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Some Arithmetic Functions Involving Exponential Divisors

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Abstract

In this paper we study several arithmetic functions connected with the exponential divisors of integers. We establish some asymptotic formulas under the Riemann hypothesis, which improve previous results. We also prove some asymptotic lower bounds.

1 Introduction and results

Suppose $n > 1$ is an integer with prime factorization $n = p_1^{a_1} \cdots p_r^{a_r}$. An integer d is called an exponential divisor (e-divisor) of n if $d = p_1^{b_1} \cdots p_r^{b_r}$ with $b_j | a_j (1 \le j \le r)$, which is denoted by the notation $d|_e n$. For convenience $1|_e 1$. The properties of the exponential divisors attract the interests of many authors(see, for example, [4, 6, 7, 9, 11, 12, 14, 15, 16, 19, 20, 21, 22]).

An integer $n = p_1^{a_1}\cdots p_r^{a_r}$ is called exponentially squarefree (e-squarefree) if all the exponents $a_1, \cdots, a_r$ are squarefree. The integer 1 is also considered to be an e-squarefree number.

Suppose f and g are two arithmetic functions and $n = p_1^{a_1}\cdots p_r^{a_r}$. Subbarao [17] first introduced the exponential convolution (e-convolution) by

$$(f \odot g)(n) = \sum_{b_1c_1=a_1} \cdots \sum_{b_rc_r=a_r} f(p_1^{b_1}\cdots p_r^{b_r})g(p_1^{c_1}\cdots p_r^{c_r}),$$

which is an analogue of the classical Dirichlet convolution. The e-convolution $\odot$ is commutative, associative and has the identity element μ^2, where μ is the Möbius function. Furthermore, a function f has an inverse with respect to $\odot$ iff $f(1) \neq 0$ and $f(p_1 \cdots p_r) \neq 0$ for any distinct primes $p_1, \cdots, p_r$.

The inverse of the constant function $f(n) \equiv 1$ with respect to $\odot$ is called the exponential analogue of the Möbius function and it is denoted by $\mu^{(e)}$. Hence $\sum_{d|_e n} \mu^{(e)}(d) = \mu^2(n)$ for $n \geq 1$, $\mu^{(e)}(1) = 1$, and $\mu^{(e)}(n) = \mu(a_1)\cdots\mu(a_r)$ for $n = p_1^{a_1}\cdots p_r^{a_r} > 1$. Note that $|\mu^{(e)}(n)| = 1$ or 0, according as n is e-squarefree or not.

An integer d is called an exponential squarefree exponential divisor (e-squarefree e-divisor) of n if $d = p_1^{b_1}\cdots p_r^{b_r}$ with $b_j|a_j (1 \leq j \leq r)$ and $b_1, \cdots, b_r$ are squarefree. Observe that the integer 1 is an e-squarefree and it is not an e-divisor of $n > 1$. Let $t^{(e)}$ denote the number of e-sqaurefree e-divisors of n.

Now we introduce some notation for later use. As usual, let $\mu(n)$ and $\omega(n)$ denote the Möbius function , and the number of distinct prime factors of n respectively. If t is real, then $\{t\}$ denotes the fractional part of $t, \psi(t) = \{t\} - 1/2$. Throughout this paper, ε is a small fixed positive constant and $m \sim M$ means that $cM \leq m \leq CM$ for some constants $0 < c < C$. For any fixed integers a and b, define the function $d(a, b; n) := \sum_{m_1^a m_2^b = n} 1$ and let $\Delta(a, b; t)$ denote the error term in the asymptotic formula of the sum $\sum_{n \leq x} d(a, b; n)$.

Many authors have studied the properties of the above three functions; see, for example, [4, 7, 15, 17, 19, 20, 22]. Tóth [20] proved that

$$\sum_{n \leq x} \mu^{(e)}(n) = A_1 x + (x^{1/2} exp(-c_1(\log x)^{\Delta})), \tag{1}$$

where $0 < \Delta < 9/25$ and $c_1 > 0$ are constants and

$$A_1 := m(\mu^{(e)}) = \prod_p \left(1 + \sum_{k=2}^{\infty} \frac{\mu(k) - \mu(k-1)}{p^k}\right).$$

Tóth [20] proved that if the Riemann hypothesis (RH) is true, then

$$\sum_{n \leq x} \mu^{(e)}(n) = A_1 x + O(x^{91/202+\varepsilon}). \tag{2}$$

Subbarao [17] and Wu [22] studied the asymptotic properties of the sum $\sum_{n \leq x} |\mu^e(n)|$. Tóth [20] proved that if RH is true, then

$$\sum_{n \leq x} |\mu^{(e)}(n)| = B_1 x + O(x^{1/5+\varepsilon}), \tag{3}$$

where

$$B_1 := \prod_p (1 + \sum_{\alpha=4}^{\infty} \frac{\mu^2(a) - \mu^2(a-1)}{p^a}).$$

For the function $t^{(e)}(n)$, Tóth [20] proved the asymptotic formula

$$\sum_{n\le x} t^{(e)}(n) = C_1 x + C_2 x^{1/2} + O(x^{1/4+\varepsilon}), \tag{4}$$

where

$$C_1 := \prod_p \left(1 + \sum_{k=2}^{\infty} \frac{2^{\omega(k)} - 2^{\omega(k-1)}}{p^k}\right),$$
$$C_2 := \zeta(\frac{1}{2}) \prod_p \left(1 + \sum_{k=4}^{\infty} \frac{2^{\omega(k)} - 2^{\omega(k-1)} - 2^{\omega(k-2)} + 2^{\omega(k-3)}}{p^k}\right).$$

Pétermann [13] proved the formula (4) with a better error term $O(x^{1/4})$, which is the best unconditional result up to date. In order to further reduce the exponent 1/4, we have to know more information about the distribution of the non-trivial zeros of the Riemann zeta-function.

In this short paper we shall prove the following theorem.

Theorem 1. *If RH is true, then*

$$\sum_{n\le x} \mu^{(e)}(n) = A_1 x + O\left(x^{\frac{37}{94}+\varepsilon}\right), \tag{5}$$
$$\sum_{n\le x} |\mu^{(e)}(n)| = B_1 x + B_2 x^{\frac{1}{5}} + O(x^{\frac{38}{193}+\varepsilon}), \tag{6}$$
$$\sum_{n\le x} t^{(e)}(n) = C_1 x + C_2 x^{1/2} + O\left(x^{\frac{3728}{15469}+\varepsilon}\right), \tag{7}$$

where B_2 is a computable constant.

Remark 1. Numerically, we have

$$\frac{37}{94} = 0.39361\cdots, \frac{91}{202} = 0.45049\cdots,$$
$$\frac{38}{193} = 0.1968\cdots < 1/5, \frac{3728}{15469} = 0.240\cdots < 1/4.$$

Let $\Delta_{\mu^{(e)}}(x), \Delta_{|\mu^{(e)}|}(x), \Delta_{t^{(e)}}(x)$ denote the error terms in (5), (6) and (7) respectively. We have the following result.

Theorem 2. *We have*

$$\Delta_{\mu^{(e)}}(x) = \Omega(x^{1/4}), \tag{8}$$
$$\Delta_{|\mu^{(e)}|}(x) = \Omega(x^{1/8}), \tag{9}$$
$$\Delta_{t^{(e)}}(x) = \Omega(x^{1/6}). \tag{10}$$

2 Some generating functions

In this section we shall study the generating Dirichlet series of the functions $\mu^{(e)}(n), |\mu^{(e)}(n)|$ and $t^{(e)}(n)$, respectively. We consider only $|\mu^{(e)}(n)|$ and $t^{(e)}(n)$, since Tóth [20] already proved the following formula, which is enough for our purpose,

$$\sum_{n=1}^{\infty} \frac{\mu^{(e)}(n)}{n^s} = \frac{\zeta(s)}{\zeta^2(2s)} U(s) \ (\Re s > 1), \tag{11}$$

where $U(s) := \sum_{n=1}^{\infty} \frac{u(n)}{n^s}$ is absolutely convergent for $\Re s > 1/5$.

We first consider the function $|\mu^{(e)}(n)|$. The function $\mu^{(e)}$ is multiplicative and $\mu^{(e)}(p^a) = \mu(a)$ for every prime power p^a, namely for every prime p, $\mu^{(e)}(p) = 1, \mu^{(e)}(p^2) = -1, \mu^{(e)}(p^3) = -1, \mu^{(e)}(p^4) = 0, \mu^{(e)}(p^5) = -1, \mu^{(e)}(p^6) = 1, \mu^{(e)}(p^7) = 1, \mu^{(e)}(p^8) = 0, \mu^{(e)}(p^9) = 0, \mu^{(e)}(p^{10}) = 1, \mu^{(e)}(p^{11}) = -1, \cdots$. Hence by the Euler product we have for $\Re s > 1$ that

$$\sum_{n=1}^{\infty} \frac{|\mu^{(e)}(n)|}{n^s} = \prod_p \left(1 + \sum_{m=1}^{\infty} \frac{|\mu(m)|}{p^{ms}}\right). \tag{12}$$

Applying the product representation of Riemann zeta-function

$$\zeta(s) = \prod_p (1 + p^{-s} + p^{-2s} + \cdots) = \prod_p (1 - p^{-s})^{-1} \ (\Re s > 1), \tag{13}$$

we have for $\Re s > 1$

$$\zeta(s)\zeta(5s) = \prod_p \left((1 - p^{-s})(1 - p^{-5s})\right)^{-1}. \tag{14}$$

Let

$$\begin{aligned} f_{|\mu^{(e)}|}(z) : \quad &= 1 + \sum_{m=1}^{\infty} |\mu(m)| z^m \\ &= 1 + z + z^2 + z^3 + z^5 + z^6 + z^7 + z^{10} + z^{11} + \sum_{m=12}^{\infty} |\mu(m)| z^m. \end{aligned} \tag{15}$$

For $|z| < 1$, it is easy to verify that

$$\begin{aligned} & f_{|\mu^{(e)}|}(z)(1 - z)(1 - z^5) \\ &= \left(1 + z + z^2 + z^3 + z^5 + z^6 + z^7 + z^{10} + z^{11} + \sum_{m=12}^{\infty} |\mu(m)| z^m\right)(1 - z - z^5 + z^6) \\ &= 1 - z^4 - z^8 + z^9 + \sum_{m=12}^{\infty} c_m^{(1)} z^m, \end{aligned} \tag{16}$$

where

$$c_m^{(1)} := \quad |\mu(m)| - |\mu(m-1)| - |\mu(m-5)| + |\mu(m-6)|. \ (m \geq 12)$$

Furthermore, we have

$$
\begin{aligned}
&f_{|\mu^{(e)}|}(z)(1-z)(1-z^5)(1-z^4)^{-1}(1-z^8)^{-1} \\
&= \left(1 - z^4 - z^8 + z^9 + \sum_{m=12}^{\infty} c_m^{(1)} z^m\right)(1+z^4+z^8+\cdots)(1+z^8+z^{16}+\cdots) \\
&= 1 + z^9 + \sum_{m=12}^{\infty} C_m^{(1)} z^m.
\end{aligned} \tag{17}
$$

We get from (12), (13), (15) and (17) by taking $z = p^{-s}$ that

$$
\sum_{n=1}^{\infty} \frac{|\mu^{(e)}(n)|}{n^s} = \frac{\zeta(s)\zeta(5s)}{\zeta(4s)\zeta(8s)} V(s) \ (\Re s > 1), \tag{18}
$$

where $V(s) := \sum_{n=1}^{\infty} \frac{v(n)}{n^s}$ is absolutely convergent for $\Re s > \frac{1}{9}$.

We now consider the function $t^{(e)}(n)$, which is also multiplicative and $t^{(e)}(p^a) = 2^{\omega(a)}$ for every prime power p^a. Therefore for every prime p, $t^{(e)}(p) = 1, t^{(e)}(p^2) = t^{(e)}(p^3) = t^{(e)}(p^4) = t^{(e)}(p^5) = 2, t^{(e)}(p^6) = 4, t^{(e)}(p^7) = t^{(e)}(p^8) = t^{(e)}(p^9) = 2, t^{(e)}(p^{10}) = 4, \cdots$.

By the Euler product we have for $\Re s > 1$ that

$$
\sum_{n=1}^{\infty} \frac{t^{(e)}(n)}{n^s} = \prod_p \left(1 + \sum_{m=1}^{\infty} \frac{2^{\omega(m)}}{p^{ms}}\right). \tag{19}
$$

Let

$$
\begin{aligned}
f_{t^{(e)}}(z) &:= 1 + \sum_{m=1}^{\infty} 2^{\omega(m)} z^m \\
&= 1 + z + 2z^2 + 2z^3 + 2z^4 + 2z^5 + 4z^6 + 2z^7 + 2z^8 + 2z^9 + 4z^{10} + 2z^{11} + \sum_{m=12}^{\infty} 2^{\omega(m)} z^m.
\end{aligned} \tag{20}
$$

For $|z| < 1$, it is easy to check that

$$
\begin{aligned}
&f_{t^{(e)}}(z)(1-z)(1-z^2)(1-z^6)^2 \\
&= 1 - z^4 - 2z^7 - 2z^8 + 2z^9 + \sum_{m=10}^{\infty} c_m^{(2)} z^m
\end{aligned}
$$

and

$$
\begin{aligned}
&f_{t^{(e)}}(z)(1-z)(1-z^2)(1-z^6)^2(1-z^4)^{-1} \\
&= \left(1 - z^4 - 2z^7 - 2z^8 + 2z^9 + \sum_{m=10}^{\infty} c_m^{(2)} z^m\right)(1+z^4+z^8+\cdots) \\
&= 1 - 2z^7 + \sum_{m=8}^{\infty} C_m^{(2)} z^m.
\end{aligned} \tag{21}
$$

Combining (13) and (19)–(21) we get

$$\sum_{n=1}^{\infty}\frac{t^{(e)}(n)}{n^s}=\frac{\zeta(s)\zeta(2s)\zeta^2(6s)}{\zeta(4s)}W(s)\ (\Re s>1), \tag{22}$$

where $W(s):=\sum_{n=1}^{\infty}\frac{w(n)}{n^s}$ is absolutely convergent for $\Re s>\frac{1}{7}$.

3 Proof of Theorem 2

We see from (11) that the generating Dirichlet series of the function $\mu^{(e)}(n)$ is $\frac{\zeta(s)}{\zeta^2(2s)}U(s)$, which has infinitely many poles on the line $\Re s=\frac{1}{4}$, whence the estimate (8) follows. From the expression (18) we see that the generating Dirichlet series of the function $|\mu^{(e)}(n)|$ is $\frac{\zeta(s)\zeta(5s)}{\zeta(4s)}V(s)$, which has infinitely many poles on the line $\Re s=\frac{1}{8}$, whence the estimate (9) follows. Via (22), we get the estimate (10) with the help of Theorem 2 of Küleitner and Nowak [8] (or by Balasubramanian, Ramachandra and Subbarao's method in [1]).

4 The Proofs of (5) and (7)

We follow the approach of Theorem 1 in Montgomery and Vaughan [10]. Throughout this section, we assume RH.

We first prove (5). It is well-known that the characteristic function of the set of squarefree integers is

$$\mu^2(n)=|\mu(n)|=\sum_{d^2|n}\mu(d), \tag{23}$$

We write

$$\Delta(x):=\sum_{n\le x}\mu^2(d)-\frac{x}{\zeta(2)}:=D(x)-\frac{x}{\zeta(2)}. \tag{24}$$

Define the function $a_1(n)$ by

$$a_1(n)=\sum_{md^2=n}\mu^2(m)\mu(d), \tag{25}$$

it is easy to see that

$$\frac{\zeta(s)}{\zeta^2(2s)}=\frac{\sum_{n=1}^{\infty}\frac{\mu^2(n)}{n^s}}{\zeta(2s)}=\sum_{n=1}^{\infty}\frac{a_1(n)}{n^s},\Re s>1, \tag{26}$$

Thus

$$T_1(x):=\sum_{n\le x}a_1(n)=\sum_{md^2\le x}\mu^2(m)\mu(d)=\sum_{d\le x^{\frac{1}{2}}}\mu(d)D(\frac{x}{d^2}). \tag{27}$$

Suppose $1 < y < x^{1/2}$ is a parameter to be determined. We now write $T_1(x)$ in the form

$$T_1(x) = S_1(x) + S_2(x), \tag{28}$$

where

$$S_1(x) = \sum_{d\leq y} \mu(d) D\left(\frac{x}{d^2}\right), \tag{29}$$

and

$$S_2(x) = \sum_{y<d\leq x^{\frac{1}{2}}} \mu(d) D\left(\frac{x}{d^2}\right) = \sum_{\substack{md^2\leq x\\ d>y}} \mu^2(m)\mu(d). \tag{30}$$

We first evaluate $S_1(x)$. From (24) we get

$$\begin{aligned} S_1(x) \quad &= \sum_{d\leq y} \mu(d)\left(\frac{x}{d^2\zeta(2)} + \Delta\left(\frac{x}{d^2}\right)\right) \\ &= \frac{x}{\zeta(2)} \sum_{d\leq y} \frac{\mu(d)}{d^2} + \sum_{d\leq y} \mu(d)\Delta\left(\frac{x}{d^2}\right). \end{aligned} \tag{31}$$

To treat $S_2(x)$, we let

$$g_y(s) = \zeta^{-1}(s) - \sum_{d\leq y} \frac{\mu(d)}{d^s}, (s = \sigma + it), \tag{32}$$

so that for $\Re s = \sigma > 1$

$$g_y(s) = \sum_{d>y} \frac{\mu(d)}{d^s}. \tag{33}$$

Hence

$$\frac{g_y(2s)\zeta(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{b_1(n)}{n^s}, \tag{34}$$

for $\sigma > 1$, where

$$b_1(n) = \sum_{\substack{md^2=n\\ d>y}} \mu^2(m)\mu(d) \tag{35}$$

Let $\frac{1}{2} < \sigma < 2$, $\delta = \frac{\varepsilon}{10}$. Assume RH, from 14.25 in Titchmarsh [18] we have

$$\sum_{d\leq y} \frac{\mu(d)}{d^s} = \zeta^{-1}(s) + O\left(y^{\frac{1}{2}-\sigma+\delta}(|t|^{\delta} + 1)\right). \tag{36}$$

In addition, RH implies that $\zeta(s) \ll |t|^{\delta} + 1$ and $\zeta^{-1}(s) \ll (|t|^{\delta} + 1)$ uniformly for $\sigma > \frac{1}{2} + \delta$ and $|s-1| > \varepsilon$, and $\sum_{n \le y} \mu(n) \ll y^{\frac{1}{2}+\delta}$. From (32), (33) and (36), we have

$$g_y(2s) \ll y^{-\frac{1}{2}}(|t|^{\delta} + 1), (\sigma \ge \frac{1}{2} + \delta).$$

Thus

$$g_y(2s)\frac{\zeta(s)}{\zeta(2s)} \ll y^{-\frac{1}{2}}(|t|^{3\delta} + 1), (\sigma \ge \frac{1}{2} + \delta,\ |s-1| > \varepsilon). \tag{37}$$

From (30), (34), (35) and Perron's formula([18], Lemma 3.19), we obtain

$$S_2(x) = \sum_{n \le x} b_1(n) = \frac{1}{2\pi i}\int_{1+\varepsilon-ix^2}^{1+\varepsilon+ix^2} g_y(2s)\frac{\zeta(s)}{\zeta(2s)}x^s s^{-1}\mathrm{d}s + O(x^{\delta}), \tag{38}$$

since $b(n) \ll n^{\delta}$ by a divisor argument. If we move the line of integration to $\sigma = \frac{1}{2} + \delta$, then by the residue theorem

$$\begin{aligned} &\frac{1}{2\pi i}\int_{1+\varepsilon-ix^2}^{1+\varepsilon+ix^2} g_y(2s)\frac{\zeta(s)}{\zeta(2s)}x^s s^{-1}\mathrm{d}s \\ &= I_1 + I_2 - I_3 + \operatorname*{Res}_{s=1}\ g_y(2s)\frac{\zeta(s)}{\zeta(2s)}x^s s^{-1}, \end{aligned} \tag{39}$$

where

$$\begin{aligned} I_1 &= \frac{1}{2\pi i}\int_{\frac{1}{2}+\delta+ix^2}^{1+\varepsilon+ix^2} g_y(2s)\frac{\zeta(s)}{\zeta(2s)}x^s s^{-1}\mathrm{d}s, I_2 = \frac{1}{2\pi i}\int_{\frac{1}{2}+\delta-ix^2}^{\frac{1}{2}+\delta+ix^2} g_y(2s)\frac{\zeta(s)}{\zeta(2s)}x^s s^{-1}\mathrm{d}s, \\ I_3 &= \frac{1}{2\pi i}\int_{\frac{1}{2}+\delta-ix^2}^{1+\varepsilon-ix^2} g_y(2s)\frac{\zeta(s)}{\zeta(2s)}x^s s^{-1}\mathrm{d}s. \end{aligned}$$

From (37) it is not difficult to see that

$$I_j \ll y^{-\frac{1}{2}}x^{\frac{1}{2}+8\delta}, (j = 1, 2, 3). \tag{40}$$

Combining (38)–(40), we get

$$S_2(x) = \frac{x}{\zeta(2)}\sum_{d>y}\frac{\mu(d)}{d^2} + O\left(y^{-\frac{1}{2}}x^{\frac{1}{2}+\varepsilon} + x^{\varepsilon}\right). \tag{41}$$

Finally, combining (27), (28), (31) and (41) we obtain

$$T_1(x) = \frac{x}{\zeta^2(2)} + \sum_{d \le y}\mu(d)\Delta\left(\frac{x}{d^2}\right) + O\left(x^{\frac{1}{2}+\varepsilon}y^{-\frac{1}{2}} + x^{\varepsilon}\right). \tag{42}$$

In [5], Jia proved the estimate $\Delta(u) \ll u^{\frac{17}{54}+\varepsilon}$. Inserting this estimate into (42) and on taking $y = x^{\frac{10}{57}}$, we get

$$T_1(x) = \frac{x}{\zeta^2(2)} + O\left(x^{\frac{37}{94}+\varepsilon}\right). \tag{43}$$

The asymptotic formula (5) follows form (25)-(27) and (43) by the well-known convolution method.

Now we prove the asymptotic formula (7). We define the functions $a_2(n)$ by the following identity

$$\frac{\zeta(s)\zeta(2s)}{\zeta(4s)} = \sum_{n=1}^{\infty} \frac{a_2(n)}{n^s}, \ \Re s > 1. \tag{44}$$

Hence

$$a_2(n) = \sum_{d^4 m = n} \mu(d) d(1, 2; m).$$

Thus by the same approach as (42), we easily get that for $1 \le y \le x^{\frac{1}{4}}$

$$\begin{aligned} T_2(x) &:= \sum_{n \le x} a_2(n) \qquad (45) \\ &= \frac{\zeta(2)}{\zeta(4)} x + \frac{\zeta(\frac{1}{2})}{\zeta(2)} x^{\frac{1}{2}} + \sum_{d \le y} \mu(d) \Delta\left(1, 2; \frac{x}{d^4}\right) + O\left(x^{\frac{1}{2}+\varepsilon} y^{-\frac{3}{2}} + x^{\varepsilon}\right). \end{aligned}$$

Graham and Kolesnik [2] proved that $\Delta(1, 2; u) \ll u^{\frac{1057}{4785}+\varepsilon}$. Inserting this estimate into (45) and on taking $y = ^{\frac{2671}{15469}}$, we get

$$T_2(x) = \sum_{n \le x} a_2(n) = \frac{\zeta(2)}{\zeta(4)} x + \frac{\zeta(\frac{1}{2})}{\zeta(2)} x^{\frac{1}{2}} + O\left(x^{\frac{3728}{15469}+\varepsilon}\right). \tag{46}$$

The asymptotic formula (7) follows from (22), (44) and (46) by the convolution method.

5 Proof of (6)

Throughout this section we assume RH.

Define the function $a_3(n)$ by

$$\frac{\zeta(s)\zeta(5s)}{\zeta(4s)} = \sum_{n=1}^{\infty} \frac{a_3(n)}{n^s}, \ \Re s > 1. \tag{47}$$

Similar to (45) we have for some $1 \le y \le x^{\frac{1}{4}}$ that

$$\begin{aligned} T_3(x) &:= \sum_{n \le x} a_3(n) \\ &= \frac{\zeta(5)}{\zeta(4)} x + \frac{\zeta(\frac{1}{5})}{\zeta(\frac{4}{5})} x^{\frac{1}{5}} + \sum_{d \le y} \mu(d) \Delta\left(1, 5; \frac{x}{d^4}\right) + O\left(x^{\frac{1}{2}+\varepsilon} y^{-\frac{3}{2}} + x^{\frac{1}{5}+\varepsilon} y^{-\frac{3}{10}} + x^{\varepsilon}\right). \end{aligned}$$

Unfortunately, we can not improve Tóth's exponent $\frac{1}{5}$ in (3) by the above formula even if we use the conjectural bound $\Delta(1, 5; t) \ll t^{1/12+\varepsilon}$. So we use a different approach to prove (6).

Let $q_4(n)$ denote the characteristic function of the set of 4-free numbers, then

$$\sum_{n=1}^{\infty}\frac{q_4(n)}{n^s}=\frac{\zeta(s)}{\zeta(4s)},\Re s>1.$$

We write

$$\Delta_4(x):=\sum_{n\leq 1}q_4(n)-\frac{x}{\zeta(4)}. \tag{48}$$

From (47) and (48), it follows from the Dirichlet hyperbolic approach that for some $1\leq y\leq x^{\frac{1}{5}}$,

$$\begin{aligned}
B_3(x)&=\sum_{n\leq x}a_3(n)=\sum_{md^5\leq x}q_4(m) \qquad (49)\\
&=\sum_{d\leq y}\sum_{m\leq\frac{x}{d^5}}q_4(m)+\sum_{m\leq\frac{x}{y^5}}q_4(m)\sum_{y<d\leq(\frac{x}{m})^{\frac{1}{5}}}1\\
&=\frac{x}{\zeta(4)}\sum_{d\leq y}\frac{1}{d^5}+x^{\frac{1}{5}}\sum_{m\leq\frac{x}{y^5}}\frac{q_4(m)}{m^{\frac{1}{5}}}-y\sum_{m\leq\frac{x}{y^5}}q_4(m)\\
&+\sum_{d\leq y}\Delta_4\left(\frac{x}{d^5}\right)-\sum_{m\leq\frac{x}{y^5}}q_4(m)\psi\left((\frac{x}{m})^{\frac{1}{5}}\right)+\psi(y)\sum_{m\leq\frac{x}{y^5}}q_4(m).
\end{aligned}$$

Applying partial summation formula, we get

$$\begin{aligned}
&\sum_{m\leq\frac{x}{y^5}}\frac{q_4(m)}{m^{\frac{1}{5}}} \qquad (50)\\
=\quad&\frac{\sum_{m\leq\frac{x}{y^5}}q_4(m)}{(\frac{x}{y^5})^{\frac{1}{5}}}+\frac{1}{5}\int_1^{\frac{x}{y^5}}\frac{\sum_{m\leq t}q_4(m)}{t^{\frac{6}{5}}}\mathrm{d}t\\
=\quad&\frac{y}{x^{\frac{1}{5}}}\sum_{m\leq\frac{x}{y^5}}q_4(m)+\frac{1}{5}\int_1^{\frac{x}{y^5}}\frac{\frac{t}{\zeta(4)}+\Delta_4(t)}{t^{\frac{6}{5}}}\mathrm{d}t\\
=\quad&\frac{y}{x^{\frac{1}{5}}}\sum_{m\leq\frac{x}{y^5}}q_4(m)+\frac{1}{4\zeta(4)}\left(\frac{x^{\frac{4}{5}}}{y^4}-1\right)+\frac{1}{5}\int_1^{\frac{x}{y^5}}\frac{\Delta_4(t)}{t^{\frac{6}{5}}}\mathrm{d}t.
\end{aligned}$$

In addition, we have by Euler-Maclaurin formula

$$\sum_{d\leq y}\frac{1}{d^5}=\zeta(5)-\frac{1}{4y^4}-\psi(y)y^{-5}+O(y^{-6}). \tag{51}$$

If RH is true, Graham and Pintz [3] showed that

$$\Delta_4(x)=O(x^{\frac{7}{38}+\varepsilon}), \tag{52}$$

and so

$$\int_1^{\frac{x}{y^5}} \frac{\Delta_4(t)}{t^{\frac{6}{5}}} dt = \int_1^\infty \frac{\Delta_4(t)}{t^{\frac{6}{5}}} \mathrm{d}t - \int_{\frac{x}{y^5}}^\infty \frac{\Delta_4(t)}{t^{\frac{6}{5}}} \mathrm{d}t \tag{53}$$
$$= \int_1^\infty \frac{\Delta_4(t)}{t^{\frac{6}{5}}} \mathrm{d}t + O\left(x^{-\frac{3}{190}+\varepsilon} y^{\frac{3}{38}}\right).$$

Combining (49)–(53) and taking on $y = x^{31/193}$, we obtain

$$\begin{aligned} T_3(x) &= \sum_{md^5 \le x} q_4(m) \qquad (54) \\ &= \frac{\zeta(5)}{\zeta(4)} x + \left(\frac{1}{5}\int_1^\infty \frac{\Delta_4(t)}{t^{\frac{6}{5}}} \mathrm{d}t - \frac{1}{4\zeta(4)}\right) x^{\frac{1}{5}} \\ &\quad - \sum_{m \le \frac{x}{y^5}} q_4(m) \psi\left((\frac{x}{m})^{\frac{1}{5}} \right) + O\left(x^{\frac{7}{38}+\varepsilon} y^{\frac{3}{38}} + xy^{-6}\right) \\ &= \frac{\zeta(5)}{\zeta(4)} x + C_3 x^{1/5} + O(x^{\frac{7}{38}+\varepsilon} y^{\frac{3}{38}} + xy^{-5}), \\ &= \frac{\zeta(5)}{\zeta(4)} x + C_3 x^{1/5} + O(x^{\frac{38}{193}+\varepsilon}), \end{aligned}$$

where we used the trivial estimate

$$\sum_{m \le \frac{x}{y^5}} q_4(m) \psi\left((\frac{x}{m})^{\frac{1}{5}} \right) \ll \frac{x}{y^5}$$

and where

$$C_3 := \frac{1}{5} \int_1^\infty \frac{\Delta_4(t)}{t^{\frac{6}{5}}} \mathrm{d}t - \frac{1}{4\zeta(4)}.$$

Now the asymptotic formula (6) follows from (18), (47), (49) and (54) by the convolution approach.

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