Journal of Integer Sequences, Vol. 14 (2011),
Article 11.2.5

Formulas for Odd Zeta Values and Powers of π

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Abstract

Plouffe conjectured fast converging series formulas for π^{2n+1} and $\zeta(2n+1)$ for small values of n. We find the general pattern for all integer values of n and offer a proof.

1 Introduction

It took nearly one hundred years for the Basel Problem — finding a closed form solution to $\sum_{k=1}^{\infty} 1/k^2$ — to see a solution. Euler solved this in 1735 and essentially solved the problem where the power of two is replaced with any even power. This formula is now usually written as

$$\zeta(2n) = (-1)^{n+1} \frac{B_{2n}(2\pi)^{2n}}{2(2n)!},$$

where $\zeta(s)$ is the Riemann zeta function and B_k is the k^{th} Bernoulli number uniquely defined by the generating function

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}, \quad |x| < 2\pi.$$

and whose first few values are $0, -1/2, 1/6, 0, -1/30, \ldots$. However, finding a closed form for $\zeta(2n+1)$ has remained an open problem. Only in 1979 did Apéry show that $\zeta(3)$ is irrational. His proof involved the snappy acceleration

$$\zeta(3) = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}}.$$

This tidy formula does not generalize to the other odd zeta values, but other representations, such as nested sums or integrals, have been well-studied. The hunt for a clean result like Euler's has largely been abandoned, leaving researchers with the goal of finding formulas which either converge quickly or have an elegant form.

Following his success in discovering a new formula for π, Simon Plouffe[3] postulated several identities which relate either π^m or $\zeta(m)$ to three infinite series. Letting

$$S_n(r) = \sum_{k=1}^{\infty} \frac{1}{k^n(e^{\pi r k} - 1)},$$

the first few examples are

$$\begin{aligned}
\pi &= 72S_1(1) - 96S_1(2) + 24S_1(4) \\
\pi^3 &= 720S_3(1) - 900S_3(2) + 180S_3(4) \\
\pi^5 &= 7056S_5(1) - 6993S_5(2) + 63S_5(4) \\
\pi^7 &= \frac{907200}{13}S_7(1) - 70875S_7(2) + \frac{14175}{13}S_5(4)
\end{aligned}$$

and

$$\begin{aligned}
\zeta(3) &= 28S_3(1) - 37S_3(2) + 7S_3(4) \\
\zeta(5) &= 24S_5(1) - \frac{259}{10}S_5(2) + \frac{1}{10}S_5(4) \\
\zeta(7) &= \frac{304}{13}S_7(1) - \frac{103}{4}S_7(2) + \frac{19}{52}S_7(4).
\end{aligned}$$

Plouffe conjectured these formulas by first assuming, for example, that there exist constants a, b, and c such that

$$\pi = aS_1(1) + bS_1(2) + cS_1(4).$$

By obtaining accurate approximations of each the three series, he wrote some computer code to postulate rational values for a, b, c. Today, such integer relations algorithms have been used to discover many formulas. The widely used PSLQ algorithm, developed by Ferguson and Bailey[2], is implemented in Maple. The following Maple code (using Maple 14) solves the above problem:

```
> with(IntegerRelations):
> Digits := 100;
> S := r -> sum( 1/k/( exp(Pi*r*k)-1 ), k=1..infinity );
> PSLQ( [ Pi, S(1), S(2), S(4) ] );
```

The PSLQ command returns the vector $[-1, 72, -96, 24]$, producing the first formula.

While the computer can be used to conjecture the coefficients of a specific power, finding the general sequence of rationals has remained an open problem. This note finds the sequences and offers formal proofs.

2 Exact Formulas

While it does not seem that $\zeta(2n+1)$ is a rational multiple of π^{2n+1}, a result in Ramanujan's notebooks gives a relationship with infinite series which converge quickly.

Theorem 1. *(Ramanujan) If $\alpha > 0$, $\beta > 0$, and $\alpha\beta = \pi^2$, then*

$$\alpha^{-n}\left\{\frac{1}{2}\zeta(2n+1) + S_{2n+1}(2\alpha)\right\} =$$
$$(-\beta)^{-n}\left\{\frac{1}{2}\zeta(2n+1) + S_{2n+1}(2\beta)\right\} - 4^n\sum_{k=0}^{n+1}(-1)^k\frac{B_{2k}B_{2n+2-2k}}{(2k)!(2n+2-2k)!}\alpha^{n+1-k}\beta^k.$$

Using $\alpha = \beta = \pi$ in Proposition 1 and defining

$$F_n = \sum_{k=0}^{n+1}(-1)^k\frac{B_{2k}B_{2n+2-2k}}{(2k)!(2n+2-2k)!},$$

we have

$$\left(\pi^{-n} - (-\pi)^{-n}\right)\left(\frac{1}{2}\zeta(2n+1) + S_{2n+1}(2\pi)\right) = -4^n\pi^{n+1}F_n.$$

To find formulas for the odd zeta values and powers of π, we will divide these into two classes: $\zeta(4m-1)$ and $\zeta(4m+1)$. Such distinctions can be seen in other studies; see [1, pp. 137–139].

First we find the formulas for π^{4m-1} and $\zeta(4m-1)$. If n is odd, then

$$\frac{1}{2}\zeta(2n+1) + S_{2n+1}(2\pi) = \frac{-4^n}{2}\pi^{2n+1}F_n \tag{1}$$

Using $\alpha = \pi/2$ and $\beta = 2\pi$ in Proposition 1 and defining

$$G_n = \sum_{k=0}^{n+1}(-4)^k\frac{B_{2k}B_{2n+2-2k}}{(2k)!(2n+2-2k)!},$$

one has

$$\frac{1}{2}\zeta(2n+1) = \frac{S_{2n+1}(4\pi) + 4^nS_{2n+1}(\pi) + \frac{4^n}{2}\pi^{2n+1}G_n}{-(4^n+1)}.$$

Combining this with equation (1) yields

$$\frac{4^nS_{2n+1}(\pi) - (4^n+1)S_{2n+1}(2\pi) + S_{2n+1}(4\pi)}{\frac{4^n}{2}(4^n+1)F_n - \frac{4^n}{2}G_n} = \pi^{2n+1}$$

Substituting $n = 2m-1$ and defining

$$D_m = 4^{2m-1}[(4^{2m-1}+1)F_{2m-1} - G_{2m-1}]/2$$

produces

$$\pi^{4m-1} = \frac{4^{2m-1}}{D_m} S_{4m-1}(\pi) - \frac{4^{2m-1}+1}{D_m} S_{4m-1}(2\pi) + \frac{1}{D_m} S_{4m-1}(4\pi)$$

This identity may be used in conjunction with equation (1) to obtain

$$\zeta(4m-1) = -\frac{F_{2m-1}4^{4m-2}}{D_m} S_{4m-1}(\pi) + \frac{G_{2m-1}4^{2m-1}}{D_m} S_{4m-1}(2\pi) - \frac{F_{2m-1}4^{2m-1}}{D_m} S_{4m-1}(4\pi).$$

To obtain formulas for the $4m+1$ cases, use $\alpha = \pi/2$ and $\beta = 2\pi$ in Theorem 1 with $n = 2m$ to obtain

$$\zeta(4m+1) = \frac{\frac{4^{2m}}{2}\pi^{4m+1} G_{2m} - S_{4m+1}(4\pi) + 4^{2m} S_{4m+1}(\pi)}{\frac{1}{2}(1-4^{2m})}. \tag{2}$$

Define $T_n(r)$ (similar to $S_n(r)$) by

$$T_n(r) = \sum_{k=1}^{\infty} \frac{1}{k^n(e^{rk}+1)}$$

and another finite sum of Bernoulli numbers by

$$H_n = \sum_{k=0}^{n} (-4)^{n+k} \frac{B_{4k} B_{4n+2-4k}}{(4k)!(4n+2-4k)!}.$$

Vepstas [4] cites an expression credited to Ramanujan:

$$\begin{aligned}&(1+(-4)^m - 2^{4m+1})\zeta(4m+1) = \\ &\quad 2T_{4m+1}(2\pi) + 2(2^{4m+1} - (-4)^m) S_{4m+1}(2\pi) + 2^{4m+1}\pi^{4m+1} H_m + 2^{4m}\pi^{4m+1} G_{2m}.\end{aligned}$$

Vepstas also produces a formula to show the relationship between T_k and S_k:

$$T_k(x) = S_k(s) - 2S_k(2x)$$

Combining the last two equations produces

$$\begin{aligned}&\frac{1+(-4)^m - 2^{4m+1}}{\frac{1}{2}(1-4^{2m})} \left(\frac{4^{2m}}{2}\pi^{4m+1} G_{2m} - S_{4m+1}(4\pi) + 4^{2m} S_{4m+1}(\pi) \right) = \\ &\quad 2[2^{4m+1} - (-4)^m + 1] S_{4m+1}(2\pi) - 4S_{4m+1}(4\pi) + 2^{4m+1}\pi^{4m+1} H_m + 2^{4m}\pi^{4m+1} G_{2m}\end{aligned}$$

Letting

$$K_m = \frac{\frac{1}{2}(1-4^{2m})}{1+(-4)^m - 2^{4m+1}}$$

and

$$E_m = \frac{4^{2m}}{2} G_{2m} - 2^{4m+1} K_m H_m - 2^{4m} K_m G_{2m},$$

one eventually finds

$$\pi^{4m+1} = -\frac{4^{2m}}{E_m} S_{4m+1}(\pi) + \frac{2K_m[2^{4m+1} - (-4)^m + 1]}{E_m} S_{4m+1}(2\pi) + \frac{(1-4K_m)}{E_m} S_{4m+1}(4\pi)$$

Substituting this into equation (2) produces

$$\begin{aligned}\zeta(4m+1) \quad = \quad & -\frac{16^m(G(2m)16^m + 2E_m)}{(-1+16^m)E_m} S_{4m+1}(\pi) \\ & -\frac{2G_{2m}K_m 16^m(2\cdot 16^m - (-4)^m + 1)}{(-1+16^m)E_m} S_{4m+1}(2\pi) \\ & -\frac{G(2m)16^m + 4G_{2m}16^m K_m + 2E_m}{(-1+16^m)E_m} S_{4m+1}(4\pi).\end{aligned}$$

References

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2010 *Mathematics Subject Classification*: Primary 11Y60.
Keywords: π, Riemann zeta function.

Received January 24 2011; revised version received February 7 2011. Published in *Journal of Integer Sequences*, February 20 2011.

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