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CLARKSON-MCCARTHY INTERPOLATED INEQUALITIES IN FINSLER NORMS

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ABSTRACT. We apply the complex interpolation method to prove that, given two spaces $B^{(n)}_{p_0,a;s_0}$, $B^{(n)}_{p_1,b;s_1}$ of n-tuples of operators in the p-Schatten class of a Hilbert space H, endowed with weighted norms associated to positive and invertible operators a and b of $B(H)$ then, the curve of interpolation $(B^{(n)}_{p_0,a;s_0}, B^{(n)}_{p_1,b;s_1})_{[t]}$ of the pair is given by the space of n-tuples of operators in the p_t-Schatten class of H, with the weighted norm associated to the positive invertible element $\gamma_{a,b}(t) = a^{1/2}(a^{-1/2}ba^{-1/2})^t a^{1/2}$.

Key words and phrases: p-Schatten class, Complex method, Clarkson-McCarthy inequalities.

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1. INTRODUCTION

In [6], J. Clarkson introduced the concept of uniform convexity in Banach spaces and obtained that spaces L_p (or l_p) are uniformly convex for $p > 1$ throughout the following inequalities

$$2\left(\|f\|_p^p + \|g\|_p^p\right) \le \|f-g\|_p^p + \|f+g\|_p^p \le 2^{p-1}\left(\|f\|_p^p + \|g\|_p^p\right),$$

Let $(B(H), \|\cdot\|)$ denote the algebra of bounded operators acting on a complex and separable Hilbert space H, $Gl(H)$ the group of invertible elements of $B(H)$ and $Gl(H)^+$ the set of all positive elements of $Gl(H)$.

If $X \in B(H)$ is compact we denote by $\{s_j(X)\}$ the sequence of singular values of X (decreasingly ordered). For $0 < p < \infty$, let

$$\|X\|_p = \left(\sum s_j(X)^p\right)^{\frac{1}{p}},$$

and the linear space

$$B_p(H) = \{X \in B(H) : \|X\|_p < \infty\}.$$

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For $1 \leq p < \infty$, this space is called the $p-$*Schatten class* of $B(H)$ (to simplify notation we use B_p) and by convention $\|X\| = \|X\|_\infty = s_1(X)$. A reference for this subject is [9].

C. McCarthy proved in [14], among several other results, the following inequalities for p-Schatten norms of Hilbert space operators:

$$\begin{aligned} 2(\|A\|_p^p + \|B\|_p^p) &\leq \|A-B\|_p^p + \|A+B\|_p^p \\ &\leq 2^{p-1}(\|A\|_p^p + \|B\|_p^p), \end{aligned} \tag{1.1}$$

for $2 \leq p < \infty$, and

$$\begin{aligned} 2^{p-1}(\|A\|_p^p + \|B\|_p^p) &\leq \|A-B\|_p^p + \|A+B\|_p^p \\ &\leq 2(\|A\|_p^p + \|B\|_p^p), \end{aligned} \tag{1.2}$$

for $1 \leq p \leq 2$.

These are non-commutative versions of Clarkson's inequalities. These estimates have been found to be very powerful tools in operator theory (in particular they imply the uniform convexity of B_p for $1 < p < \infty$) and in mathematical physics (see [16]).

M. Klaus has remarked that there is a simple proof of the Clarkson-McCarthy inequalities which results from mimicking the proof that Boas [4] gave of the Clarkson original inequalities via the complex interpolation method.

In a previous work [7], motivated by [1], we studied the effect of the complex interpolation method on $B_p^{(n)}$ (this set will be defined below) for $p, s \geq 1$ and $n \in \mathbb{N}$ with a Finsler norm associated with $a \in Gl(H)^+$:

$$\|X\|_{p,a;s} := \|a^{-1/2}Xa^{-1/2}\|_p^s.$$

From now on, for the sake of simplicity, we denote with lower case letters the elements of $Gl(H)^+$.

As a by-product, we obtain Clarkson type inequalities using the Klaus idea with the linear operator $T_n : B_p^{(n)} \longrightarrow B_p^{(n)}$ given by

$$T_n(\bar{X}) = (T_n(X_1, \ldots, X_n) = \left(\sum_{j=1}^n X_j, \sum_{j=1}^n \theta_j^1 X_j, \ldots, \sum_{j=1}^n \theta_j^{n-1} X_j \right),$$

where $\theta_1, \ldots, \theta_n$ are the n roots of unity.

Recently, Kissin in [12], motivated by [3], obtained analogues of the Clarkson-McCarthy inequalities for n-tuples of operators from Schatten ideals. In this work the author considers H^n, the orthogonal sum of n copies of the Hilbert space H, and each operator $R \in B(H^n)$ can be represented as an $n \times n$ block-matrix operator $R = (R_{jk})$ with $R_{jk} \in B(H)$, and the linear operator $T_R : B_p^{(n)} \to B_p^{(n)}$ is defined by $T_R(\overline{A}) = R\overline{A}$. Finally we remark that the works [3] and [11] are generalizations of [10].

In these notes we obtain inequalities for the linear operator T_R in the Finsler norm $\|\cdot\|_{p,a;s}$ as by-products of the complex interpolation method and Kissin's inequalities.

2. Geometric Interpolation

We follow the notation used in [2] and we refer the reader to [13] and [5] for details on the complex interpolation method. For completeness, we recall the classical Calderón-Lions theorem.

Theorem 2.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two compatible couples. Assume that T is a linear operator from $\mathcal{X}_j$ to $\mathcal{Y}_j$ bounded by M_j, $j = 0, 1$. Then for $t \in [0, 1]$*

$$\|T\|_{\mathcal{X}_{[t]} \to \mathcal{Y}_{[t]}} \leq M_0^{1-t} M_1^t.$$

Here and subsequently, let $1 \le p < \infty$, $n \in \mathbb{N}$, $s \ge 1$, $a \in Gl(H)^+$ and

$$B_p^{(n)} = \{\overline{A} = (A_1, \ldots, A_n)^t : A_i \in B_p\},$$

(where with t we denote the transpose of the n-tuple) endowed with the norm

$$\|\overline{A}\|_{p,a;s} = (\|A_1\|_{p,a}^s + \cdots + \|A_n\|_{p,a}^s)^{1/s},$$

and $\mathbb{C}^n$ endowed with the norm

$$|(a_0, \ldots, a_{n-1})|_s = (|a_0|^s + \cdots + |a_{n-1}|^s)^{1/s}.$$

From now on, we denote with $B_{p,a;s}^{(n)}$ the space $B_p^{(n)}$ endowed with the norm $\|(\cdot, \ldots, \cdot)\|_{p,a;s}$.

From Calderón-Lions interpolation theory we get that for $p_0, p_1, s_0, s_1 \in [1, \infty)$

$$\left(B_{p_0,1;s_0}^{(n)}, B_{p_1,1;s_1}^{(n)}\right)_{[t]} = B_{p_t,1;s_t}^{(n)}, \tag{2.1}$$

where

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} \qquad \text{and} \qquad \frac{1}{s_t} = \frac{1-t}{s_0} + \frac{t}{s_1}.$$

Note that for $p = 2$, (1.1) and (1.2) both reduce to the *parallelogram law*

$$2(\|A\|_2^2 + \|B\|_2^2) = \|A - B\|_2^2 + \|A + B\|_2^2\,,$$

while for the cases $p = 1, \infty$ these inequalities follow from the triangle inequality for B_1 and $B(H)$ respectively. Then the inequalities (1.1) and (1.2) can be proved (for $n = 2$ via Theorem 2.1) by interpolation between the previous elementary cases with the linear operator $T_2 : B_{p,1;p}^{(2)} \longrightarrow B_{p,1;p}^{(2)}$, $T_2(\overline{A}) = (A_1 + A_2, A_1 - A_2)^t$ as observed by Klaus.

In this section, we generalize (2.1) for the Finsler norms $\|(\cdot, \ldots, \cdot)\|_{p,a;s}$. In [7], we have obtained this extension for the particular case when $p_0 = p_1 = p$ and $s_0 = s_1 = s$. For sake of completeness, we recall this result

Theorem 2.2 ([7, Th. 3.1]). *Let $a, b \in Gl(H)^+, 1 \le p, s < \infty$, $n \in \mathbb{N}$ and $t \in (0, 1)$. Then*

$$\left(B_{p,a;s}^{(n)}, B_{p,b;s}^{(n)}\right)_{[t]} = B_{p,\gamma_{a,b}(t);s}^{(n)},$$

where $\gamma_{a,b}(t) = a^{1/2}(a^{-1/2}ba^{-1/2})^t a^{1/2}$.

Remark 1. Note that when a and b commute the curve is given by $\gamma_{a,b}(t) = a^{1-t}b^t$. The previous corollary tells us that the interpolating space, $B_{p,\gamma_{a,b}(t);s}$ can be regarded as a weighted p-Schatten space with weight $a^{1-t}b^t$ (see [2, Th. 5.5.3]).

We observe that the curve $\gamma_{a,b}$ looks formally equal to the geodesic (or shortest curve) between positive definitive matrices ([15]), positive invertible elements of a C^*-algebra ([8]) and positive invertible operators that are perturbations of the p-Schatten class by multiples of the identity ([7]).

There is a natural action of $Gl(H)$ on $B_p^{(n)}$, defined by

$$l : Gl(H) \times B_p^{(n)} \longrightarrow B_p^{(n)}, \quad l_g(\overline{A}) = (gA_1g^*, \ldots, gA_ng^*)^t. \tag{2.2}$$

Proposition 2.3 ([7, Prop. 3.1]). *The norm in $B_{p,a;s}^{(n)}$ is invariant for the action of the group of invertible elements. By this we mean that for each $\overline{A} \in B_p^{(n)}$, $a \in Gl(H)^+$ and $g \in Gl(H)$, we have*

$$\left\|\overline{A}\right\|_{p,a;s} = \left\|l_g(\overline{A})\right\|_{p,gag^*;s}.$$

Now, we state the main result of this paper, the general case $1 \le p_0, p_1, s_0, s_1 < \infty$.

Theorem 2.4. *Let* $a, b \in Gl(H)^+, 1 \leq p_0, p_1, s_0, s_1 < \infty$, $n \in \mathbb{N}$ *and* $t \in (0,1)$. *Then*

$$\left(B^{(n)}_{p_0,a;s_0}, B^{(n)}_{p_1,b;s_1}\right)_{[t]} = B^{(n)}_{p_t,\gamma_{a,b}(t);s_t},$$

where

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} \qquad \textit{and} \qquad \frac{1}{s_t} = \frac{1-t}{s_0} + \frac{t}{s_1}.$$

Proof. For the sake of simplicity, we will only consider the case $n = 2$ and omit the transpose. The proof below works for n-tuples ($n \geq 3$) with obvious modifications.

By the previous proposition, $\|(X_1, X_2)\|_{[t]}$ is equal to the norm of $a^{-1/2}(X_1, X_2)a^{-1/2}$ interpolated between the norms $\|(\cdot,\cdot)\|_{p_0,1;s_0}$ and $\|(\cdot,\cdot)\|_{p_1,c;s_1}$. Consequently it is sufficient to prove our statement for these two norms.

Let $t \in (0,1)$ and $(X_1, X_2) \in B^{(2)}_{p_t}$ such that $\|(X_1, X_2)\|_{p_t,c^t;s_t} = 1$, and define

$$\begin{aligned} g(z) &= \left(U_1 \left|c^{\frac{z}{2}} c^{-\frac{t}{2}} X_1 c^{\frac{-t}{2}} c^{\frac{z}{2}}\right|^{\lambda(z)}, U_2 \left|c^{\frac{z}{2}} c^{-\frac{t}{2}} X_2 c^{-\frac{t}{2}} c^{\frac{z}{2}}\right|^{\lambda(z)}\right) \\ &= (g_1(z), g_2(z)), \end{aligned}$$

where

$$\lambda(z) = p_t\left(\frac{1-z}{p_0} + \frac{z}{p_1}\right) s_t\left(\frac{1-z}{s_0} + \frac{z}{s_1}\right)$$

and $X_i = U_i|X_i|$ is the polar decomposition of X_i for $i = 1, 2$.

Then for each $z \in S$, $g(z) \in B^{(2)}_{p_0} + B^{(2)}_{p_1}$ and

$$\begin{aligned} \|g(iy)\|^{s_0}_{p_0,1;s_0} &= \left(\sum_{k=1}^{2} \left\|U_k \left|c^{\frac{iy}{2}} c^{-\frac{t}{2}} X_k c^{-\frac{t}{2}} c^{\frac{iy}{2}}\right|^{\lambda(iy)}\right\|^{s_0}_{p_0}\right) \\ &\leq \left(\sum_{k=1}^{2} \left\|c^{\frac{iy}{2}} c^{-\frac{t}{2}} X_k c^{-\frac{t}{2}} c^{\frac{iy}{2}}\right\|^{p_t}_{p_t}\right) \\ &\leq \left(\sum_{k=1}^{2} \|X_k\|^{p_t}_{p_t,c^t}\right) = 1 \end{aligned}$$

and

$$\|g(1+iy)\|^{s_1}_{p_1,c;s_1} \leq \left(\sum_{k=1}^{2} \|X_k\|^{p_t}_{p_t,c^t}\right) = 1.$$

Since $g(t) = (X_1, X_2)$ and $g = (g_1, g_2) \in \mathcal{F}\left(B^{(2)}_{p_0,1;s_0}, B^{(2)}_{p_1,c;s_1}\right)$, we have $\|(X_1, X_2)\|_{[t]} \leq 1$. Thus we have shown that

$$\|(X_1, X_2)\|_{[t]} \leq \|(X_1, X_2)\|_{p_t,c^t;s_t}.$$

To prove the converse inequality, let $f = (f_1, f_2) \in \mathcal{F}\left(B^{(2)}_{p_0,1;s_0}, B^{(2)}_{p_1,c;s_1}\right)$; $f(t) = (X_1, X_2)$ and $Y_1, Y_2 \in B_{0,0}(H)$ (the set of finite-rank operators) with $\|Y_k\|_{q_t} \leq 1$, where q_t is the conjugate exponent for $1 < p_t < \infty$ (or a compact operator and $q = \infty$ if $p = 1$). For $k = 1, 2$, let

$$g_k(z) = c^{-\frac{z}{2}} Y_k c^{-\frac{z}{2}}.$$

Consider the function $h : S \to \left(\mathbb{C}^2, |(\cdot,\cdot)|_{s_t}\right)$,

$$h(z) = (tr(f_1(z)g_1(z)), tr(f_2(z)g_2(z))).$$

Since $f(z)$ is analytic in $\overset{\circ}{S}$ and bounded in S, then h is analytic in $\overset{\circ}{S}$ and bounded in S, and

$$h(t)=\left(tr\left(c^{-\frac{t}{2}}X_1c^{-\frac{t}{2}}Y_1\right),tr\left(c^{-\frac{t}{2}}X_2c^{-\frac{t}{2}}Y_2\right)\right)=(h_1(t),h_2(t)).$$

By Hadamard's three line theorem applied to h and the Banach space $(\mathbb{C}^2,|(\cdot,\cdot)|_{s_t})$, we have

$$|h(t)|_{s_t}\le\max\left\{\sup_{y\in\mathbb{R}}|h(iy)|_{s_t},\sup_{y\in\mathbb{R}}|h(1+iy)|_{s_t}\right\}.$$

For $j=0,1$,

$$\begin{aligned}\sup_{y\in\mathbb{R}}\ |h(j+iy)|_{s_t}&=\sup_{y\in\mathbb{R}}\left(\sum_{k=1}^{2}|tr(f_k(j+iy)g_k(j+iy))|^{s_t}\right)^{\frac{1}{s_t}}\\&=\sup_{y\in\mathbb{R}}\left(\sum_{k=1}^{2}|tr(c^{-j/2}f_k(j+iy)c^{-j/2}g_k(iy))|^{s_t}\right)^{\frac{1}{s_t}}\\&\le\sup_{y\in\mathbb{R}}\left(\sum_{k=1}^{2}\|f_k(j+iy)\|_{p,c^j}^{s_t}\right)^{\frac{1}{s_t}}\\&\le\|f\|_{\mathcal{F}\left(B^{(2)}_{p_0,1;s_0},B^{(2)}_{p_1,c;s_1}\right)},\end{aligned}$$

then

$$\begin{aligned}\|X_1\|_{p_t,c^t}^{s_t}+\|X_2\|_{p_t,c^t}^{s_t}&=\sup_{\substack{\|Y_1\|_{q_t}\le1,Y_1\in B_{00}(H)\\\|Y_2\|_{q_t}\le1,Y_2\in B_{00}(H)}}\{|h_1(t)|^{s_t}+|h_2(t)|^{s_t}\}\\&=\sup_{\substack{\|Y_1\|_{q_t}\le1,Y_1\in B_{00}(H)\\\|Y_2\|_{q_t}\le1,Y_2\in B_{00}(H)}}|h(t)|_{s_t}^{s_t}\le\|f\|^{s_t}_{\mathcal{F}\left(B^{(2)}_{p_0,1;s_0},B^{(2)}_{p_1,c;s_1}\right)}.\end{aligned}$$

Since the previous inequality is valid for each $f\in\mathcal{F}\left(B^{(2)}_{p_0,1;s_0},B^{(2)}_{p_1,c;s_1}\right)$ with $f(t)=(X_1,X_2)$, we have

$$\|(X_1,X_2)\|_{p_t,c^t;s_t}\le\|(X_1,X_2)\|_{[t]}.$$

□

In the special case that $p_0=p_1=p$ and $s_0=s_1=s$ we obtain Theorem 2.2.

3. CLARKSON-KISSIN TYPE INEQUALITIES

Bhatia and Kittaneh [3] proved that if $2\le p<\infty$, then

$$n^{\frac{2}{p}}\sum_{j=1}^{n}\|A_j\|_p^2\le\sum_{j=1}^{n}\|B_j\|_p^2\le n^{2-\frac{2}{p}}\sum_{j=1}^{n}\|A_j\|_p^2.$$

$$n\sum_{j=1}^{n}\|A_j\|_p^p\le\sum_{j=1}^{n}\|B_j\|_p^p\le n^{p-1}\sum_{j=1}^{n}\|A_j\|_p^p.$$

(for $0<p\le2$, these two inequalities are reversed) where $B_j=\sum_{k=1}^{n}\theta_k^jA_k$ with $\theta_1,\dots,\theta_n$ the n roots of unity.

If we interpolate these inequalities we obtain that

$$n^{\frac{1}{p}}\left(\sum_{j=1}^{n}\|A_j\|_p^{s_t}\right)^{\frac{1}{s_t}} \leq \left(\sum_{j=1}^{n}\|B_j\|_p^{s_t}\right)^{\frac{1}{s_t}} \leq n^{\left(1-\frac{1}{p}\right)}\left(\sum_{j=1}^{n}\|A_j\|_p^{s_t}\right)^{\frac{1}{s_t}},$$

where

$$\frac{1}{s_t} = \frac{1-t}{2} + \frac{t}{p}.$$

Dividing by n^{s_t}, we obtain

$$\begin{aligned} n^{\frac{1}{p}}\left(\frac{1}{n}\sum_{j=1}^{n}\|A_j\|_p^{s_t}\right)^{\frac{1}{s_t}} &\leq \left(\frac{1}{n}\sum_{j=1}^{n}\|B_j\|_p^{s_t}\right)^{\frac{1}{s_t}} \\ &\leq n^{\left(1-\frac{1}{p}\right)}\left(\frac{1}{n}\sum_{j=1}^{n}\|A_j\|_p^{s_t}\right)^{\frac{1}{s_t}}. \end{aligned}$$

This inequality can be rephrased as follows, if $\mu \in [2,p]$ then

$$\begin{aligned} n^{\frac{1}{p}}\left(\frac{1}{n}\sum_{j=1}^{n}\|A_j\|_p^{\mu}\right)^{\frac{1}{\mu}} &\leq \left(\frac{1}{n}\sum_{j=1}^{n}\|B_j\|_p^{\mu}\right)^{\frac{1}{\mu}} \\ &\leq n^{\left(1-\frac{1}{p}\right)}\left(\frac{1}{n}\sum_{j=1}^{n}\|A_j\|_p^{\mu}\right)^{\frac{1}{\mu}}. \end{aligned}$$

In each of the following statements $R \in Gl(H^n)$ and we denote by T_R the linear operator

$$T_R : B_p^{(n)} \longrightarrow B_p^{(n)} \qquad T_R(\overline{A}) = R\overline{A} = (B_1,\ldots,B_n)^t,$$

with $B_j = \sum_{k=1}^{n} R_{jk}A_k$ and $\alpha = \|R^{-1}\|$, $\beta = \|R\|$ (we use the same symbol to denote the norm in $B(H)$ and $B(H^n)$).

We observe that if the norm of T_R is at most M when

$$T_R : \left(B_p^{(n)}, \|(\cdot,\ldots,\cdot)\|_{p,1,s}\right) \to \left(B_p^{(n)}, \|(\cdot,\ldots,\cdot)\|_{p,1,r}\right),$$

then if we consider the operator T_R between the spaces

$$T_R : \left(B_p^{(n)}, \|(\cdot,\ldots,\cdot)\|_{p,a,s}\right) \to \left(B_p^{(n)}, \|(\cdot,\ldots,\cdot)\|_{p,b,r}\right),$$

its norm is at most $F(a,b)M$ with

$$F(a,b) = \begin{cases} \min\{\|b^{-1}\|\|a\|, \|a^{1/2}b^{-1}a^{1/2}\|\|a^{-1}\|\|a\|\} & \text{if } a \neq b, \\ \|a^{-1}\|\|a\| & \text{if } \ a = b. \end{cases}$$

Remark 2. If $a^{-1/2} \in Gl(H)$ commutes with $R \in B(H^n)$, that is, if $a^{-1/2}$ commutes with R_{jk} for all $1 \leq j,k \leq n$, then F is reduced to

$$F(a,b) = \begin{cases} \min\{\|b^{-1}\|\|a\|, \|a^{1/2}b^{-1}a^{1/2}\|\} = \|a^{1/2}b^{-1}a^{1/2}\| & \text{if } a \neq b, \\ 1 & \text{if } \ a = b. \end{cases}$$

In [12], Kissin proved the following Clarkson type inequalities for the n-tuples $\overline{A} \in B_p^{(n)}$. If $2 \le p < \infty$ and $\lambda, \mu \in [2, p]$, or if $0 < p \le 2$ and $\lambda, \mu \in [p, 2]$, then

$$n^{-f(p)} \alpha^{-1} \left(\frac{1}{n} \sum_{j=1}^{n} \|A_j\|_p^{\mu} \right)^{\frac{1}{\mu}} \le \left(\frac{1}{n} \sum_{j=1}^{n} \|B_j\|_p^{\lambda} \right)^{\frac{1}{\lambda}} \le n^{f(p)} \beta \left(\frac{1}{n} \sum_{j=1}^{n} \|A_j\|_p^{\mu} \right)^{\frac{1}{\mu}}, \tag{3.1}$$

where $f(p) = \left| \frac{1}{p} - \frac{1}{2} \right|$.

Remark 3. This result extends the results of Bhatia and Kittaneh proved for $\mu = \lambda = 2$ or p and $R = (R_{jk})$ where

$$R_{jk} = e^{\left(i \frac{2\pi(j-1)(k-1)}{n} \right)} 1.$$

We use the inequalities (3.1) and the interpolation method to obtain the following inequalities.

Theorem 3.1. *Let* $a, b \in Gl(H)^+, \overline{A} \in B_p^{(n)}, 1 \le p < \infty$ *and* $t \in [0, 1]$, *then*

$$\tilde{k} \left(\sum_{j=1}^{n} \|A_j\|_{p,a}^{\mu} \right)^{\frac{1}{\mu}} \le \left(\sum_{j=1}^{n} \|B_j\|_{p,\gamma_{a,b}(t)}^{\lambda} \right)^{\frac{1}{\lambda}} \le \tilde{K} \left(\sum_{j=1}^{n} \|A_j\|_{p,a}^{\mu} \right)^{\frac{1}{\mu}} \tag{3.2}$$

where

$$\tilde{k} = \tilde{k}(p, a, b, t) = F(a, a)^{t-1} F(b, a)^{-t} n^{\frac{1}{\lambda} - \frac{1}{\mu} - \left| \frac{1}{p} - \frac{1}{2} \right|} \alpha^{-1},$$

and

$$\tilde{K} = \tilde{K}(p, a, b, t) = F(a, a)^{1-t} F(a, b)^{t} n^{\frac{1}{\lambda} - \frac{1}{\mu} + \left| \frac{1}{p} - \frac{1}{2} \right|} \beta,$$

if $2 \le p$ *and* $\lambda, \mu \in [2, p]$ *or if* $1 \le p \le 2$ *and* $\lambda, \mu \in [p, 2]$.

Proof. We will denote by $\gamma(t) = \gamma_{a,b}(t)$, when no confusion can arise.

Consider the space $B_p^{(n)}$ with the norm:

$$\|\overline{A}\|_{p,a;s} = (\|A_1\|_{p,a}^{s} + \cdots + \|A_n\|_{p,a}^{s})^{1/s},$$

where $a \in Gl(H)^+$.

By (3.1), the norm of T_R is at most $F(a, a) n^{\frac{1}{\lambda} - \frac{1}{\mu} + \left| \frac{1}{p} - \frac{1}{2} \right|} \beta$ when

$$T_R : \left(B_p^{(n)}, \|(\cdot, \dots, \cdot)\|_{p,a;\mu} \right) \longrightarrow \left(B_p^{(n)}, \|(\cdot, \dots, \cdot)\|_{p,a;\lambda} \right),$$

and the norm of T_R is at most $F(a, b) n^{\frac{1}{\lambda} - \frac{1}{\mu} + \left| \frac{1}{p} - \frac{1}{2} \right|} \beta$ when

$$T_R : \left(B_p^{(n)}, \|(\cdot, \dots, \cdot)\|_{p,a;\mu} \right) \longrightarrow \left(B_p^{(n)}, \|(\cdot, \dots, \cdot)\|_{p,b;\lambda} \right).$$

Therefore, using the complex interpolation, we obtain the following diagram of interpolation for $t \in [0, 1]$

$$\begin{array}{ccc} & & (B_p^{(n)}, \|(\cdot, \dots, \cdot)\|_{p,a;\lambda}) \\ & \overset{T_R}{\nearrow} & \\ (B_p^{(n)}, \|(\cdot, \dots, \cdot)\|_{p,a;\mu}) & \xrightarrow{T_R} & (B_p^{(n)}, \|(\cdot, \dots, \cdot)\|_{p,\gamma(t);\lambda}) \\ & \underset{T_R}{\searrow} & \\ & & (B_p^{(n)}, \|(\cdot, \dots, \cdot)\|_{p,b;\lambda}). \end{array}$$

By Theorem 2.1, T_R satisfies

$$\|T_R(\overline{A})\|_{p,\gamma(t);\lambda} \leq F(a,a)^{1-t}F(a,b)^t n^{\frac{1}{\lambda}-\frac{1}{\mu}+\left|\frac{1}{p}-\frac{1}{2}\right|}\beta\|\overline{A}\|_{p,a;\mu}. \tag{3.3}$$

Now applying the Complex method to

$$\begin{array}{ccc}
(B_p^{(n)}, \|(\cdot,\dots,\cdot)\|_{p,a;\lambda}) & \searrow^{T_{R^{-1}}} & \\
(B_p^{(n)}, \|(\cdot,\dots,\cdot)\|_{p,\gamma(t);\lambda}) & \xrightarrow{T_{R^{-1}}} & (B_p^{(n)}, \|(\cdot,\dots,\cdot)\|_{p,a;\mu}) \\
(B_p^{(n)}, \|(\cdot,\dots,\cdot)\|_{p,b;\lambda}) & \nearrow_{T_{R^{-1}}} &
\end{array}$$

one obtains

$$\|T_{R^{-1}}(\overline{A})\|_{p,a;\mu} \leq F(a,a)^{1-t}F(b,a)^t n^{\frac{1}{\mu}-\frac{1}{\lambda}+\left|\frac{1}{p}-\frac{1}{2}\right|}\alpha\|\overline{A}\|_{p,\gamma(t);\lambda}. \tag{3.4}$$

Replacing in (3.4) $\overline{A}$ by $R\overline{A}$ we obtain

$$\|\overline{A}\|_{p,a;\mu} \leq F(a,a)^{1-t}F(b,a)^t n^{\frac{1}{\mu}-\frac{1}{\lambda}+\left|\frac{1}{p}-\frac{1}{2}\right|}\alpha\|R\overline{A}\|_{p,\gamma(t);\lambda}, \tag{3.5}$$

or equivalently

$$F(a,a)^{t-1}F(b,a)^{-t} n^{\frac{1}{\lambda}-\frac{1}{\mu}-\left|\frac{1}{p}-\frac{1}{2}\right|}\alpha^{-1}\|\overline{A}\|_{p,a;\mu} \leq \|T_R(\overline{A})\|_{p,\gamma(t);\lambda}. \tag{3.6}$$

Finally, the inequalities (3.3) and (3.6) complete the proof. ☐

We remark that the previous statement is a generalization of Th. 4.1 in [7] where $T_n = T_R$ with $R = \left(e^{(i\frac{2\pi(j-1)(k-1)}{n})}1\right)_{1\leq j,k\leq n}$ and $a^{-1/2}$ commutes with R for all $a \in Gl(H)^+$.

On the other hand, it is well known that if $x_1,\dots,x_n$ are non-negative numbers, $s \in \mathbb{R}$ and we denote $\mathcal{M}_s(\overline{x}) = \left(\frac{1}{n}\sum_{i=1}^n x_i^s\right)^{1/s}$ then for $0 < s < s'$, $\mathcal{M}_s(\overline{x}) \leq \mathcal{M}_{s'}(\overline{x})$.

If we denote $\|\overline{B}\| = (\|B_1\|_p,\dots,\|B_n\|_p)$ and we consider $1 < p \leq 2$, then it holds for $t \in [0,1]$ and $\frac{1}{s_t} = \frac{1-t}{p} + \frac{t}{q}$ that

$$\mathcal{M}_{s_t}(\|\overline{B}\|) \leq \mathcal{M}_q(\|\overline{B}\|) \leq r^{\frac{2}{p}-1}\beta^{\frac{2}{q}}n^{\frac{-1}{q}}\left(\sum_{j=1}^n \|A_j\|_p^p\right)^{\frac{1}{p}},$$

or equivalently

$$\left(\sum_{j=1}^n \|B_j\|_p^{s_t}\right)^{\frac{1}{s_t}} \leq r^{\frac{2}{p}-1}\beta^{\frac{2}{q}}n^{\frac{1}{s_t}-\frac{1}{q}}\left(\sum_{j=1}^n \|A_j\|_p^p\right)^{\frac{1}{p}}. \tag{3.7}$$

Analogously, for $2 \leq p < \infty$ we get

$$\left(\sum_{j=1}^n \|A_j\|_p^p\right)^{\frac{1}{p}} \leq \rho^{1-\frac{2}{p}}\alpha^{\frac{2}{p}}n^{\frac{1}{q}-\frac{1}{s_t}}\left(\sum_{j=1}^n \|B_j\|_p^{s_t}\right)^{\frac{1}{s_t}} \tag{3.8}$$

where $\frac{1}{s_t} = \frac{1-t}{q} + \frac{t}{p}$.

Now we can use the interpolation method with the inequalities (3.7) and (3.1) (or (3.8) and (3.1)).

If we consider the following diagram of interpolation with $1 < p \leq 2$ and $t \in [0,1]$,

$$\begin{array}{ccc} & & (B_p^{(n)}, \|(\cdot,\ldots,\cdot)\|_{p,1;p}) \\ & \overset{T_R}{\nearrow} & \\ (B_p^{(n)}, \|(\cdot,\ldots,\cdot)\|_{p,1;p}) & \xrightarrow{T_R} & (B_p^{(n)}, \|(\cdot,\ldots,\cdot)\|_{p,1;s_t}) \\ & \underset{T_R}{\searrow} & \\ & & (B_p^{(n)}, \|(\cdot,\ldots,\cdot)\|_{p,1;q}). \end{array}$$

By Theorem 2.1 and (3.1), T_R satisfies

$$\|T_R(\overline{A})\|_{p,1;s_t} \leq \left(n^{f(p)}\beta\right)^{1-t}\left(r^{\frac{2}{p}-1}\beta^{\frac{2}{q}}\right)^t \|\overline{A}\|_{p,1;p}. \tag{3.9}$$

Finally, from the inequalities (3.7) and (3.9) we obtain

$$\left(\sum_{j=1}^n \|B_j\|_p^{s_t}\right)^{\frac{1}{s_t}} \leq \min\left\{r^{\frac{2}{p}-1}\beta^{\frac{2}{q}}n^{\frac{1}{s_t}-\frac{1}{q}}, n^{f(p)(1-t)}\beta^{1+t(\frac{2}{q}-1)}r^{(\frac{2}{p}-1)t}\right\}\left(\sum_{j=1}^n \|A_j\|_p^p\right)^{\frac{1}{p}}.$$

We can summarize the previous facts in the following statement.

Theorem 3.2. *Let* $\overline{A} \in B_p^{(n)}$ *and* $B = R\overline{A}$, *where* $R = (R_{jk})$ *is invertible. Let* $r = \max \|R_{jk}\|$, $\rho = \max \|(R^{-1})_{jk}\|$ *and* q *be the conjugate exponent of* p. *Then, for* $t \in [0,1]$ *we get*

$$\left(\sum_{j=1}^n \|A_j\|_p^p\right)^{\frac{1}{p}} \leq \min\left\{\rho^{1-\frac{2}{p}}\alpha^{\frac{2}{p}}n^{\frac{1}{q}-\frac{1}{s_t}}, n^{f(p)t}\alpha^{t+(1-t)\frac{2}{p}}\rho^{(1-\frac{2}{p})(1-t)}\right\}\left(\sum_{j=1}^n \|B_j\|_p^{s_t}\right)^{\frac{1}{s\,t}}$$

if $2 \leq p$ *and* $\frac{1}{s_t} = \frac{1-t}{q} + \frac{t}{p}$, *or*

$$\left(\sum_{j=1}^n \|B_j\|_p^{s_t}\right)^{\frac{1}{s_t}} \leq \min\left\{r^{\frac{2}{p}-1}\beta^{\frac{2}{q}}n^{\frac{1}{s_t}-\frac{1}{q}}, n^{f(p)(1-t)}\beta^{1+t(\frac{2}{q}-1)}r^{(\frac{2}{p}-1)t}\right\}\left(\sum_{j=1}^n \|A_j\|_p^p\right)^{\frac{1}{p}},$$

if $1 < p \leq 2$ *and* $\frac{1}{s_t} = \frac{1-t}{p} + \frac{t}{q}$.

Finally using the Finsler norm $\|(\cdot,\ldots,\cdot)\|_{p,a;s}$, Calderón's method and the previous inequalities we obtain:

Corollary 3.3. *Let* $a, b \in Gl(H)^+$, $\overline{A} \in B_p^{(n)}$ *and* $B = R\overline{A}$, *where* $R = (R_{jk})$ *is invertible. Let* $r = \max \|R_{jk}\|$, $\rho = \max \|(R^{-1})_{jk}\|$ *and* q *be the conjugate exponent of* p. *Then, for* $t, u \in [0,1]$ *we get*

$$\left(\sum_{j=1}^n \|A_j\|_{p,a}^p\right)^{\frac{1}{p}} \leq F(a,a)^{1-u}F(b,a)^u M_1 \left(\sum_{j=1}^n \|B_j\|_{p,\gamma_{a,b}(u)}^{s_t}\right)^{\frac{1}{s_t}},$$

if $2 \leq p$, $\frac{1}{s_t} = \frac{1-t}{q} + \frac{t}{p}$ *and*

$$M_1 = M_1(R,p,t) = \min\left\{\rho^{1-\frac{2}{p}}\alpha^{\frac{2}{p}}n^{\frac{1}{q}-\frac{1}{s_t}}, n^{f(p)t}\alpha^{t+(1-t)\frac{2}{p}}\rho^{(1-\frac{2}{p})(1-t)}\right\}$$

or

$$\left(\sum_{j=1}^n \|B_j\|_{p,\gamma_{a,b}(u)}^{s_t}\right)^{\frac{1}{s_t}} \leq F(a,a)^{1-u}F(a,b)^u M_2 \left(\sum_{j=1}^n \|A_j\|_p^p\right)^{\frac{1}{p}},$$

if $1 < p \leq 2$, $\frac{1}{s_t} = \frac{1-t}{p} + \frac{t}{q}$ *and*

$$M_2 = M_2(R,p,t) = \min\left\{ r^{\frac{2}{p}-1}\beta^{\frac{2}{q}} n^{\frac{1}{s_t}-\frac{1}{q}}, n^{f(p)(1-t)}\beta^{1+t(\frac{2}{q}-1)} r^{(\frac{2}{p}-1)t} \right\}.$$

REFERENCES

[1] E. ANDRUCHOW, G. CORACH, M. MILMAN AND D. STOJANOFF, Geodesics and interpolation, *Revista de la Unión Matemática Argentina*, **40**(3-4) (1997), 83–91.

[2] J. BERGH AND J. LÖFSTRÖM, *Interpolation Spaces. An Introduction*, Springer-Verlag, New York, 1976.

[3] R. BHATIA AND F. KITTANEH, Clarkson inequalities with several operators, *Bull. London Math. Soc.*, **36**(6) (2004), 820–832.

[4] R. BOAS, Some uniformly convex spaces, *Bull. Amer. Math. Soc.*, **46** (1940), 304–311.

[5] A. CALDERÓN, Intermediate spaces and interpolation, the complex method, *Studia Math.*, **24** (1964), 113–190.

[6] J.A. CLARKSON, Uniformly convex spaces, *Trans. Amer. Math. Soc.*, **40** (1936), 396–414.

[7] C. CONDE, Geometric interpolation in p-Schatten class, *J. Math. Anal. Appl.*, **340**(2) (2008), 920–931.

[8] G. CORACH, H. PORTA AND L. RECHT, The geometry of the space of selfadjoint invertible elements of a C*-algebra, *Integra. Equ. Oper. Theory*, **16**(3) (1993), 333–359.

[9] I. GOHBERG AND M. KREIN, *Introduction to the Theory of Linear Nonselfadjoint Operators*, American Mathematical Society, Vol. 18, Providence, R. I.,1969.

[10] O. HIRZALLAH AND F. KITTANEH, Non-commutative Clarkson inequalities for unitarily invariants norms, *Paciic J. Math.*, **202** (2002), 363–369.

[11] O. HIRZALLAH AND F. KITTANEH, Non-commutative Clarkson inequalities for n-tuples of operators, *Integr. Equ. Oper. Theory*, **60** (2008), 369–379.

[12] E. KISSIN, On Clarkson-McCarthy inequalities for n-tuples of operators, *Proc. Amer. Math. Soc.*, **135**(8) (2007), 2483–2495.

[13] S. KREIN, J. PETUNIN AND E. SEMENOV, *Interpolation of linear operators*. Translations of Mathematical Monographs, Vol. 54, American Mathematical Society, Providence, R. I., 1982.

[14] C. MCCARTHY, c_p, *Israel J. Math.*, **5** (1967), 249–271.

[15] G. MOSTOW, Some new decomposition theorems for semi-simple groups, *Mem. Amer. Math. Soc.*, **14** (1955), 31–54.

[16] B. SIMON, *Trace Ideals and their Applications. Second edition*, Mathematical Surveys and Monographs, 120. American Mathematical Society, Providence, RI, 2005.