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A Note on the Lonely Runner Conjecture

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Abstract

Suppose n runners having nonzero distinct constant speeds run laps on a unit-length circular track. The *Lonely Runner Conjecture* states that there is a time at which all the n runners are simultaneously at least $1/(n+1)$ units from their common starting point. The conjecture has been already settled up to six ($n \leq 6$) runners and it is open for seven or more runners. In this paper the conjecture has been proved for two or more runners provided the speed of the $(i+1)$th runner is more than double the speed of the ith runner for each i, arranged in an increasing order.

1 Introduction and Summary

The conjecture in its original form stated by Wills [10] and also independently by Cusick [6] is as follows:

> For any n positive integers $w_1, w_2, \ldots, w_n$, there is a real number x such that
>
> $$\|w_i x\| \geq \frac{1}{n+1},$$
>
> for each $i = 1, 2, \ldots, n$, where for a real number x, $\|x\|$ is the distance of x from the nearest integer.

Due to the interpretation by Goddyn [4], the conjecture is now known as the "Lonely Runner Conjecture".

> Suppose n runners having nonzero distinct constant speeds run laps on a unit-length circular track. Then there is a time at which all the n runners are simultaneously at least $1/(n+1)$ units from their common starting point.

The term "lonely runner" reflects an equivalent formulation in which there are $n+1$ runners with distinct speeds.

> Suppose $n+1$ runners having nonzero distinct constant speeds run laps on a unit-length circular track. A runner is called lonely if the distance (on the circular track) between him (or her) and every other runner is at least $1/(n+1)$. The conjecture is equivalent to asserting that for each runner there is a time when he (or she) is lonely.

The case $n = 2$ is very simple. For $n = 3$, Betke and Wills [3] settled the conjecture while Wills was dealing with some Diophantine approximation problem and also independently by Cusick [6] while Cusick was considering n-dimensional "view-obstruction" problem. The case $n = 4$ was first proved by Cusick and Pomerence [7] with a proof that requires a work of electronic case checking. Later, Bienia et al. [4] gave a simpler proof for $n = 4$. The case $n = 5$ was proved by Bohman, Holzman and Kleitman [5]. A simpler proof for the case $n = 5$ was given by Renault [9]. Recently, Barajas and Serra ([1], [2]) proved the conjecture for $n = 6$. Goddyn and Wong [8] gave some tight instances of the lonely runner. For $n \geq 7$ the conjecture is still open. We prove the conjecture for two or more runners provided the speed of the $(i+1)$th runner is more than double the speed of the ith runner for each i, with the speeds arranged in an increasing order.

2 Main Result

Theorem 1. *Let* $M = \{m_1, m_2, \ldots, m_n\}$ *where* $n \geq 2$, *and* $(\frac{m_{j+1}}{m_j})(\frac{n-1}{n+1}) \geq 2$ *for each* $j = 1, 2, \ldots, n-1$. *Then there exists a real number* x *such that*

$$\|m_j x\| \geq \frac{1}{n+1},$$

for each $j = 1, 2, \ldots, n$.

Proof. Consider an interval $I = [u, v] = [\frac{1}{m_1(n+1)}, \frac{n}{m_1(n+1)}]$. Clearly, for $x \in I$, we have $\|m_1 x\| \geq \frac{1}{n+1}$, and $v - u = \frac{1}{m_1}(\frac{n-1}{n+1})$. Let us denote the interval I by I_1. We now construct the intervals $I_2, I_3, \ldots I_n$ satisfying the following properties:

(a) $I_1 \supset I_2 \supset I_3 \supset \ldots \supset I_n$

(b) For $I_j = [u_j, v_j]$, $v_j - u_j = \frac{1}{m_j}(\frac{n-1}{n+1})$

(c) For each $x \in I_j$ we have $\|m_j x\| \geq \frac{1}{n+1}$

Clearly, I_1 satisfies (b) and (c). Inductively, we now define the jth interval $I_j = [u_j, v_j]$. We have

$$m_j v_{j-1} - m_j u_{j-1} = \frac{m_j}{m_{j-1}} \left(\frac{n-1}{n+1} \right) \geq 2.$$

Therefore, there exists an integer $\ell(j)$ such that

$$m_j u_{j-1} \leq \ell(j) < \ell(j) + 1 \leq m_j v_{j-1} \Rightarrow u_{j-1} \leq \frac{\ell(j)}{m_j} < \frac{\ell(j)+1}{m_j} \leq v_{j-1}.$$

Define,

$$I_j = [u_j, v_j] = \left[\frac{\ell(j) + \frac{1}{n+1}}{m_j}, \frac{\ell(j) + \frac{n}{n+1}}{m_j} \right].$$

It can be seen easily that the interval I_j satisfies all (a), (b) and (c). Since the intersection of the intervals $I_1, I_2, \dots I_n$ is nonempty therefore, we have the theorem. $\square$

In the theorem we have seen that the n runners having their speeds $r_1, r_2, \dots, r_n$ with $(\frac{r_{j+1}}{r_j})(\frac{n-1}{n+1}) \geq 2$ satisfy the Lonely Runner Conjecture.

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