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Counting Binary Words Avoiding Alternating Patterns

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Abstract

Let $F^{[\mathfrak{p}]}$ denote the set of binary words, with no more 0's than 1's, that do not contain the pattern $\mathfrak{p} = (10)^j1$ as a factor for any fixed $j \geq 1$. We give the generating function for the integer sequence enumerating, according to the number of 1's, the words in $F^{[\mathfrak{p}]}$.

1 Introduction

In this paper we consider the set $F \subset \{0,1\}^*$ of binary words ω such that $|\omega|_0 \leq |\omega|_1$, for any $\omega \in F$, where $|\omega|_0$ and $|\omega|_1$ denote the number of 0's and 1's in the word ω, respectively. We study the enumeration of the subset $F^{[\mathfrak{p}]} \subset F$ of binary words excluding a given pattern $\mathfrak{p} = p_0 \cdots p_{\ell-1} \in \{0,1\}^\ell$, i.e., a word ω is contained in $F^{[\mathfrak{p}]}$ if and only if there is no sequence of consecutive indices $i, i+1, \ldots, i+\ell-1$ such that $\omega_i\omega_{i+1}\cdots\omega_{i+\ell-1} = p_0p_1\cdots p_{\ell-1}$.

If we consider the set of binary words without any restriction, the defined language is regular, and it can be enumerated on the basis of the number of bits 1 and 0 by using classical results (see, e.g., [7, 8, 9]). With the additional restriction that the words have no more 0's than 1's, the language $F^{[\mathfrak{p}]}$ is no longer regular, and it becomes more difficult to deal

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with. For example, in order to generate the language $F^{[\mathfrak{p}]}$ for each forbidden pattern $\mathfrak{p}$ an "ad hoc" grammar (from which the generating function can be obtained) must be defined. Consequently, for each pattern $\mathfrak{p}$ a different generating function enumerating the words in $F^{[\mathfrak{p}]}$ must be computed. Our aim is to determine a constructive algorithm suggesting a more unified approach, which makes it possible to generate all binary words in the class $F^{[\mathfrak{p}]}$. Furthermore, this approach allows us to obtain a generating function for the enumeration of the class, according to the number of 1's, which applies to each forbidden pattern $\mathfrak{p}$.

In this paper we compute an explicit formula for the generating function which counts the words of $F^{[\mathfrak{p}]}$ according to the number of 1's where $\mathfrak{p} = (10)^j1$, for any fixed $j \geq 1$.

In order to obtain the enumeration of the class $F^{[\mathfrak{p}]}$ according to the number of 1's, we use a standard method, called the ECO-method, for the enumeration of combinatorial objects which admit recursive descriptions in terms of generating trees (see, e.g., [1]). So, we first construct all binary words belonging to F and avoiding the pattern $\mathfrak{p} = (10)^j1$, for any fixed $j \geq 1$. Then, we obtain the succession rule [2, 5, 6], describing the generating algorithm, from which we can compute the generating function enumerating the words in $F^{[\mathfrak{p}]}$.

We [4] introduced an algorithm for the construction of all binary words in F having a fixed number of 1's and excluding those containing the forbidden pattern $1^{j+1}0^j$, for any fixed $j \geq 1$. That algorithm first generates all the words in F, and then it eliminates those containing the forbidden pattern. Basically, the construction marks in an appropriate way the forbidden patterns in the words and generates 2^C copies of each word having C forbidden patterns such that the 2^{C-1} instances containing an odd number of marked forbidden pattern are annihilated by the other 2^{C-1} instances containing an even number of marked forbidden patterns. For example, the words $00110\overline{110}$ and $00\overline{110}110$, containing two copies of the forbidden pattern $\mathfrak{p} = 110$, (the marked forbidden patterns are over-lined) are eliminated by the words 00110110 and $00\overline{110110}$, respectively.

This is possible since no prefix of $\mathfrak{p} = 1^{j+1}0^j$ is also a suffix of $\mathfrak{p}$, that is the forbidden patterns do not overlap and so they are uniquely identified inside the words.

However, the algorithm in [4] cannot be used to generate the words in $F^{[\mathfrak{p}]}$ when $\mathfrak{p} = (10)^j1$ since now the forbidden patterns may overlap inside the words. For example, in $\omega = 110101010$ there are two overlapping copies of the forbidden pattern $\mathfrak{p} = (10)^21$. So, we propose a new algorithm that generates (all and) only the words in F avoiding the forbidden pattern $\mathfrak{p} = (10)^j1$, for any fixed $j \geq 1$.

The paper is organized as follows. In Section 2 we give some basic definitions and notation. In particular, we recall how every binary word can be represented as a path on the Cartesian plane.

In Section 3 we give a construction, according to the number of 1's, for the set of binary words in F excluding the pattern $\mathfrak{p} = (10)^j1$, for any fixed $j \geq 1$. The generating function enumerating such words according to the number of 1's is given in Section 4.

2 Basic definitions and notation

We begin with some definitions. Recall that $F \subset \{0,1\}^*$ is the set of binary words ω such that $|\omega|_0 \leq |\omega|_1$, for any $\omega \in F$, where $|\omega|_0$ and $|\omega|_1$ denote the number of 0's and 1's in the word ω, respectively.

Given $|\omega| = |\omega|_0 + |\omega|_1$ the *length* of $\omega \in F$, we let ω^h, $(h > 0)$ denote the word with length $h \cdot |\omega|$ obtained by linking ω to itself h times, that is, $\omega^h = \underbrace{\omega\,\omega \cdots \omega}_{h}$ and $\omega^0 = \varepsilon$, ε being the empty word.

Each word $\omega \in F$ can be naturally represented as a path on the Cartesian plane by associating a *rise* (or *up*) *step*, defined by (1,1) and indicated by x, with each bit 1 in ω and a *fall* (or *down*) *step*, defined by (1,-1) and indicated by $\bar{x}$, with each bit 0 in ω. For example, the word $\omega = 11011010010000101111$ is represented by the path $\gamma = xx\bar{x}xx\bar{x}x\bar{x}\bar{x}x\bar{x}\bar{x}\bar{x}\bar{x}x\bar{x}xxxx$ (see Figure 1). An *up-down* step is the sequence $x\bar{x}$.

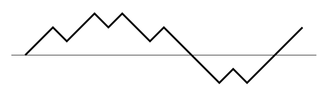

Figure 1: The path representing $\omega = 11011010010000101111$

Hereafter we refer interchangeably to words or their graphical representation on the Cartesian plane, i.e., paths. So we let F denote both the set of patterns $\mathfrak{p}$ avoiding binary words, and the set of corresponding paths.
In the rest of this paper, a path is

- *primitive*, if it begins and ends at ordinate 0 and remains strictly above the x-axis;
- *positive*, if it begins at ordinate 0 and remains above or on the x-axis;
- *negative*, if it begins and ends at ordinate 0 and remains below or on the x-axis (remark that a negative path in $F^{[\mathfrak{p}]}$ necessarily ends at ordinate 0);
- *strongly negative*, if it begins and ends at ordinate -1 and remains below or on the line $y = -1$;
- *underground*, if it ends with a negative suffix.

The *complement* of a path φ is the path φ^c obtained from φ by switching rise and fall steps.

3 A construction for the set $F^{[\mathfrak{p}]}$

In this section we show the constructive algorithm to generate the set $F^{[\mathfrak{p}]}$, $\mathfrak{p} = (x\bar{x})^j x = (10)^j 1$ for any fixed $j \geq 1$, according to the number of rise steps, or equivalently to the number of 1's.

The proof that the construction given in this section allows to generate all the words in $F^{[\mathfrak{p}]}$ with n 1's for any fixed forbidden pattern $\mathfrak{p} = (x\bar{x})^j x = (10)^j 1$, $j \geq 1$, is given in [3].

Given a path $\omega \in F^{[\mathfrak{p}]}$ with n rise steps, we generate a given number of paths in $F^{[\mathfrak{p}]}$ with $n + h$ rise steps, $1 \leq h \leq j$, by means of constructive rules. The number and the shape of the generated paths depend on the ordinate k of the endpoint of ω and on its suffix. With regard to k, we can point out three cases: $k = 0$, $k = 1$ and $k \geq 2$, while as for the suffix we consider whether it is equal to $(x\bar{x})^j$ or not. When $k = 0$, we must pay attention also to the case in which ω is an underground path ending with the pattern $(x\bar{x})^{j-1}x$.

As we will show further on, for each $\omega \in F^{[\mathfrak{p}]}$ such that $k = 0$ or $k \geq 2$, the generating algorithm produces two or more positive paths and one underground path with $n + h$ rise steps, $1 \leq h \leq j$, while, when $k = 1$, it produces only one positive path with $n+h$ rise steps.

The generating algorithm of the class $F^{[\mathfrak{p}]}$ with $\mathfrak{p} = (x\bar{x})^j x = (10)^j 1$, for any fixed $j \geq 1$, is described in the following sections. The constructive rules related to the special cases in which the suffix of ω is $(x\bar{x})^j$ or $(x\bar{x})^{j-1}x$ are described in Sections 3.2 and 3.3, respectively. In Section 3.1 we examine all the other simple cases.
The starting point of the algorithm is the empty word ε. $\omega_{|k}$ denotes a path with endpoint at ordinate k.

3.1 Simple cases

In this section we describe the constructive rules to be applied when the suffix of ω is neither $(x\bar{x})^j$ nor $(x\bar{x})^{j-1}x$. We point out three cases for the ordinate k of the endpoint of ω: $k = 0$, $k = 1$ and $k \geq 2$.

$k = 0$. A path $\omega \in F^{[\mathfrak{p}]}$, with n rise steps and such that its endpoint has ordinate 0, generates, for any h, $1 \leq h \leq j$, three paths with $n + h$ rise steps: a path ending at ordinate 1 by adding to ω a rise step and a sequence of $h - 1$ up-down steps; a path ending at ordinate 0 by adding to ω a rise step, a sequence of $h - 1$ up-down steps and a fall step, and an underground path obtained by the one generated in the previous step and mirroring on x-axis its rightmost primitive suffix. Figure 2 shows the above described operations; the number above the right arrow corresponds to the value of h. Both in this and in the following figures we consider $j = 4$, that is, $\mathfrak{p} = (x\bar{x})^4 x = (10)^4 1$.

Therefore,

$$\omega_{|0} \Rightarrow \begin{cases} \omega_{|0}x\,(x\bar{x})^{h-1}; \\ \omega_{|0}x(x\bar{x})^{h-1}\bar{x}; \\ \omega_{|0}\bar{x}(\bar{x}x)^{h-1}x. \end{cases} \tag{1}$$

$k = 1$. A path $\omega \in F^{[\mathfrak{p}]}$, with n rise steps and such that its endpoint has ordinate 1, generates, for any h, $1 \leq h \leq j$, a path with $n+h$ rise steps with endpoint at ordinate 2 obtained by adding to ω a rise step and a sequence of $h - 1$ up-down steps (see Figure 3).

Therefore,

$$\omega_{|1} \Rightarrow \omega_{|1}x(x\bar{x})^{h-1}. \tag{2}$$

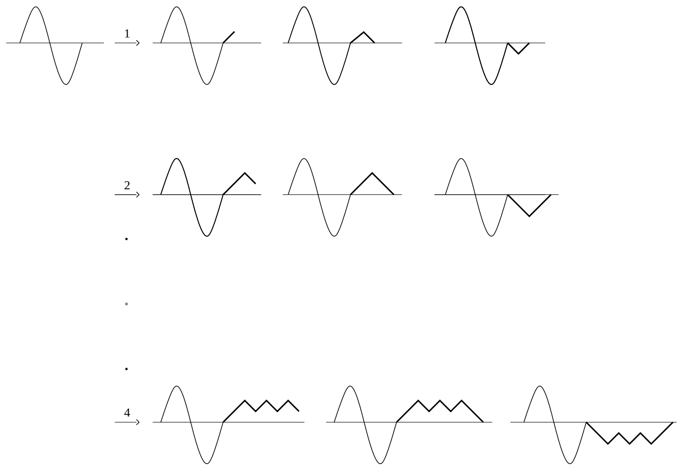

Figure 2: The paths generated by $\omega_{|0}$

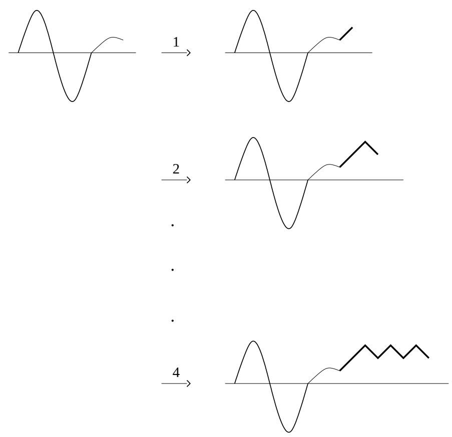

Figure 3: The paths generated by $\omega_{|1}$

$k \geq 2$**.** A path $\omega \in F^{[\mathfrak{p}]}$, with n rise steps and such that its endpoint has ordinate k, $k \geq 2$, generates, for any h, $1 \leq h \leq j$, $k+2$ paths with $n+h$ rise steps: a path ending at ordinate $(k+1)$ by adding to ω a rise step and a sequence of $h-1$ up-down steps; $k-1$ paths ending at ordinate $(k-1), (k-2), \ldots, (1)$, respectively, by adding to ω a rise step, a sequence of m, $2 \leq m \leq k$, fall steps and a sequence of $h-1$ up-down steps; a path ending at ordinate 0 by adding to ω a rise step, a sequence of k fall steps, a sequence of $h-1$ up-down steps and a fall step, and an underground path which will be described in Section 3.4. Figure 4 shows the above described operations.

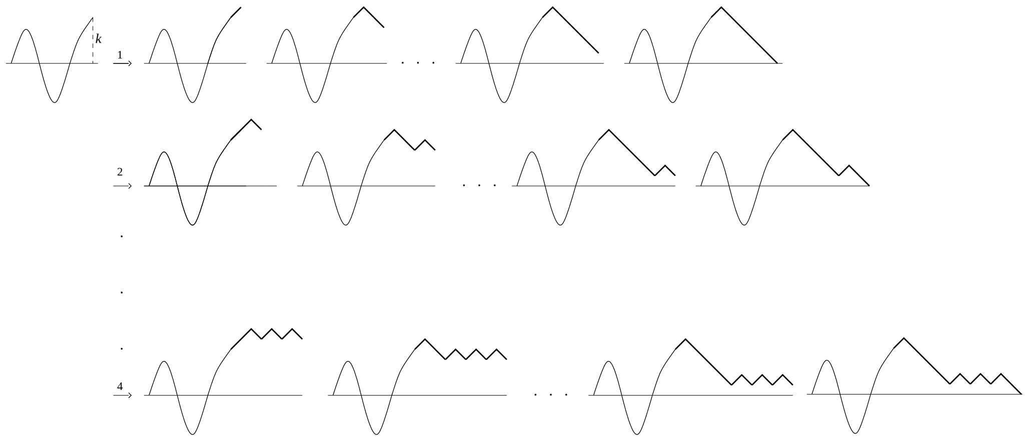

Figure 4: The paths generated by $\omega_{|k}, k \geq 2$

Therefore,

$$\omega_{|k} \Rightarrow \begin{cases} \omega_{|k} x (x\bar{x})^{h-1}; \\ \omega_{|k} x (\bar{x})^m (x\bar{x})^{h-1}, \quad 2 \leq m \leq k; \\ \omega_{|k} x (\bar{x})^k (x\bar{x})^{h-1} \bar{x}. \end{cases} \tag{3}$$

At this point it is clear that:

1. when the path ω ends with the suffix $(x\bar{x})^j$ the paths obtained by means of the constructions (1), (2) and (3) contain the forbidden pattern $\mathfrak{p} = (x\bar{x})^j x$. So, we will act as described in Section 3.2;

2. when ω is an underground path ending with the pattern $(x\bar{x})^{j-1}x$, some paths generated by means of the above constructions might contain the forbidden pattern $\mathfrak{p} = (x\bar{x})^j x$. So, we will follow the procedure described in Section 3.3.

3.2 Paths ending with $(x\bar{x})^j$

Even when the path ω ends with the suffix $(x\bar{x})^j$, the number and the shape of the generated paths depend on the ordinate k of the endpoint of ω. Let $\varrho = (x\bar{x})^j$ be the suffix of ω.

$k = 0$**.** A path $\omega \in F^{[\mathfrak{p}]}$, with n rise steps and such that its endpoint has ordinate 0, generates, for any h, $1 \leq h \leq j$, three paths with $n + h$ rise steps (see Figure 5): a path ending at ordinate 1, by inserting a sequence of $h-1$ up-down steps and a rise step on the left of ϱ; a path ending at ordinate 0, by inserting a sequence of $h-1$ up-down steps and a rise step on the left of ϱ and adding a fall step at the end of ω, and an underground path, obtained by mirroring on x-axis the rightmost primitive suffix of the path generated in the previous step.

Therefore,

$$\omega_{|0}\varrho \Rightarrow \begin{cases} \omega_{|0}(x\bar{x})^{h-1}x\varrho; \\ \omega_{|0}(x\bar{x})^{h-1}x\varrho\,\bar{x}; \\ \omega_{|0}(x\bar{x})^{h-1}\bar{x}\bar{x}(x\bar{x})^{j-1}xx. \end{cases} \tag{4}$$

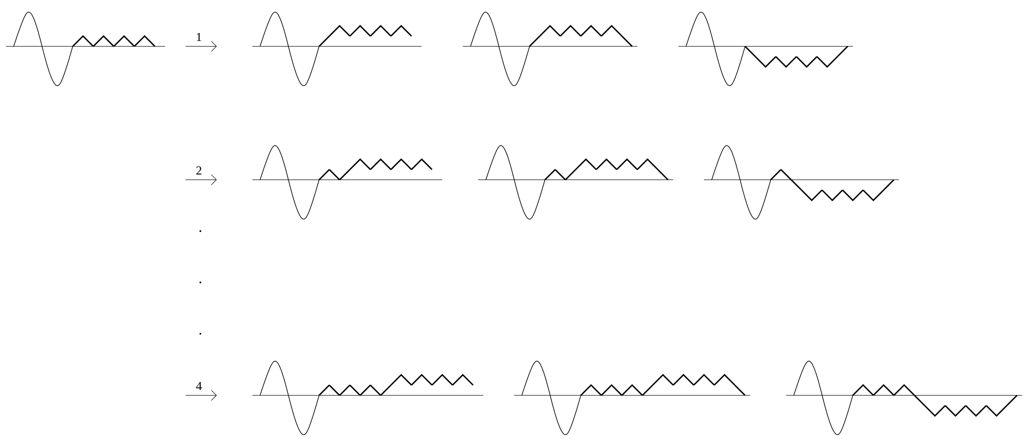

Figure 5: The paths generated by $\omega_{|0}(x\bar{x})^j$

$k = 1$. A path $\omega \in F^{[\mathfrak{p}]}$, with n rise steps and such that its endpoint has ordinate 1, generates, for any h, $1 \leq h \leq j$, a path with $n+h$ rise steps with endpoint at ordinate 2, obtained by inserting a sequence of $h-1$ up-down steps and a rise step on the left of the suffix ϱ (see Figure 6). Therefore,

$$\omega_{|1}\varrho \Rightarrow \omega_{|1}(x\bar{x})^{h-1}x\,\varrho. \tag{5}$$

$k \geq 2$. A path $\omega \in F^{[\mathfrak{p}]}$, with n rise steps and such that its endpoint has ordinate k, $k \geq 2$, generates, for any h, $1 \leq h \leq j$, $k+2$ paths with $n+h$ rise steps (see Figure 7): a path ending at ordinate $(k+1)$, by inserting a sequence of $h-1$ up-down steps and a rise step on the left of the suffix ϱ; $k-1$ paths ending at ordinate $(k-1), (k-2), \ldots, (1)$, respectively, by inserting a sequence of $h-1$ up-down steps, a rise step and a sequence of m, $2 \leq m \leq k$, fall steps on the left of ϱ; a path ending at ordinate 0, by first inserting a sequence of $h-1$ up-down steps, a rise step and a sequence of k fall steps on the left of ϱ, and then adding a fall step at the end of ω, and an underground path which will be described in Section 3.4.

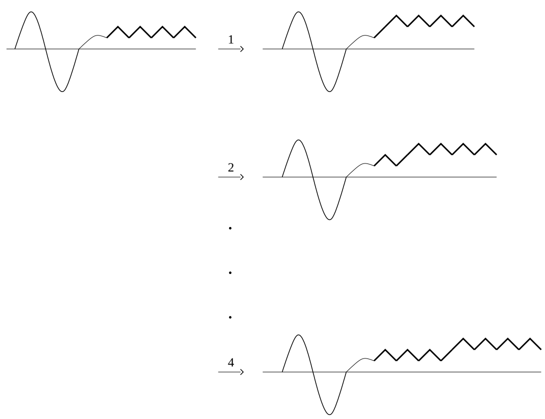

Figure 6: The paths generated by $\omega_{|1}(x\bar{x})^j$

Therefore,

$$\omega_{|k}\varrho \Rightarrow \begin{cases} \omega_{|k}(x\bar{x})^{h-1}x\varrho; \\ \omega_{|k}(x\bar{x})^{h-1}x(\bar{x})^m\,\varrho, \qquad 2 \leq m \leq k; \\ \omega_{|k}(x\bar{x})^{h-1}x\,(\bar{x})^k\,\varrho\,\bar{x}. \end{cases} \tag{6}$$

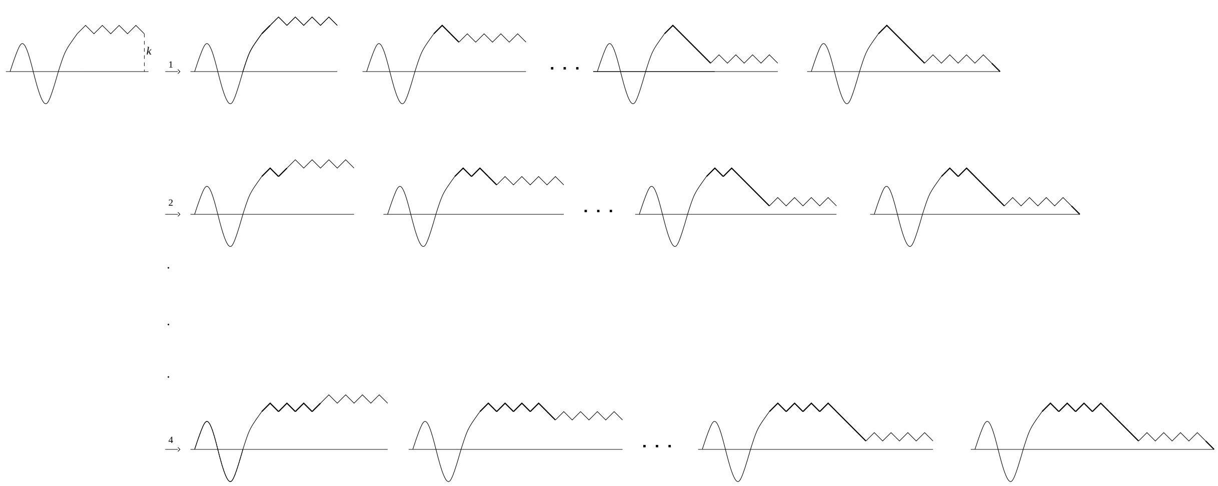

Figure 7: The paths generated by $\omega_{|k}(x\bar{x})^j, k \geq 2$

3.3 Paths ending with $(x\bar{x})^{j-1}x$

The paths $\omega \in F^{[\mathfrak{p}]}$ ending on the x-axis with the sequence $(x\bar{x})^{j-1}x$ have the following shape

$$\omega_{|0} = \mu\,\bar{x}\,\eta\,x\,(\bar{x}x)^{j-1}$$

where μ is a path ending on the x-axis and η is either the empty path ε or a strongly negative path.

The constructions applied to paths ending at ordinate 0 described in (1) (see Figure 2) can be used even for the paths ending with the sequence $(x\bar{x})^{j-1}x$, when $h \geq 2$, or to generate the paths ending at ordinate 1 or on the x-axis with a positive suffix, when $h = 1$. Nevertheless, when $h = 1$, by applying the construction, we obtain an underground path which contains the forbidden pattern $\mathfrak{p} = (x\bar{x})^j x$.

Therefore if the path ends with the sequence $(x\bar{x})^{j-1}x$ and $h = 1$, in order to generate the underground path we proceed as follows. Two cases must be taken into consideration.

1) μ does not end with a peak $x\bar{x}$.

The underground path is obtained by adding the path $\bar{x}x$ to $\omega_{|0}$, mirroring on x-axis the rightmost suffix $(\bar{x}x)^j$ of $\omega_{|0}\bar{x}x$ and shifting the sequence $(x\bar{x})^j$ between μ and the sub-path $\bar{x}\,\eta\,x$. Therefore, the path $\omega_{|0} = \mu\,\bar{x}\,\eta\,x\,(\bar{x}x)^{j-1}$ generates the underground path $\mu\,(x\bar{x})^j\,\bar{x}\,\eta\,x$ (see Figure 8). Note that this construction applies to ω even if $\mu = \varepsilon$.

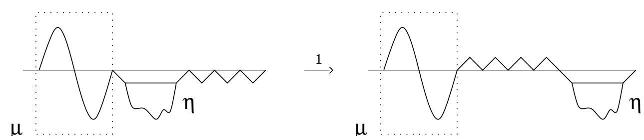

Figure 8: The underground path generated by $\omega_{|0}$ in the case 1)

2) μ ends with a peak $x\bar{x}$.

When the path μ ends with a peak $x\bar{x}$, that is, $\mu = \mu' x\bar{x}$, the insertion of the sequence $(x\bar{x})^j$ between μ and the sequence $\bar{x}\,\eta\,x$ produces the forbidden pattern $\mathfrak{p} = (x\bar{x})^j x$. Let us consider the following subcases: $\eta \neq \varepsilon$ and $\eta = \varepsilon$.

2.1) $\eta \neq \varepsilon$.

The underground path is obtained by shifting the rightmost peak $x\bar{x}$ of μ to the right of the sub-path $\bar{x}\,\eta\,x$, mirroring on x-axis the sequence $(\bar{x}x)^{j-1}$ and adding to such path the steps $\bar{x}\,x$. So, when $h = 1$, the underground path with negative suffix generated by $\omega_{|0} = \mu'\,x\,\bar{x}\,\bar{x}\,\eta\,x\,(\bar{x}x)^{j-1}$ is $\mu'\,\bar{x}\,\eta\,x\,(x\bar{x})^j\,\bar{x}\,x$ (see Figure 9).

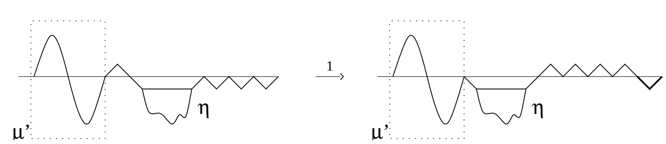

Figure 9: The underground path generated by $\omega_{|0}$ in the case 2.1)

2.2) $\eta = \varepsilon$.

In this case, the underground path obtained by means of the construction described in 2.1) is $\omega' = \mu' \bar{x}\, x\, (x\bar{x})^j\, \bar{x}\, x$ and it contains the forbidden pattern $\mathfrak{p} = (x\bar{x})^j x$ if μ' ends with the sequence $(\bar{x}x)^j$ or with the sequence $\bar{x}\, \eta'\, x\, (\bar{x}x)^{j-1}$, where η' is a non-empty strongly negative path. Let us take the longest suffix of $\omega_{|0} = \mu'\, x\, \bar{x}\, (\bar{x}x)^j$ into account so that $\omega_{|0} = \varphi\, \nu_1\, \nu_2 \ldots \nu_k$, where

$$\begin{cases} \nu_1 = \bar{x}\, \lambda\, x\, (\bar{x}x)^{j-1}\, x\, \bar{x}; & \\ \nu_i = (\bar{x}x)^j\, x\, \bar{x}, & 1 < i < k; \\ \nu_k = (\bar{x}x)^j, & \end{cases}$$

and λ is the empty path or is a strongly negative path. Every sequence ν_i, $1 \leq i \leq k$, will be changed into $\bar{\nu}_i$ in the following way.

2.2.1) If φ is a path that does not end with a peak $x\,\bar{x}$, then

$$\begin{cases} \bar{\nu}_1 = (x\bar{x})^j\, \bar{x}\, \lambda\, x; & \\ \bar{\nu}_i = (x\bar{x})^j\, \bar{x}\, x, & 1 < i < k; \\ \bar{\nu}_k = (x\bar{x})^j\, \bar{x}\, x. & \end{cases}$$

The underground path generated by $\omega_{|0}$ is $\varphi\, \bar{\nu}_1 \ldots \bar{\nu}_k$ (see Figure 10).

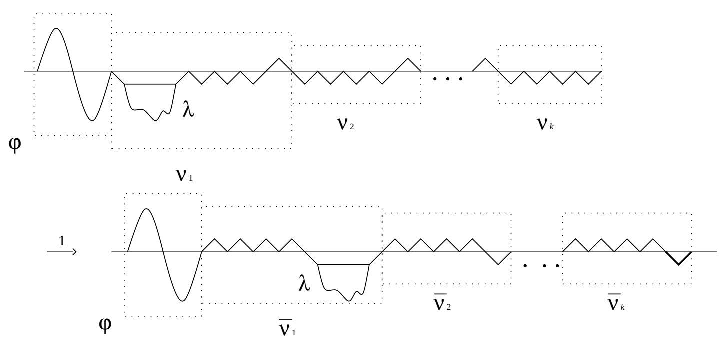

Figure 10: The underground paths generated by $\omega_{|0}$ in the case 2.2.1)

2.2.2) If φ ends with a peak $x\bar{x}$, that is, $\varphi = \varphi'\, x\bar{x}$, then

$$\begin{cases} \bar{\nu}_1 = \bar{x}\, \lambda\, x(x\bar{x})^j\, \bar{x}\, x; & \\ \bar{\nu}_i = (x\bar{x})^j\, \bar{x}\, x, & 1 < i < k; \\ \bar{\nu}_k = (x\bar{x})^j\, \bar{x}\, x. & \end{cases}$$

The underground path generated by $\omega_{|0}$ is $\varphi'\, \bar{\nu}_1 \ldots \bar{\nu}_k$ (see Figure 11).

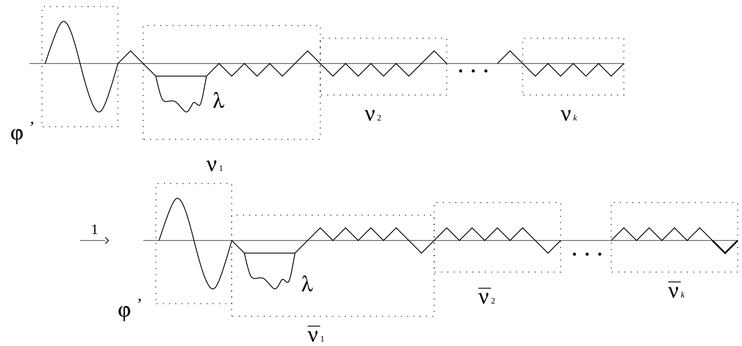

Figure 11: The underground paths generated by $\omega_{|0}$ in the case 2.2.2)

3.4 The underground path generated by $\omega_{|k}$

In this section, we describe how to obtain the underground path generated by $\omega_{|k}, k \geq 2$. For any h, $1 \leq h \leq j$, let $\omega' = \nu\varphi$ be the path obtained from $\omega_{|k}$ and ending on the x-axis with a positive suffix; φ is the rightmost suffix in ω' which is primitive.

If the path φ^c does not contain the forbidden pattern $\mathfrak{p}$, the underground path generated by $\omega_{|k}$ is $\nu\varphi^c$.

If the path φ^c contains the forbidden pattern $\mathfrak{p}$, we must apply a *swap* operation Φ in order to obtain a path $\varphi_1 = \Phi(\varphi^c)$ avoiding the forbidden pattern. The underground path generated by $\omega_{|k}$ is $\nu\varphi_1$.

Before describing the Φ operation on φ^c, let us consider the following proposition.

Proposition 1. *Let $\mu \in F^{[\mathfrak{p}]}$ be a primitive path; μ^c contains the forbidden pattern $\mathfrak{p} = (x\bar{x})^j x$ if and only if μ contains the pattern $\mathfrak{p}' = (\bar{x})^2(x\bar{x})^j\bar{x}$.*

From Proposition 1 it follows that, if φ^c contains the forbidden pattern $\mathfrak{p}$, then it is preceded and followed by at least a rise step.

Operation Φ must generate a path φ_1 avoiding the forbidden pattern $\mathfrak{p} = (x\bar{x})^j x$ and such that $\varphi_1^c \in F \backslash F^{[\mathfrak{p}]}$; in this way φ_1 is not the complement of any path in $F^{[\mathfrak{p}]}$. The path $\varphi_1 = \Phi(\varphi^c)$ is obtained as follows:

i) consider the straight line r from the beginning of the pattern $\mathfrak{p} = (x\bar{x})^j x$ and let t_1 be the rightmost point in which r intersects φ^c on the left of $\mathfrak{p}$ such that t_1 is preceded by at least two fall steps. Let $\delta_2 = (x\bar{x})^m$, $0 \leq m < j$, be the subsequence on the right of t_1 and followed by at least a fall step;

ii) *swap* the initial subsequence $\delta_1 = (x\bar{x})^j$ of $\mathfrak{p}$ and δ_2. It is straightforward to see that δ_2 can not be equal to $(x\bar{x})^j$ since φ does not contain the forbidden pattern $\mathfrak{p} = (x\bar{x})^j x$ (see Figure 12.a)). When $m = 0$, that is, δ_2 is the empty word, we simply insert δ_1 into t_1 (see Figure 12.b)).

Operation Φ is applied to each forbidden pattern in φ^c.

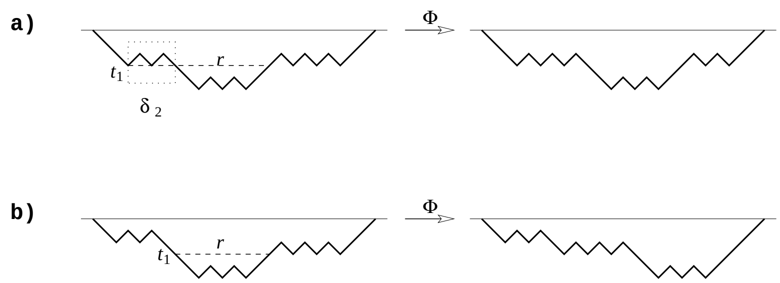

Figure 12: Some examples of the Φ operation, $\mathfrak{p} = (x\bar{x})^3 x$

Proposition 2. *Let $\varphi_1 = \Phi(\varphi^c)$, then $\varphi_1^c \in F \backslash F^{[\mathfrak{p}]}$.*

Proof. The Φ operation transforms the subsequence $\varrho_1 = (\bar{x})^m \, \delta_2 \, \bar{x}$, $(m \geq 2)$, of φ^c into the subsequence $\varrho_2 = (\bar{x})^m \, \delta_1 \, \bar{x} = (\bar{x})^m \, (x\bar{x})^j \, \bar{x}$ of φ_1. The complement of ϱ_2 is

$$\varrho_2^c = (x)^m \, (\bar{x}x)^j \, x \; = (x)^{m-1} \, (x\bar{x})^j \, x \, x.$$

So, φ_1^c contains the forbidden pattern $\mathfrak{p} = (x\bar{x})^j x$. □

Proposition 3. *Let $\mu \in F \backslash F^{[\mathfrak{p}]}$ be a primitive path such that $\mu^c \in F^{[\mathfrak{p}]}$. Then there exists a path $\eta \in F^{[\mathfrak{p}]}$ such that $\mu^c = \Phi(\eta^c)$.*

Proof. If $\mu \in F \backslash F^{[\mathfrak{p}]}$ and $\mu^c \in F^{[\mathfrak{p}]}$ then μ^c contains the pattern $\bar{x}\bar{x}(x\bar{x})^j\bar{x}$; we apply to μ^c the following operation Φ^{-1}.

i) Consider the straight line r from the end of the pattern $(x\bar{x})^j$ and let t_2 be the leftmost point where r intersects μ^c on the right of $(x\bar{x})^j$ such that t_2 is followed by at least two rise steps. Let $\delta_2 = (x\bar{x})^m$, $0 \leq m < j$, be the subsequence on the left of t_2 and preceded by at least a rise step.

ii) Swap the subsequence $(x\bar{x})^j$ and δ_2. When $m = 0$, that is, δ_2 is the empty word, we simply insert $(x\bar{x})^j$ into t_2.

□

Figure 13 shows the initial steps of the generating algorithm of the paths corresponding to words in $F^{[\mathfrak{p}]}$, with $\mathfrak{p} = (x\bar{x})^2 x = (10)^2 1$.

Recall that, following the above constructions, given a path ω, the number of generated paths depends only on the ordinate of endpoint of ω.

Therefore, the complete generating algorithm can be briefly described by the succession rule (7). For more details on succession rules, the reader is invited to see [2, 5, 6].

$$\begin{cases} (0) & \\ (0) \overset{h}{\rightsquigarrow} (0)(0)(1) & 1 \leq h \leq j \\ (1) \overset{h}{\rightsquigarrow} (2) & 1 \leq h \leq j \\ (k) \overset{h}{\rightsquigarrow} (0)(0)(1) \cdots (k-1)(k+1) & 1 \leq h \leq j, \, k \geq 2 \end{cases} \tag{7}$$

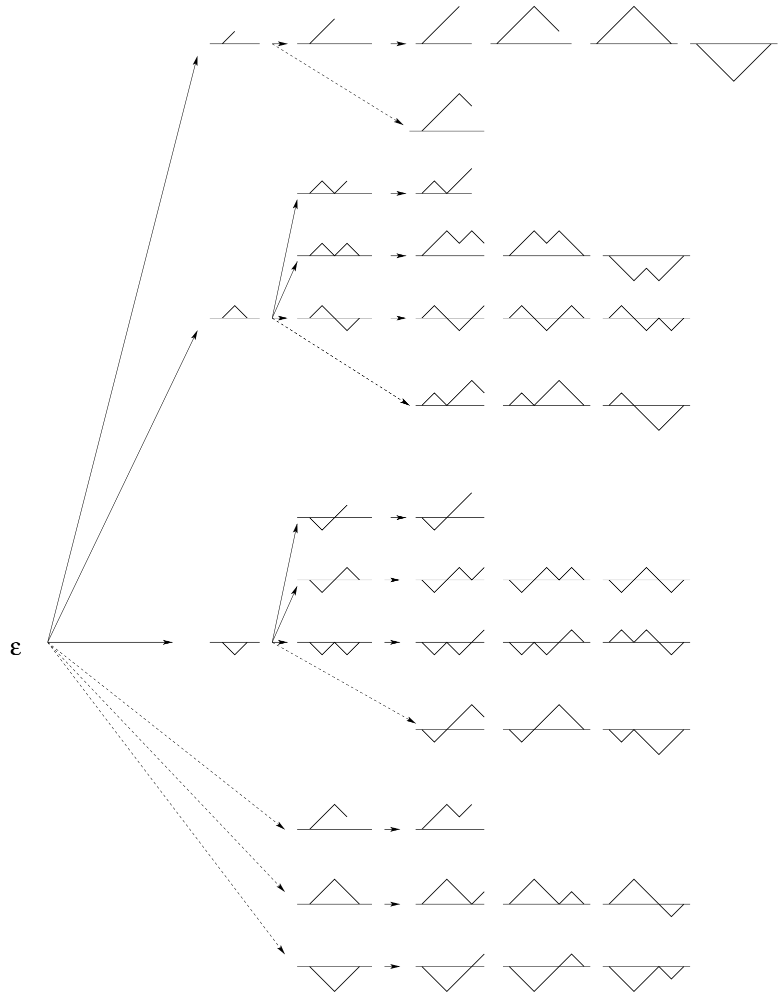

Figure 13: The initial steps of the generating algorithm of the paths corresponding to words in $F^{[\mathfrak{p}]}$, $\mathfrak{p} = (x\bar{x})^2x = (10)^2 1$. Dotted lines are related to $h = 2$

In the succession rule (7) each number corresponds to the ordinate of the endpoint of a path. The zero in the first line in (7) is associated with the empty path. Given the pattern $\mathfrak{p} = (x\bar{x})^j x = (10)^j 1$, the second line in (7) says that a path with n rise steps and ending at ordinate 0 generates two paths with $n + h$ rise steps, $1 \leq h \leq j$, and ending at ordinate 0, and one path with $n+h$ rise steps and ending at ordinate 1, i.e., the second line is associated with operations (1) and (4). Similarly, the third line is associated with operations (2) and (5). The last line describes the construction when the endpoint of the path has ordinate $k \geq 2$, underground path included.

4 Enumeration of $F^{[\mathfrak{p}]}$

With reference to the succession rule (7), let A_0 be the set of paths whose endpoints have ordinate 0, let A_1 be the set of paths whose endpoints have ordinate 1 and let A_k be the set of paths whose endpoints have ordinate k, $k \geq 2$. Then $F^{[\mathfrak{p}]} = A_0 \cup A_1 \cup A_k$.

The paths in A_0 with n rise steps are obtained from the paths in A_0 with $n-h$, $1 \leq h \leq j$, rise steps by means of the first production of (7) and from those in A_k with $n-h$ rise steps by means of the last production of (7).

The paths in A_1 with n rise steps are obtained from the paths in A_0 with $n-h$, $1 \leq h \leq j$, rise steps by means of the first production of (7) and from those in A_k with $n-h$ rise steps by means of the last production of (7).

The paths in A_k with n rise steps are obtained from the paths in A_1 with $n-h$, $1 \leq h \leq j$, rise steps by means of the second production of (7) and from those in A_k with $n-h$ rise steps by means of the last production of (7).

Let $n(\omega)$ be the number of rise steps of a path $\omega \in F^{[\mathfrak{p}]}$ and let $f(\omega)$ be the ordinate of the last point of ω itself. From the succession rule (7) we obtain

$$\begin{aligned} A_0(x,1) &= 1 + 2\sum_{h=1}^{j}\sum_{\omega\in A_0} x^{n(\omega)+h}y^0 + 2\sum_{h=1}^{j}\sum_{\omega\in A_k} x^{n(\omega)+h}y^0, && (8)\\ A_1(x,y) &= \sum_{h=1}^{j}\sum_{\omega\in A_0} x^{n(\omega)+h}y + \sum_{h=1}^{j}\sum_{\omega\in A_k} x^{n(\omega)+h}y, && (9)\\ A_k(x,y) &= \sum_{h=1}^{j}\sum_{\omega\in A_1} x^{n(\omega)+h}y^2 + \sum_{h=1}^{j}\sum_{\omega\in A_k}\sum_{i=2}^{f(\omega)-1} x^{n(\omega)+h}y^i + \\ &\quad + \sum_{h=1}^{j}\sum_{\omega\in A_k} x^{n(\omega)+h}y^{f(\omega)+1}. && (10) \end{aligned}$$

Since

$$\sum_{h=1}^{j}\sum_{w\in A_0} x^{n(\omega)+h}y^0 = \sum_{h=1}^{j} x^h \sum_{w\in A_0} x^{n(\omega)}y^0 = \sum_{h=1}^{j} x^h A_0(x,1) = \frac{(x-x^{j+1})}{1-x}A_0(x,1),$$

going on in the same way with the other terms, (8), (9) and (10) can rewritten

$$\begin{aligned}
A_0(x,1) &= 1+\frac{2(x-x^{j+1})}{1-x}A_0(x,1)+\frac{2(x-x^{j+1})}{1-x}A_k(x,1), \qquad (11)\\
A_1(x,y) &= \frac{y(x-x^{j+1})}{1-x}A_0(x,1)+\frac{y(x-x^{j+1})}{1-x}A_k(x,1), \qquad (12)\\
A_k(x,y) &= \frac{y^2(x-x^{j+1})}{1-x}A_1(x,1)+\frac{(x-x^{j+1})}{(1-x)(y-1)}A_k(x,y)-\\
&\quad -\frac{y^2(x-x^{j+1})}{(1-x)(y-1)}A_k(x,1)+\frac{y(x-x^{j+1})}{1-x}A_k(x,y). \qquad (13)
\end{aligned}$$

From (13) we obtain

$$(y^2(x-x^{j+1})-y(1-x^{j+1})+1-x^{j+1})A_k(x,y)=$$
$$=y^2(x-x^{j+1})A_k(x,1)-y^2(y-1)(x-x^{j+1})A_1(x,1).$$

Let

$$y_0=\frac{1-x^{j+1}-\sqrt{(1-x^{j+1})^2-4(x-x^{j+1})(1-x^{j+1})}}{2(x-x^{j+1}}$$

be a solution of

$$y^2(x-x^{j+1})-y(1-x^{j+1})+1-x^{j+1}=0.$$

By using the kernel method [1] we have

$$A_k(x,1)=(y_0-1)A_1(x,1).$$

By solving (11) and (12) we obtain

$$A_0(x,1)=\frac{1-x+2(x-x^{j+1})(y_0-1)A_1(x,1)}{(1-x)-2(x-x^{j+1})},$$

$$A_1(x,1)=\frac{x-x^{j+1}}{(1-x)-(x-x^{j+1})(y_0+1)}.$$

Hence, we can state the following result.

Proposition 4. *The generating function $F_j(x)$, $j\geq 1$, for the words $\omega\in F^{[\mathfrak{p}]}$ according to the number of 1's is given by*

$$F_j(x)=F_j(x,1)=A_0(x,1)+A_1(x,1)+A_k(x,1)=$$
$$=\frac{2(1-x^{j+1})}{3x^{j+1}-4x+1+\sqrt{(1-x^{j+1})^2-4(x-x^{j+1})(1-x^{j+1})}}.$$

The first numbers of the sequences enumerating the binary words in $F^{[\mathfrak{p}]}$, with $\mathfrak{p}=(10)^j1$, according to the number n of 1's, $1\leq n\leq 10$, and the value of j, $1\leq j\leq 10$, are shown in Table 1.

	n									
j	1	2	3	4	5	6	7	8	9	10
1	3	7	18	48	131	363	1017	2873	8169	23349
2	3	10	32	109	377	1324	4697	16795	60425	218485
3	3	10	35	123	445	1631	6036	22511	84460	318438
4	3	10	35	126	459	1699	6350	23911	90572	344737
5	3	10	35	126	462	1713	6418	24225	91979	350910
6	3	10	35	126	462	1716	6432	24293	92293	352317
7	3	10	35	126	462	1716	6435	24307	92361	352631
8	3	10	35	126	462	1716	6435	24310	92375	352699
9	3	10	35	126	462	1716	6435	24310	92378	352713
10	3	10	35	126	462	1716	6435	24310	92378	352716

Table 1: The sequences enumerating the binary words in $F^{[\mathfrak{p}]}$, with $\mathfrak{p} = (10)^j 1$, according to the number n of 1's

5 Conclusions and further developments

In this paper we give the generating function which counts particular binary words, according to the number of 1's, excluding a fixed pattern $\mathfrak{p} = (10)^j 1$, $j \geq 1$.

Successive studies should take into consideration binary words avoiding different forbidden patterns both from an enumerative and a constructive point of view.

Moreover, it would be interesting to study words avoiding patterns which have a different shape, that is, not just patterns consisting of a sequence of rise and fall steps.

Another interesting field of study consists of the determination of a sort of invariant class of avoiding patterns that is the paths $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_l$ such that $|F^{[\mathfrak{p}_1]}| = |F^{[\mathfrak{p}_2]}| = \cdots = |F^{[\mathfrak{p}_l]}|$ with consequent bijective problems.

One could also consider a forbidden pattern on an arbitrary alphabet and investigate words avoiding that pattern, or study words avoiding more than one pattern and the related combinatorial objects, considering various parameters.

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