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TURÁN-TYPE INEQUALITIES FOR SOME SPECIAL FUNCTIONS

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ABSTRACT. We use a generalization of the Schwarz inequality to give a short proof of new Turán-type inequalities for polygamma and Riemann zeta functions.

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1. INTRODUCTION

P. Turán [8] proved that the Legendre polynomials $P_n(x)$ satisfy the determinantal inequality

$$\begin{vmatrix} P_n(x) & P_{n+1}(x) \\ P_{n+1}(x) & P_{n+2}(x) \end{vmatrix} \le 0, \qquad -1 \le x \le 1 \tag{1.1}$$

where $n = 0, 1, 2, \dots$ and equality occurs only if $x = \pm 1$. This classical result has been extended in several directions: ultraspherical polynomials, Laguerre and Hermite polynomials, Bessel functions of the first kind, modified Bessel functions, etc. In view of the interest in inequalities of the type (1.1), Karlin and Szegö named determinants such as (1.1) *Turánians*. The proof given by Turán is based on the recurrence relation [7, p. 81],

$$\begin{cases} (n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), & n = 1, 2, \dots \\ P_{-1}(x) = 0, \qquad P_0(x) = 1. \end{cases} \tag{1.2}$$

and on the differential relation [7, p. 83],

$$\left(1 - x^2\right) P_n'(x) = nP_{n-1}(x) - nxP_n(x). \tag{1.3}$$

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L. Lorch [6] established Turán-type inequalities for the positive zeros $c_{\nu k}$, $k = 1, 2, \ldots$ of the general Bessel function

$$C_\nu(x) = J_\nu(x)\cos\alpha - Y_\nu(x)\sin\alpha, \qquad 0 \le \alpha < \pi,$$

where $J_\nu(x)$ and $Y_\nu(x)$ denote the Bessel functions of the first and the second kind respectively.

Finally, the corresponding results for the positive zeros $c'_{\nu k}$ $\nu \ge 0$, $k = 1, 2, \ldots$ of the derivative $C'_\nu(x) = \frac{d}{dx}\, C_\nu(x)$ and for the zeros of ultraspherical, Laguerre and Hermite polynomials have been established in [5], [3] and [2], respectively.

The aim of this paper is to prove new Turán-type inequalities for the polygamma and Riemann zeta functions. The approach used in the present paper is different from that used in the above mentioned papers and based, prevalently, on Sturm theory. Here our main tool is the following generalization of the Schwarz inequality

$$\int_a^b g(t)\,[f(t)]^m\,dt \cdot \int_a^b g(t)\,[f(t)]^n\,dt \ge \left[\int_a^b g(t)\,[f(t)]^{\frac{m+n}{2}}\,dt\right]^2 \tag{1.4}$$

where f and g are two nonnegative functions of a real variable and m and n belonging to a set S of real numbers, such that the integrals in (1.4) exist.

2. The Results

Theorem 2.1. *For $n = 1, 2, \ldots$ we denote by $\psi_n(x) = \psi^{(n)}(x)$ the polygamma functions defined as the n-th derivative of psi function*

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \qquad x > 0$$

with the usual notation for the gamma function. Then

$$\psi_m(x)\psi_n(x) \ge \psi^2_{\frac{m+n}{2}}(x), \tag{2.1}$$

where $\frac{m+n}{2}$ is an integer.

Proof. The polygamma functions have the following integral representation

$$\psi_n(x) = (-1)^{n+1}\int_0^\infty \frac{t^n}{1-e^{-t}}e^{-xt}dt, \qquad x > 0, \;\; n = 1, 2, \ldots \tag{2.2}$$

We choose the integers m and n both even or odd, in such a way that $(m+n)/2$ is an integer. By (1.4) with $g(t) = \frac{e^{-xt}}{1-e^{-t}}$, $f(t) = t$ and $a = 0$, $b = +\infty$, we get

$$\int_0^\infty \frac{e^{-xt}}{1-e^{-t}}t^n dt \cdot \int_0^\infty \frac{e^{-xt}}{1-e^{-t}}t^m dt \ge \left[\int_0^\infty \frac{e^{-xt}}{1-e^{-t}}t^{\frac{m+n}{2}}dt\right]^2, \tag{2.3}$$

that is

$$\psi_m(x)\,\psi_n(x) \ge \psi^2_{\frac{m+n}{2}}(x), \qquad m, n = 1, 3, 5, \ldots \text{ or } m, n = 2, 4, 6, \ldots. \tag{2.4}$$

The proof is complete. □

Remark 2.2. When $m = n + 2$ we find

$$\frac{\psi_n(x)}{\psi_{n+1}(x)} \ge \frac{\psi_{n+1}(x)}{\psi_{n+2}(x)}, \qquad n = 1, 2, \ldots, \;\; x > 0. \tag{2.5}$$

Theorem 2.3. *We denote by $\zeta(s)$ the Riemann zeta function. Then*

$$(s+1)\frac{\zeta(s)}{\zeta(s+1)} \ge s\frac{\zeta(s+1)}{\zeta(s+2)}, \qquad \forall s > 1. \tag{2.6}$$

Proof. For $s > 1$ the Riemann zeta function satisfies the integral relation

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt, \qquad s > 1. \tag{2.7}$$

By (1.4) with $g(t) = \frac{1}{e^t-1}$, $f(t) = t$ and $a = 0$, $b = +\infty$, we get

$$\int_0^\infty \frac{t^{s-1}}{e^t - 1} dt \cdot \int_0^\infty \frac{t^{s+1}}{e^t - 1} dt \geq \left[\int_0^\infty \frac{t^s}{e^t - 1} dt \right]^2. \tag{2.8}$$

Further, using (2.7) this inequality becomes

$$\zeta(s)\Gamma(s)\zeta(s+2)\Gamma(s+2) \geq \zeta^2(s+1)\Gamma^2(s+1) \tag{2.9}$$

or, by the functional relation $\Gamma(x+1) = x\Gamma(x)$,

$$(s+1)\zeta(s)\zeta(s+2) \geq s\zeta^2(s+1) \tag{2.10}$$

which is equivalent to the conclusion of Theorem 2.3. □

Concluding remark. *Many other Turán-type inequalities can be obtained for the functions which admit integral representations of the type (2.2). For example starting from the integral representation for the exponential integral function* [1, p. 228, 5.1.4],

$$E_n(x) = \int_1^\infty e^{-xt}\, t^{-n}\, dt, \qquad n = 0, 1, \ldots, \quad x > 0,$$

and using inequality (1.4) we find

$$E_n(x)\, E_m(x) \geq E_{\frac{n+m}{2}}(x).$$

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