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General Eulerian Polynomials as Moments Using Exponential Riordan Arrays

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Abstract

Using the theory of exponential Riordan arrays and orthogonal polynomials, we demonstrate that the general Eulerian polynomials, as defined by Xiong, Tsao and Hall, are moment sequences for simple families of orthogonal polynomials, which we characterize in terms of their three-term recurrence. We obtain the generating functions of this polynomial sequence in terms of continued fractions, and we also calculate the Hankel transforms of the polynomial sequence. We indicate that the polynomial sequence can be characterized by the further notion of generalized Eulerian distribution first introduced by Morisita. We finish with examples of related Pascal-like triangles.

1 Introduction

The triangle of Eulerian numbers

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 4 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 11 & 11 & 1 & 0 & 0 & \ldots \\ 1 & 26 & 66 & 26 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right),$$

with its general elements

$$A_{n,k}=\sum_{i=0}^{k}(-1)^i\binom{n+1}{i}(k-i+1)^n=\sum_{i=0}^{n-k}(-1)^i\binom{n+1}{i}(n-k-i)^n,$$

along with its variants, has been studied extensively [1, 10, 14, 15, 17, 18]. It is closely associated with the family of Eulerian polynomials

$$E_n(t)=\sum_{k=0}A_{n,k}t^k.$$

The Eulerian polynomials have exponential generating function

$$\sum_{n=0}^{\infty}E_n(t)\frac{x^n}{n!}=\frac{t-1}{t-e^{x(t-1)}}.$$

It can be shown that the sequence $E_n(t)$ is the moment sequence of a family of orthogonal polynomials [3]. Recently, Xiong, Tsao and Hall have provided an "arithmetical" generalization of the Eulerian numbers and Eulerian polynomials [33].

Definition 1. *For a given arithmetic progression* $\{a, a+d, a+2d, a+3d, \ldots\}$*, the general (arithmetical) Eulerian numbers* $A_{n,k}(a,d)$ *are defined by*

$$A_{n,k}(a,d)=\sum_{i=0}^{k}(-1)^i\binom{n+1}{i}((k+1-i)d-a)^n.$$

Definition 2. *The general (arithmetical) Eulerian polynomials associated to the arithmetic progression* $\{a, a+d, a+2d, a+3d, \ldots\}$ *are defined by*

$$P_n(t;a,d)=\sum_{k=0}^{n}A_{n,k}(a,d)t^k.$$

It can be shown [33] that the generating function of the family of polynomials $P_n(t;a,d)$ is given by

$$\sum_{n\geq 0}P_n(t;a,d)\frac{x^n}{n!}=\frac{(t-1)e^{ax(t-1)}}{t-e^{dx(t-1)}}.$$

Note that since (using the language of Riordan arrays)

$$\frac{(t-1)e^{ax(t-1)}}{t-e^{dx(t-1)}}=\left[e^{ax(t-1)},x\right]\cdot\frac{(t-1)}{t-e^{dx(t-1)}}$$

we have

$$P_n(t;a,d)=\sum_{k=0}^{n}\binom{n}{k}(a(t-1))^{n-k}d^kE_n(t).$$

In this paper, we shall assume that the reader is familiar with the basic elements of the theory of exponential Riordan arrays [2, 11], orthogonal polynomials [9, 16, 31], the links between exponential Riordan arrays and orthogonal polynomials [5, 6], and such techniques as that of production matrices [12, 13, 27]. We shall calculate the Hankel transform [21, 22, 23, 28] of many of the sequences that we encounter. This often involves characterising certain generating functions as continued fractions [32]. Specific examples of the use of these techniques can be found in [3]. Where sequences encountered are documented in the On-Line Encyclopedia of Integer Sequences [29, 30] we shall refer to them by their sequence number *Annnnnn*. For instance, the binomial matrix (Pascal's triangle) with general element $\binom{n}{k}$ is A007318.

2 Main results

The main result of this note is a characterization of the general (arithmetical) Eulerian polynomials as a family of moments. We have

Theorem 3. *The family of general arithmetical Eulerian polynomials $P_n(t; a, d)$ are the moments of the family of orthogonal polynomials $Q_n(x)$ where*

$$Q_n(x) = (x - (a(t-1) + d(n + (n-1)t)))Q_{n-1}(x) - (n-1)^2 d^2 t Q_{n-2}(x),$$

where $Q_0(x) = 1$ and $Q_1(x) = x - (a(t-1) + d)$.

This is a consequence of the following proposition.

Proposition 4. *The Riordan array*

$$\left[\frac{(t-1)e^{a(t-1)x}}{t - e^{d(t-1)x}}, \frac{e^{d(t-1)x} - 1}{d(t - e^{d(t-1)x})}\right]$$

has a tri-diagonal production matrix.

Proof. We recall that the bivariate generating function of the production matrix of the exponential Riordan array $[g, f]$ is given by [12, 13]

$$e^{xy}(Z(x) + A(x)y)$$

where

$$A(x) = f'(\bar{f}(x)),$$

and

$$Z(x) = \frac{g'(\bar{f}(x))}{g(\bar{f}(x))}.$$

In our case,

$$f(x) = \frac{e^{d(t-1)x} - 1}{d(t - e^{d(t-1)x})}$$

which implies that

$$\bar{f}(x) = \frac{1}{d(t-1)} \ln\left(\frac{1+dtx}{1+dx}\right).$$

We deduce that

$$A(x) = f'(\bar{f}(x)) = (1+dx)(1+dtx).$$

We have

$$g(x) = \frac{(t-1)e^{a(t-1)x}}{t-e^{d(t-1)x}},$$

which implies that

$$Z(x) = \frac{g'(\bar{f}(x))}{g(\bar{f}(x))} = d^2tx + a(t-1) + d.$$

The production matrix is then generated by

$$e^{xy}(d^2tx + a(t-1) + d + (1+dx)(1+dtx)y).$$

Thus the production matrix is indeed tri-diagonal, beginning

$$\begin{pmatrix} a(t-1)+d & 1 & 0 & 0 & 0 & 0 & \cdots \\ d^2t & a(t-1)+d(t+2) & 1 & 0 & 0 & 0 & \cdots \\ 0 & 4d^2t & a(t-1)+d(2t+3) & 1 & 0 & 0 & \cdots \\ 0 & 0 & 9d^2t & a(t-1)+d(3t+4) & 1 & 0 & \cdots \\ 0 & 0 & 0 & 16d^2t & a(t-1)+d(4t+5) & 1 & \cdots \\ 0 & 0 & 0 & 0 & 25d^2t & a(t-1)+d(5t+6) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

□

The recurrence coefficients for the three-term recurrence that defines the orthogonal polynomials $Q_n(x)$ can now be read from the above.

Corollary 5. *The Hankel transform of the sequence of polynomials $P_n(t; a, d)$ is given by*

$$h_n = (dt^2)^{\binom{n+1}{2}} \prod_{i=0}^{n} (i!)^2.$$

Corollary 6. *The family of orthogonal polynomials $Q_n(t)$ has coefficient array given by the exponential Riordan array*

$$\left[\frac{1}{1+dx}\left(\frac{1+dx}{1+dtx}\right)^{\frac{a}{d}}, \frac{1}{d(t-1)} \ln\left(\frac{1+dtx}{1+dx}\right)\right].$$

Proof. We have

$$\left[\frac{(t-1)e^{a(t-1)x}}{t-e^{d(t-1)x}}, \frac{e^{d(t-1)x}-1}{d(t-e^{d(t-1)x})}\right]^{-1} = \left[\frac{1}{1+dx}\left(\frac{1+dx}{1+dtx}\right)^{\frac{a}{d}}, \frac{1}{d(t-1)} \ln\left(\frac{1+dtx}{1+dx}\right)\right].$$

□

Corollary 7. *The generating function $g(x)$ of the sequence of polynomials $P_n(t;a,d)$ can be expressed as the following continued fraction.*

$$g(x) = \cfrac{1}{1-(a(t-1)+d)x-\cfrac{d^2tx^2}{1-(a(t-1)+d(t+2))x-\cfrac{4d^2tx^2}{1-(a(t-1)+d(2t+3))x-\cfrac{9d^2tx^2}{1-\dots}}}}.$$

We note that it is sometimes more convenient to use the polynomials

$$\tilde{P}_n(t;a,d) = P_n(t+1;a,d).$$

We then have

$$\sum_{n\geq 0} \tilde{P}_n(t;a,d)\frac{x^n}{n!} = \frac{te^{axt}}{t+1-e^{dxt}}.$$

Evidently we have

$$\tilde{P}_n(t;a,d) = \sum_{k=0}^{n} \binom{n}{k}(at)^{n-k}d^k E_n(t+1).$$

Theorem 8. *The family of polynomials $\tilde{P}_n(t;a,d)$ are the moments of the family of orthogonal polynomials $\tilde{Q}_n(x)$ where*

$$\tilde{Q}_n(x) = (x-(at+d(n+(n-1)(t+1))))\tilde{Q}_{n-1}(x)-(n-1)^2d^2(t+1)\tilde{Q}_{n-2}(x).$$

This is a consequence of the following proposition.

Proposition 9. *The Riordan array*

$$\left[\frac{te^{atx}}{t+1-e^{dtx}}, \frac{e^{dtx}-1}{d(t+1-e^{dtx})}\right]$$

has a tri-diagonal production matrix.

In fact, the production matrix in this case takes the form

$$\begin{pmatrix} at+d & 1 & 0 & 0 & 0 & 0 & \dots \\ d^2(t+1) & at+d(t+3) & 1 & 0 & 0 & 0 & \dots \\ 0 & 4d^2(t+1) & at+d(2t+5) & 1 & 0 & 0 & \dots \\ 0 & 0 & 9d^2(t+1) & at+d(3t+7) & 1 & 0 & \dots \\ 0 & 0 & 0 & 16d^2(t+1) & at+d(4t+9) & 1 & \dots \\ 0 & 0 & 0 & 0 & 25d^2(t+1) & at+d(5t+11) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

3 Examples

In this section we look at four examples. We firstly indicate that the generalized Eulerian polynomials defined by the sequence of odd numbers are associated with a Pascal-like matrix. We secondly propose a conjecture concerning the values $P_n(1;1,r)$ and the values of the permanents of a certain family of matrices. Finally we look at the sequences defined by $P_n(2;1,2)$ and $P_n(2;2,1)$, indicating a combinatorial interpretation for each. In large measure these examples are inspired by entries in the On-Line Encyclopedia of Integer Sequences [29, 30].

Example 10. The generalized Eulerian polynomials that correspond to the odd numbers have $a = 1$ and $d = 2$. Now the sequence of polynomials $P_n(t;1,2)$ begins

$$1, t+1, t^2+6t+1, t^3+23t^2+23t+1, t^4+76t^3+230t^2+76t+1, \ldots,$$

and has coefficient array A060187

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 6 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 23 & 23 & 1 & 0 & 0 & \cdots \\ 1 & 76 & 230 & 76 & 1 & 0 & \cdots \\ 1 & 237 & 1682 & 1682 & 237 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This is the triangle of "midpoint Eulerian numbers" [25]. The row sums are equal to $2^n n! = (2n)!!$, or A000165 (this is $P_n(1;2,2)$).

Example 11. Special care must be exercised when $t = 1$, as in this case

$$\left[\frac{(t-1)e^{a(t-1)x}}{t-e^{d(t-1)x}}, \frac{e^{d(t-1)x}-1}{d(t-e^{d(t-1)x})}\right]$$

is apparently undefined. Taking the limit as $t \to 1$, we find that

$$\left[\frac{1}{1-dx}, \frac{x}{1-dx}\right]$$

is the correct expression. This is a generalized Laguerre array [4]. Starting from the observation that the inverse binomial transform of $P_n(1;1,2)$, which begins

$$1, 1, 5, 29, 233, 2329, 27949, 391285 \ldots,$$

can be interpreted as the sequence of $n \times n$ permanents of the matrix with 1's on the diagonal and 2 elsewhere (cf. A000354), we can conjecture that the $(r-1)$-st inverse binomial transform

$$\mathrm{per}(n,r) := \sum_{k=0}^{n} \binom{n}{k} (-(r-1))^{n-k} P_k(1;1,r)$$

of $P_n(1;1,r)$ is the sequence of $n \times n$ permanents of the principal minors of the matrix with 1's on the diagonal and r elsewhere. The generating function for this is

$$\frac{e^{-(r-1)x}}{1-rx},$$

and the corresponding moment matrix is the exponential Riordan array

$$\left[\frac{e^{-(r-1)x}}{1-rx}, \frac{x}{1-rx}\right].$$

This means that the inverse matrix is the coefficient array of a family of orthogonal polynomials, as is evidenced by the form of the production matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ r^2 & 2r+1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 4r^2 & 4r+1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 9r^2 & 6r+1 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 16r^2 & 8r+1 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 25r^2 & 10r+1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The numbers $\text{per}(n;r)$ then have generating function

$$\cfrac{1}{1-x-\cfrac{r^2x^2}{1-(2r+1)x-\cfrac{4r^2x^2}{1-(4r+1)x-\cfrac{9r^2x^2}{1-\cdots}}}}.$$

From this or otherwise we can deduce that

$$\text{per}(n;r) = \sum_{k=0}^{n} T_{n,n-k} r^k$$

where $T_{n,k}$ is the (n,k)-th element of the exponential array A008290

$$\left[\frac{e^{-x}}{1-x}, x\right]$$

of rencontres numbers. We deduce that

$$\text{per}(n;r) = \sum_{k=0}^{n} \frac{n!}{(n-k)!} \sum_{i=0}^{k} \frac{(-1)^i}{i!} r^k.$$

The Hankel transform of $\text{per}(n;r)$ is given by

$$h_n = r^{n(n+1)} \prod_{k=0}^{n} (k!)^2.$$

Example 12. The sequence $P_n(2;1,2)$ begins

$$1, 3, 17, 147, 1697, 24483, 423857, 8560947, \ldots$$

and coincides with A080253, or the number of elements in the Coxeter complex of type B_n (or C_n). Its generating function is

$$\frac{e^x}{2-e^{2x}}.$$

Example 13. The sequence $P_n(2;2,1)$ begins

$$1, 3, 11, 51, 299, 2163, 18731, 189171, 2183339, \ldots$$

and coincides with A007047, or the number of chains in the power set of an n-set. Its generating function is

$$\frac{e^{2x}}{2-e^x}.$$

4 Ant lions and generalized Eulerian polynomials

Morisita proposed a statistical distribution model to explain the habitat choice model of ant lions, based on the idea of environmental density [26]. Morisita showed that this distribution is governed by an Eulerian-type recurrence. This work was further refined mathematically by others [7, 8, 19, 20]. Combining this model and the generalized Eulerian polynomials discussed above, we obtain the following result.

Theorem 14. *The family of generalized Eulerian polynomials $P_n(t;\alpha,\beta,d)$ with generating function*

$$\frac{(t-1)^{\alpha+\beta} e^{\alpha x(t-1)}}{(t-e^{dx(t-1)})^{\alpha+\beta}}$$

are the moments of the family of orthogonal polynomials $Q_n(x)$ where

$$Q_n(x) = (x-(a(d+t-1)+\beta d+(n-1)d(t+1)))Q_{n-1}(x)-(n-1)d^2t(\alpha+\beta+n-2)Q_{n-2}(x),$$

with $Q_0(x)=1$ and $Q_1(x)=x-\alpha(d+t-1)-\beta d$.

This is a consequence of the following proposition.

Proposition 15. *The Riordan array*

$$\left[\frac{(t-1)^{\alpha+\beta}e^{\alpha x(t-1)}}{(t-e^{dx(t-1)})^{\alpha+\beta}}, \frac{e^{d(t-1)x}-1}{d(t-e^{d(t-1)x})}\right]$$

has a tri-diagonal production matrix.

Proof. We let

$$f(x)=\frac{e^{d(t-1)x}-1}{d(t-e^{d(t-1)x})}.$$

As before, we obtain

$$A(x)=f'(\bar{f}(x))=(1+dx)(1+dtx).$$

Now

$$g(x)=\frac{(t-1)^{\alpha+\beta}e^{\alpha x(t-1)}}{(t-e^{dx(t-1)})^{\alpha+\beta}},$$

which implies that

$$Z(x)=\frac{g'(\bar{f}(x))}{g(\bar{f}(x))}=d^2tx(\alpha+\beta)+\alpha(d+t-1)+\beta d.$$

Thus the production matrix sought is tri-diagonal, beginning

$$\begin{pmatrix} \alpha(d+t-1)+\beta d & 1 & 0 & 0 & 0 & 0 & \cdots \\ d^2t(\alpha+\beta) & \alpha(d+t-1)+\beta d+d(t+1) & 1 & 0 & 0 & 0 & \cdots \\ 0 & 2d^2t(\alpha+\beta+1) & \alpha(d+t-1)+\beta d+2d(t+1) & 1 & 0 & 0 & \cdots \\ 0 & 0 & 3d^2t(\alpha+\beta+2) & \alpha(d+t-1)+\beta d+3d(t+1) & 1 & 0 & \cdots \\ 0 & 0 & 0 & 4d^2t(\alpha+\beta+3) & \alpha(d+t-1)+\beta d+4d(t+1) & 1 & \cdots \\ 0 & 0 & 0 & 0 & . & . & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

□

We note that

$$\left[\frac{(t-1)^{\alpha+\beta}e^{\alpha x(t-1)}}{(t-e^{dx(t-1)})^{\alpha+\beta}}, \frac{e^{d(t-1)x}-1}{d(t-e^{d(t-1)x})}\right]^{-1}=\left[\frac{1}{(1+dx)^{(\alpha+\beta)}}\left(\frac{1+dx}{1+dtx}\right)^{\frac{a}{d}}, \frac{1}{d(t-1)}\ln\left(\frac{1+dtx}{1+dx}\right)\right]$$

gives the coefficient array of the orthogonal polynomials $Q_n(x)$ in this case.

Furthermore we have the following relation between the two types of generalized Eulerian polynomials discussed in this note.

$$P_n(t;a,d)=P_n(t;a,1-a,d).$$

Corollary 16. *The Hankel transform of $P_n(t;\alpha,\beta,d)$ is given by*

$$h_n(\alpha,\beta,d)=\prod_{k=1}^{n}(kd^2t(\alpha+\beta+k-1))^{n-k+1}=(td^2)^{\binom{n+1}{2}}\prod_{k=1}^{n}k!(\alpha+\beta+k-1)^{n-k+1}.$$

Example 17. The polynomials $P_n(t;2,1,1)$ begin

$$1, 2t+1, 4t^2+7t+1, 8t^3+33t^2+18t+1, 16t^4+131t^3+171t^2+41t+1, \ldots.$$

This corresponds to the generating function

$$\frac{(t-1)^3 e^{2x(t-1)}}{(t-e^{x(t-1)})^3}.$$

Note that the limit of this expression as t goes to 1 is $\frac{1}{(1-x)^3}$. This generates the values of the sequence for $t=1$, namely

$$1, 3, 12, 60, 360, 2520, 20160, 181440, \ldots,$$

or $n!\binom{n+2}{2}$ (essentially A001710).

The reversal of the coefficient array of these polynomials begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 4 & 7 & 1 & 0 & 0 & 0 & \cdots \\ 8 & 33 & 18 & 1 & 0 & 0 & \cdots \\ 16 & 131 & 171 & 41 & 1 & 0 & \cdots \\ 32 & 473 & 1208 & 718 & 88 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right),$$

and has bivariate generating function

$$\frac{(1-t)^3 e^{2x(1-t)}}{(1-te^{x(1-t)})^3}.$$

The inverse binomial transform of this matrix begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 5 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 15 & 15 & 1 & 0 & 0 & \cdots \\ 1 & 37 & 105 & 37 & 1 & 0 & \cdots \\ 1 & 82 & 523 & 523 & 82 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right),$$

with bivariate generating function

$$\frac{(1-t)^3 e^{-x} e^{2x(1-t)}}{(1-te^{x(1-t)})^3} = \frac{(1-t)^3 e^{x(1-2t)}}{(1-te^{x(1-t)})^3}.$$

Example 18. In similar fashion, consideration of $P_n(t;3,1,1)$ leads to the matrix

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 6 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 19 & 19 & 1 & 0 & 0 & \cdots \\ 1 & 48 & 150 & 48 & 1 & 0 & \cdots \\ 1 & 109 & 794 & 794 & 109 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right),$$

with bivariate generating function

$$\frac{(1-t)^4 e^{x(1-3t)}}{(1-te^{x(1-t)})^4}.$$

We can conjecture that the $(r-1)$-st inverse binomial transform of the reversal of the coefficient array of $P_n(t;r,1,1)$ is a Pascal-like matrix with bivariate generating function given by

$$\frac{(1-t)^{r+1} e^{x(1-rt)}}{(1-te^{x(1-t)})^{r+1}}.$$

The case $r=-1$ is the binomial (Pascal's) triangle A007318, $r=0$ corresponds to A046802, while $r=1$ is the triangle of Eulerian numbers $\{A_{n+1,k}\}$.

5 Pascal-like triangles and moments

We have encountered a number of Pascal-like triangles already in this note. We finish with the following remarks. The generating function

$$\frac{(t-1)e^{(2-r)x}e^{(t-1)x}}{t-e^{r(t-1)x}} = \frac{(t-1)e^{x(t-r+1)}}{t-e^{r(t-1)x}}$$

generates the sequence of polynomials

$$1, t+1, r^2t+(t+1)^2, (t+1)(r^3t+3r^2t+(t+1)^2), \ldots$$

whose coefficient array is the Pascal-like triangle

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & r^2+2 & 1 & 0 & 0 & 0 & \cdots \\ 1 & r^3+3r^2+3 & r^3+3r^2+3 & 1 & 0 & 0 & \cdots \\ 1 & r^4+4r^3+6r^2+4 & 7r^4+8r^3+12r^2+6 & r^4+4r^3+6r^2+4 & 1 & 0 & \cdots \\ 1 & r^5+5r^4+10r^3+10r^2+5 & 21r^5+40r^4+30r^3+30r^2+10 & 21r^5+40r^4+30r^3+30r^2+10 & r^5+5r^4+10r^3+10r^2+5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

For $r = 0 \ldots 4$, we get the matrices

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & 0 & \cdots \\ 1 & 5 & 10 & 10 & 5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right), \left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 7 & 7 & 1 & 0 & 0 & \cdots \\ 1 & 15 & 33 & 15 & 1 & 0 & \cdots \\ 1 & 31 & 131 & 131 & 31 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right),$$

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 6 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 23 & 23 & 1 & 0 & 0 & \cdots \\ 1 & 76 & 230 & 76 & 1 & 0 & \cdots \\ 1 & 237 & 1682 & 1682 & 237 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right), \left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 11 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 57 & 57 & 1 & 0 & 0 & \cdots \\ 1 & 247 & 897 & 247 & 1 & 0 & \cdots \\ 1 & 1013 & 9433 & 9433 & 1013 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

We now show that the sequence of polynomials above, whose coefficient array generate these Pascal-like matrices, are the moments for a family of orthogonal polynomials.

Theorem 19. *The family of polynomials $P_n(t)$ generated by*

$$\frac{(t-1)e^{(2-r)x}e^{(t-1)x}}{t-e^{r(t-1)x}}$$

are the moments of the family of orthogonal polynomials $Q_n(x)$ where

$$Q_n(x) = (x-(t+1)((n-1)r+1)Q_{n-1}(x) - (n-1)^2r^2tQ_{n-2}(x),$$

where $Q_0(x) = 1$ and $Q_1(x) = x - t - 1$.

This is a consequence of the following proposition.

Proposition 20. *The Riordan array*

$$\left[\frac{(t-1)e^{(2-r)x}e^{(t-1)x}}{t-e^{r(t-1)x}}, \frac{e^{r(t-1)x}-1}{r(t-e^{r(t-1)x})}\right]$$

has a tri-diagonal production matrix.

Indeed, we find that the production matrix has the following form.

$$\begin{pmatrix} t+1 & 1 & 0 & 0 & 0 & 0 & \dots \\ r^2t & (r+1)(t+1) & 1 & 0 & 0 & 0 & \dots \\ 0 & 4r^2t & (t+1)(2r+1) & 1 & 0 & 0 & \dots \\ 0 & 0 & 9r^2t & (t+1)(3r+1) & 1 & 0 & \dots \\ 0 & 0 & 0 & 16r^2t & (t+1)(4r+1) & 1 & \dots \\ 0 & 0 & 0 & 0 & 25r^2 & (t+1)(5r+1) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

References

[1] M. Aigner, *A Course in Enumeration*, Springer, Berlin, 2007.

[2] P. Barry, On a family of generalized Pascal triangles defined by exponential Riordan arrays, *J. Integer Seq.*, **10** (2007), Article 07.3.5.

[3] P. Barry, Eulerian polynomials as moments, via Exponential Riordan arrays, *J. Integer Seq.*,**14** (2011), Article 11.9.5.

[4] P. Barry, Some observations on the Lah and Laguerre transforms of integer sequences, *J. Integer Seq.*, **10** (2007), Article 07.4.6.

[5] P. Barry and A. Hennessy, Meixner-type results for Riordan arrays and associated integer sequences, *J. Integer Seq.*, **13** (2010), Article 10.9.4.

[6] P. Barry, Riordan arrays, orthogonal polynomials as moments, and Hankel transforms, *J. Integer Seq.*, **14** (2011), Article 11.2.2.

[7] L. Carlitz and R. Scoville, Generalized Eulerian numbers: combinatorial applications, *J. Reine Angew. Math.*, **265** (1974), 110–137.

[8] Ch. A. Charamlambides, On a generalized Eulerian distribution, *Ann. Inst. Statist. Math.*, **43** (1991), 197–206.

[9] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.

[10] L. Comtet, *Advanced Combinatorics*, Springer, 1974.

[11] E. Deutsch, L. Shapiro, Exponential Riordan arrays, Lecture Notes, Nankai University, 2004, available electronically at
http://www.combinatorics.net/ppt2004/Louis%20W.%20Shapiro/shapiro.htm.

[12] E. Deutsch, L. Ferrari, and S. Rinaldi, Production matrices, *Adv. in Appl. Math.*, **34** (2005), 101–122.

[13] E. Deutsch, L. Ferrari, and S. Rinaldi, Production matrices and Riordan arrays, *Ann. Comb.*, **13** (2009), 65–85.

[14] L. Euler, Remarques sur un beau rapport entre les séries des puissances tant directes que réciproques, 1768. E352 (Eneström Index).

[15] D. Foata, Eulerian polynomials: from Euler's time to the present, in K. Alladi, J. R. Klauder, and C. R. Rao, eds., *The Legacy of Alladi Ramakrishnan in the Mathematical Sciences*, Springer, 2010, pp. 253–273.

[16] W. Gautschi, *Orthogonal Polynomials: Computation and Approximation*, Clarendon Press, Oxford.

[17] I. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics*, Addison–Wesley, Reading, MA, 1994.

[18] F. Hirzebruch, Eulerian polynomials, *Münster J. Math.*, **1** (2008), 9–14.

[19] K. G. Janardan, Relationship between Morisita's model for estimating the environmental density and the generalized Eulerian numbers, *Ann. Inst. Statist. Math.*, **40** (1988) 439–450.

[20] M. V. Koutras, Eulerian numbers associated to sequences of polynomials, *Fibonacci Quart.*, **32** (1994), 44-57.

[21] C. Krattenthaler, Advanced determinant calculus, *Séminaire Lotharingien Combin.* **42** (1999), Article B42q, available electronically at http://www.emis.de/journals/SLC/wpapers/s42kratt.html.

[22] C. Krattenthaler, Advanced determinant calculus: A complement, *Linear Algebra Appl.*, **411** (2005), 68–166.

[23] J. W. Layman, The Hankel transform and some of its properties, *J. Integer Seq.*, **4** (2001), Article 01.1.5.

[24] P. Luschny, Eulerian polynomials, http://oeis.org/wiki/Eulerian_polynomials, 2013.

[25] P. Luschny, Eulerian polynomials generalized, http://oeis.org/wiki/User:Peter_Luschny/EulerianPolynomialsGeneralized, 2013.

[26] M. Morisita, Measuring habitat value by environmental density method, in G. P. Patil et al., eds., *Statistical Ecology*, Pennsylvania State University Press, 1971, pp. 379–401.

[27] P. Peart and W-J. Woan, Generating functions via Hankel and Stieltjes matrices, *J. Integer Seq.*, **3** (2000), Article 00.2.1.

[28] Ch. Radoux, Déterminants de Hankel et théorème de Sylvester, available electronically at `http://www.mat.univie.ac.at/~slc/opapers/s28radoux.pdf`, 2011.

[29] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*. Published electronically at `http://oeis.org`, 2013.

[30] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, *Notices Amer. Math. Soc.*, **50** (2003), 912–915.

[31] G. Szegö, *Orthogonal Polynomials*, 4th ed., American Mathematial Society, 1975.

[32] H. S. Wall, *Analytic Theory of Continued Fractions*, AMS Chelsea Publishing, 1967.

[33] T. Xiong, H.-P. Tsao, and J. I. Hall, General Eulerian numbers and Eulerian polynomials, *Journal of Mathematics*, **2013** (2013), Article 629132. Preprint available at `http://arxiv.org/abs/1207.0430`.

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