_s] \setminus \{b_k, 2b_s - b_k : 1 \le k \le s\}$$

for all $s \ge 2$. Then $P(A) = \mathbb{N} \setminus B$. By [2], we may assume that $m \ge 3$. Thus $b_m = 3b_{m-1} + 2$. Let $A(b_{m-1}) = A \cap [0, b_{m-1}] = \{a_1, \ldots, a_{m'}\}$. Then

$$a_{m'+1} + P(A(b_{m-1})) = [a_{m'+1}, a_{m'+1} + 2b_{m-1}] \setminus B_{m,1},$$

where $B_{m,1} = \{a_{m'+1} + b_k, a_{m'+1} + 2b_{m-1} - b_k : 1 \le k \le m - 1\}$. If $a_{m'+1} > 2b_{m-1} - b_{m-2}$, then $2b_{m-1} - b_{m-2} \notin P(A)$, a contradiction. Hence $a_{m'+1} \le 2b_{m-1} - b_{m-2}$. By $a_{m'+1} \notin A \cap [0, b_{m-1}]$, we have $a_{m'+1} > b_{m-1}$.

Case 1: $a_{m'+1} = b_{m-1} + 1$. Similar to the arguments in [2] and [3], we have

$$P(A(b_{m-1}) \cup \{a_{m'+1}\}) = [0, b_m - 1] \setminus B_{m,2},$$

where $B_{m,2} = \{b_k, b_m - 1 - b_k : 1 \le k \le m - 1\}$. Thus

$$a_{m'+2} + P(A(b_{m-1}) \cup \{a_{m'+1}\}) = [a_{m'+2}, a_{m'+2} + b_m - 1] \setminus B_{m,3},$$

where $B_{m,3} = \{a_{m'+2} + b_k, a_{m'+2} + b_m - 1 - b_k : 1 \le k \le m - 1\}$. If $a_{m'+2} \le b_m - 1 - b_{m-1}$, then

$$b_m \in [a_{m'+2}, a_{m'+2} + b_m - 1], \quad a_{m'+2} + b_{m-1} < b_m < a_{m'+2} + b_m - 1 - b_{m-1}.$$

Thus $b_m \in a_{m'+2} + P(A(b_{m-1}) \cup \{a_{m'+1}\})$, a contradiction. If $a_{m'+2} > b_m - 1 - b_{m-1}$, then, by $b_m - 1 - b_{m-1} \notin P(A(b_{m-1}) \cup \{a_{m'+1}\})$, we have $b_m - 1 - b_{m-1} \notin P(A)$, a contradiction.

Case 2: $b_{m-1} + 2 \le a_{m'+1} \le 2b_{m-1} - b_{m-2}$. By $b_m \in [a_{m'+1}, a_{m'+1} + 2b_{m-1}]$ and $a_{m'+1} + b_{m-1} \le 3b_{m-1} - b_{m-2} < b_m$, there exist some $u_0 (1 \le u_0 \le m - 2)$ such that $a_{m'+1} + 2b_{m-1} - b_{u_0} = b_m = 3b_{m-1} + 2$. Hence $a_{m'+1} = b_{m-1} + b_{u_0} + 2$.

If there exist $u,\ v$ with $1 \le u,\ v \le m-2$ such that $2b_{m-1} - b_u = a_{m'+1} + b_v$, then, by $a_{m'+1} = b_{m-1} + b_{u_0} + 2$, we have $b_{m-1} = b_u + b_v + b_{u_0} + 2 \le 3b_{m-2} + 2$. This contradicts the condition $b_{m-1} > 3b_{m-2} + 2$. Hence $2b_{m-1} - b_u \ne a_{m'+1} + b_v (1 \le u,\ v \le m-2)$ and then

$$P(A(b_{m-1}) \cup \{a_{m'+1}\}) = [0, b_m + b_{u_0}] \setminus \{b_k, b_m + b_{u_0} - b_k : 1 \le k \le m-1\}.$$

Thus, for $i \ge 2$, we have

$$a_{m'+i} + P(A(b_{m-1}) \cup \{a_{m'+1}\}) = [a_{m'+i}, a_{m'+i} + b_m + b_{u_0}] \setminus B_{m,4},$$

where $B_{m,4} = \{a_{m'+i} + b_k, a_{m'+i} + b_m + b_{u_0} - b_k : 1 \le k \le m-1\}$.

If $a_{m'+2} > b_m + b_{u_0} - b_{m-1}$, then $b_m + b_{u_0} - b_{m-1} \notin P(A)$, a contradiction. So $b_{m-1} + b_{u_0} + 2 = a_{m'+1} < a_{m'+2} \le b_m + b_{u_0} - b_{m-1}$. Let $b_{m-1} + b_{u_0} + 2 < a_{m'+i} \le b_m + b_{u_0} - b_{m-1}$. Then

$$a_{m'+i} + b_{m-2} < b_m < a_{m'+i} + b_m + b_{u_0} - b_{m-1}. \tag{2}$$

Since $b_m \notin P(A)$, it follows that $b_m \notin a_{m'+i} + P(A(b_{m-1}) \cup \{a_{m'+1}\})$. Hence $b_m \in B_{m,4}$. Thus, by (2), $b_m = a_{m'+i} + b_{m-1}$, i.e., $a_{m'+i} = b_m - b_{m-1}$. So $i = 2$ and $a_{m'+3} > b_m + b_{u_0} - b_{m-1}$. Thus

$$\begin{aligned} & b_m + b_{u_0} - b_{m-1} = a_{m'+i} + b_{u_0} \\ \notin \ & (P(A(b_{m-1}) \cup \{a_{m'+1}\})) \cup (a_{m'+i} + P(A(b_{m-1}) \cup \{a_{m'+1}\})) \\ = \ & P(A(b_{m-1}) \cup \{a_{m'+1}, a_{m'+2}\}). \end{aligned}$$

By $a_{m'+3} > b_m + b_{u_0} - b_{m-1}$, we have $b_m + b_{u_0} - b_{m-1} \notin P(A)$, a contradiction. This completes the proof of Theorem 1 (ii).

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References

[1] S. A. Burr, in: P. Erdős, A. Rényi, V. T. Sós, eds., *Combinatorial Theory and its Applications III*, Coll. Math. Soc. J. Bolyai, Vol. 4, North-Holland, 1970, p. 1155.

[2] Y. G. Chen and J. H. Fang, On a problem in additive number theory, *Acta Math. Hungar.* **134** (2012), 416–430.

[3] Y. G. Chen and J. D. Wu, The inverse problem on subset sums, *European J. Combin.* **34** (2013), 841–845.

[4] N. Hegyvári, On representation problems in the additive number theory, *Acta Math. Hungar.* **72** (1996), 35–44.

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