

Journal of Inequalities in Pure and Applied Mathematics

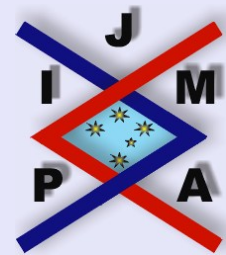
A POTPOURRI OF SCHWARZ RELATED INEQUALITIES IN INNER PRODUCT SPACES (II)

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ISSN (electronic): 1443-5756
196-05



volume 7, issue 1, article 14,
2006.

Received 28 June, 2005;
accepted 17 September, 2005.

Communicated by: A. Lupaş

Abstract

Contents



Home Page

Go Back

Close

Quit

Abstract

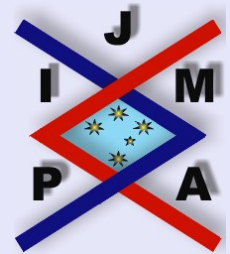
Further inequalities related to the Schwarz inequality in real or complex inner product spaces are given.

2000 Mathematics Subject Classification: 46C05, 26D15.

Key words: Schwarz inequality, Inner product spaces, Reverse inequalities.

Contents

1	Introduction	3
2	Quadratic Schwarz Related Inequalities	6
3	Other Inequalities	15
4	Applications for the Triangle Inequality	22
References		



A Potpourri of Schwarz Related Inequalities in Inner Product Spaces (II)

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 2 of 27

1. Introduction

Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field \mathbb{K} . One of the most important inequalities in inner product spaces with numerous applications is the Schwarz inequality, that may be written in two forms:

$$(1.1) \quad |\langle x, y \rangle|^2 \leq \|x\|^2 \|y\|^2, \quad x, y \in H \quad (\text{quadratic form})$$

or, equivalently,

$$(1.2) \quad |\langle x, y \rangle| \leq \|x\| \|y\|, \quad x, y \in H \quad (\text{simple form}).$$

The case of equality holds in (1.1) or in (1.2) if and only if the vectors x and y are linearly dependent.

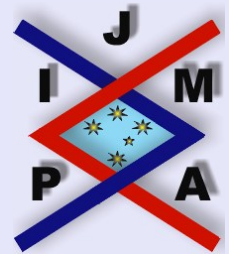
In the previous paper [6], several results related to Schwarz inequalities have been established. We recall few of them below:

1. If $x, y \in H \setminus \{0\}$ and $\|x\| \geq \|y\|$, then

$$(1.3) \quad \|x\| \|y\| - \operatorname{Re} \langle x, y \rangle \leq \begin{cases} \frac{1}{2} r^2 \left(\frac{\|x\|}{\|y\|} \right)^{r-1} \|x - y\|^2 & \text{if } r \geq 1 \\ \frac{1}{2} \left(\frac{\|x\|}{\|y\|} \right)^{1-r} \|x - y\|^2 & \text{if } r < 1. \end{cases}$$

2. If $(H; \langle \cdot, \cdot \rangle)$ is complex, $\alpha \in \mathbb{C}$ with $\operatorname{Re} \alpha, \operatorname{Im} \alpha > 0$ and $x, y \in H$ are such that

$$(1.4) \quad \left\| x - \frac{\operatorname{Im} \alpha}{\operatorname{Re} \alpha} \cdot y \right\| \leq r$$



A Potpourri of Schwarz Related
Inequalities in Inner Product
Spaces (II)

S.S. Dragomir

Title Page

Contents

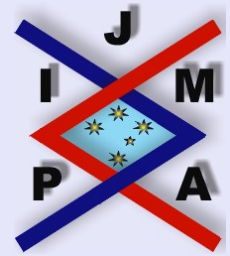


Go Back

Close

Quit

Page 3 of 27



**A Potpourri of Schwarz Related
Inequalities in Inner Product
Spaces (II)**

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 4 of 27

then

$$(1.5) \quad \|x\| \|y\| - \operatorname{Re} \langle x, y \rangle \leq \frac{1}{2} \cdot \frac{\operatorname{Re} \alpha}{\operatorname{Im} \alpha} r^2.$$

3. If $\alpha \in \mathbb{K} \setminus \{0\}$, then for any $x, y \in H$

$$(1.6) \quad \|x\| \|y\| - \operatorname{Re} \left[\frac{\alpha^2}{|\alpha|^2} \langle x, y \rangle \right] \leq \frac{1}{2} \cdot \frac{[|\operatorname{Re} \alpha| \|x - y\| + |\operatorname{Im} \alpha| \|x + y\|]^2}{|\alpha|^2}.$$

4. If $p \geq 1$, then for any $x, y \in H$ one has

$$(1.7) \quad \|x\| \|y\| - \operatorname{Re} \langle x, y \rangle \leq \frac{1}{2} \times \begin{cases} [(\|x\| + \|y\|)^{2p} - \|x + y\|^{2p}]^{\frac{1}{p}}, \\ [\|x - y\|^{2p} - \|\|x\| - \|y\|\|^{2p}]^{\frac{1}{p}}. \end{cases}$$

5. If $\alpha, \gamma > 0$ and $\beta \in \mathbb{K}$ with $|\beta|^2 \geq \alpha\gamma$ then for $x, a \in H$ with $a \neq 0$ and

$$(1.8) \quad \left\| x - \frac{\beta}{\alpha} a \right\| \leq \frac{(|\beta|^2 - \alpha\gamma)^{\frac{1}{2}}}{\alpha} \|a\|,$$

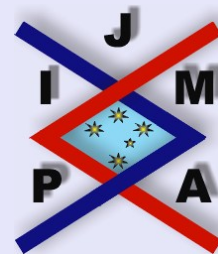
one has

$$(1.9) \quad \|x\| \|a\| \leq \frac{\operatorname{Re} \beta \operatorname{Re} \langle x, a \rangle + \operatorname{Im} \beta \operatorname{Im} \langle x, a \rangle}{\sqrt{\alpha\gamma}} \leq \frac{|\beta| |\langle x, a \rangle|}{\sqrt{\alpha\gamma}}$$

and

$$(1.10) \quad \|x\|^2 \|a\|^2 - |\langle x, a \rangle|^2 \leq \frac{|\beta|^2 - \alpha\gamma}{\alpha\gamma} |\langle x, a \rangle|^2.$$

The aim of this paper is to provide other results related to the Schwarz inequality. Applications for reversing the generalised triangle inequality are also given.



**A Potpourri of Schwarz Related
Inequalities in Inner Product
Spaces (II)**

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 5 of 27

2. Quadratic Schwarz Related Inequalities

The following result holds.

Theorem 2.1. *Let $(H; \langle \cdot, \cdot \rangle)$ be a complex inner product space and $x, y \in H$, $\alpha \in [0, 1]$. Then*

$$(2.1) \quad \begin{aligned} & [\alpha \|ty - x\|^2 + (1 - \alpha) \|ity - x\|^2] \|y\|^2 \\ & \geq \|x\|^2 \|y\|^2 - [(1 - \alpha) \operatorname{Im} \langle x, y \rangle + \alpha \operatorname{Re} \langle x, y \rangle]^2 \geq 0 \end{aligned}$$

for any $t \in \mathbb{R}$.

Proof. Firstly, recall that for a quadratic polynomial $P : \mathbb{R} \rightarrow \mathbb{R}$, $P(t) = at^2 + 2bt + c$, $a > 0$, we have that

$$(2.2) \quad \inf_{t \in \mathbb{R}} P(t) = P\left(-\frac{b}{a}\right) = \frac{ac - b^2}{a}.$$

Now, consider the polynomial $P : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$(2.3) \quad P(t) := \alpha \|ty - x\|^2 + (1 - \alpha) \|ity - x\|^2.$$

Since

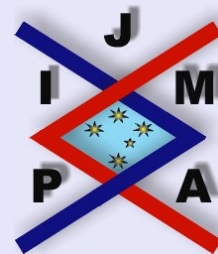
$$\|ty - x\|^2 = t^2 \|y\|^2 - 2t \operatorname{Re} \langle x, y \rangle + \|x\|^2$$

and

$$\|ity - x\|^2 = t^2 \|y\|^2 - 2t \operatorname{Im} \langle x, y \rangle + \|x\|^2,$$

hence

$$P(t) = t^2 \|y\|^2 - 2t [\alpha \operatorname{Re} \langle x, y \rangle + (1 - \alpha) \operatorname{Im} \langle x, y \rangle] + \|x\|^2.$$



A Potpourri of Schwarz Related
Inequalities in Inner Product
Spaces (II)

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 6 of 27

By the definition of P (see (2.3)), we observe that $P(t) \geq 0$ for every $t \in \mathbb{R}$, therefore $\frac{1}{4}\Delta \leq 0$, i.e.,

$$[(1 - \alpha) \operatorname{Im} \langle x, y \rangle + \alpha \operatorname{Re} \langle x, y \rangle]^2 - \|x\|^2 \|y\|^2 \leq 0,$$

proving the second inequality in (2.1).

The first inequality follows by (2.2) and the theorem is proved. □

The following particular cases are of interest.

Corollary 2.2. *For any $x, y \in H$ one has the inequalities:*

$$(2.4) \quad \|ty - x\|^2 \|y\|^2 \geq \|\alpha\|^2 \|y\|^2 - [\operatorname{Re} \langle x, y \rangle]^2 \geq 0;$$

$$(2.5) \quad \|ity - x\|^2 \|y\|^2 \geq \|\alpha\|^2 \|y\|^2 - [\operatorname{Im} \langle x, y \rangle]^2 \geq 0;$$

and

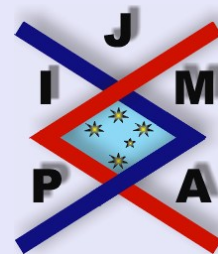
$$(2.6) \quad \frac{1}{2} [\|ty - x\|^2 + \|ity - x\|^2] \|y\|^2 \geq \|x\|^2 \|y\|^2 - \left(\frac{\operatorname{Im} \langle x, y \rangle + \operatorname{Re} \langle x, y \rangle}{2} \right)^2 \geq 0,$$

for any $t \in \mathbb{R}$.

The following corollary may be stated as well:

Corollary 2.3. *Let $x, y \in H$ and $M_i, m_i \in \mathbb{R}$, $i \in \{1, 2\}$ such that $M_i \geq m_i > 0$, $i \in \{1, 2\}$. If either*

$$(2.7) \quad \operatorname{Re} \langle M_1 y - x, x - m_1 y \rangle \geq 0 \quad \text{and} \quad \operatorname{Re} \langle M_2 i y - x, x - i m_2 y \rangle \geq 0,$$



**A Potpourri of Schwarz Related
Inequalities in Inner Product
Spaces (II)**

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 7 of 27

or, equivalently,

$$(2.8) \quad \left\| x - \frac{M_1 + m_1}{2} y \right\| \leq \frac{1}{2} (M_1 - m_1) \|y\| \quad \text{and}$$

$$\left\| x - \frac{M_2 + m_2}{2} iy \right\| \leq \frac{1}{2} (M_2 - m_2) \|y\|$$

hold, then

$$(2.9) \quad (0 \leq) \|x\|^2 \|y\|^2 - [(1 - \alpha) \operatorname{Im} \langle x, y \rangle + \alpha \operatorname{Re} \langle x, y \rangle]^2$$

$$\leq \frac{1}{4} \|y\|^4 [\alpha (M_1 - m_1)^2 + (1 - \alpha) (M_2 - m_2)^2]$$

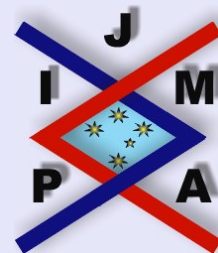
for any $\alpha \in [0, 1]$.

Proof. It is easy to see that, if $x, z, Z \in H$, then the following statements are equivalent:

- (i) $\operatorname{Re} \langle Z - x, x - z \rangle \geq 0$,
- (ii) $\left\| x - \frac{z+Z}{2} \right\| \leq \frac{1}{2} \|Z - z\|$.

Utilising this property one may simply realize that the statements (2.7) and (2.8) are equivalent.

Now, on making use of (2.8) and (2.1), one may deduce the desired inequality (2.9). □



**A Potpourri of Schwarz Related
Inequalities in Inner Product
Spaces (II)**

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 8 of 27

Remark 1. *If one assumes that $M_1 = M_2 = M$, $m_1 = m_2 = m$ in either (2.7) or (2.8), then*

$$(2.10) \quad (0 \leq) \|x\|^2 \|y\|^2 - [(1 - \alpha) \operatorname{Im} \langle x, y \rangle + \alpha \operatorname{Re} \langle x, y \rangle]^2 \\ \leq \frac{1}{4} \|y\|^4 (M - m)^2$$

for each $\alpha \in [0, 1]$.

Remark 2. *Corollary 2.3 may be seen as a potential source of some reverse results for the Schwarz inequality. For instance, if $x, y \in H$ and $M \geq m > 0$ are such that either*

$$(2.11) \quad \operatorname{Re} \langle My - x, x - my \rangle \geq 0 \quad \text{or} \quad \left\| x - \frac{M + m}{2} y \right\| \leq \frac{1}{2} (M - m) \|y\|$$

hold, then

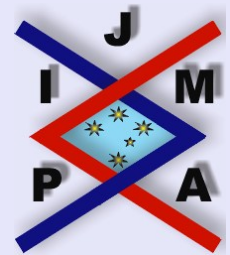
$$(2.12) \quad (0 \leq) \|x\|^2 \|y\|^2 - [\operatorname{Re} \langle x, y \rangle]^2 \leq \frac{1}{4} (M - m)^2 \|y\|^4.$$

If $x, y \in H$ and $N \geq n > 0$ are such that either

$$(2.13) \quad \operatorname{Re} \langle Niy - x, x - niy \rangle \geq 0 \quad \text{or} \quad \left\| x - \frac{N + n}{2} iy \right\| \leq \frac{1}{2} (N - n) \|y\|$$

hold, then

$$(2.14) \quad (0 \leq) \|x\|^2 \|y\|^2 - [\operatorname{Im} \langle x, y \rangle]^2 \leq \frac{1}{4} (N - n)^2 \|y\|^4.$$



A Potpourri of Schwarz Related
Inequalities in Inner Product
Spaces (II)

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 9 of 27

We notice that (2.12) is an improvement of the inequality

$$(0 \leq) \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \leq \frac{1}{4} (M - m)^2 \|y\|^4$$

that has been established in [4] under the same condition (2.11) given above.

The following result may be stated as well.

Theorem 2.4. Let $(H; \langle \cdot, \cdot \rangle)$ be a real or complex inner product space and $x, y \in H, \alpha \in [0, 1]$. Then

$$(2.15) \quad \begin{aligned} & [\alpha \|ty - x\|^2 + (1 - \alpha) \|y - tx\|^2] [\alpha \|y\|^2 + (1 - \alpha) \|x\|^2] \\ & \geq [(1 - \alpha) \|x\|^2 + \alpha \|y\|^2] [\alpha \|x\|^2 + (1 - \alpha) \|y\|^2] \\ & \quad - [\operatorname{Re} \langle x, y \rangle]^2 \geq 0 \end{aligned}$$

for any $t \in \mathbb{R}$.

Proof. Consider the polynomial $P : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$(2.16) \quad P(t) := \alpha \|ty - x\|^2 + (1 - \alpha) \|y - tx\|^2.$$

Since

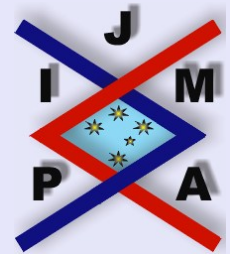
$$\|ty - x\|^2 = t^2 \|y\|^2 - 2t \operatorname{Re} \langle x, y \rangle + \|x\|^2$$

and

$$\|y - tx\|^2 = t^2 \|x\|^2 - 2t \operatorname{Re} \langle x, y \rangle + \|y\|^2,$$

hence

$$P(t) = [\alpha \|y\|^2 + (1 - \alpha) \|x\|^2] t^2 - 2t \operatorname{Re} \langle x, y \rangle + [\alpha \|x\|^2 + (1 - \alpha) \|y\|^2]$$



**A Potpourri of Schwarz Related
Inequalities in Inner Product
Spaces (II)**

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 10 of 27

for any $t \in \mathbb{R}$.

By the definition of P (see (2.16)), we observe that $P(t) \geq 0$ for every $t \in \mathbb{R}$, therefore $\frac{1}{4}\Delta \leq 0$, i.e., the second inequality in (2.15) holds true.

The first inequality follows by (2.2) and the theorem is proved. \square

Remark 3. We observe that if either $\alpha = 0$ or $\alpha = 1$, then (2.15) will generate the same reverse of the Schwarz inequality as (2.4) does.

Corollary 2.5. If $x, y \in H$, then

$$(2.17) \quad \frac{\|ty - x\|^2 + \|y - tx\|^2}{2} \cdot \frac{\|x\|^2 + \|y\|^2}{2} \geq \left(\frac{\|x\|^2 + \|y\|^2}{2} \right)^2 - [\operatorname{Re} \langle x, y \rangle]^2 \geq 0$$

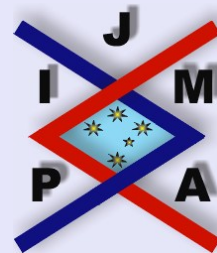
for any $t \in \mathbb{R}$ and

$$(2.18) \quad \|x \pm y\|^2 [\alpha \|y\|^2 + (1 - \alpha) \|x\|^2] \geq [(1 - \alpha) \|x\|^2 + \alpha \|y\|^2] [\alpha \|x\|^2 + (1 - \alpha) \|y\|^2] - [\operatorname{Re} \langle x, y \rangle]^2 \geq 0$$

for any $\alpha \in [0, 1]$.

In particular, we have

$$(2.19) \quad \|x \pm y\|^2 \cdot \left(\frac{\|x\|^2 + \|y\|^2}{2} \right) \geq \left(\frac{\|x\|^2 + \|y\|^2}{2} \right)^2 - [\operatorname{Re} \langle x, y \rangle]^2 \geq 0.$$



A Potpourri of Schwarz Related
Inequalities in Inner Product
Spaces (II)

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 11 of 27

In [7, p. 210], C.S. Lin has proved the following reverse of the Schwarz inequality in real or complex inner product spaces $(H; \langle \cdot, \cdot \rangle)$:

$$(0 \leq) \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \leq \frac{1}{r^2} \|x\|^2 \|x - ry\|^2$$

for any $r \in \mathbb{R}, r \neq 0$ and $x, y \in H$.

The following slightly more general result may be stated:

Theorem 2.6. *Let $(H; \langle \cdot, \cdot \rangle)$ be a real or complex inner product space. Then for any $x, y \in H$ and $\alpha \in \mathbb{K} (\mathbb{C}, \mathbb{R})$ with $\alpha \neq 0$ we have*

$$(2.20) \quad (0 \leq) \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \leq \frac{1}{|\alpha|^2} \|x\|^2 \|x - \alpha y\|^2.$$

The case of equality holds in (2.20) if and only if

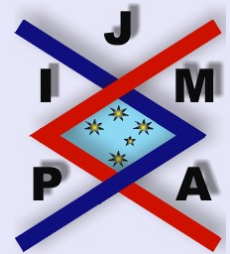
$$(2.21) \quad \operatorname{Re} \langle x, \alpha y \rangle = \|x\|^2.$$

Proof. Observe that

$$\begin{aligned} I(\alpha) &:= \|x\|^2 \|x - \alpha y\|^2 - |\alpha|^2 [\|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2] \\ &= \|x\|^2 [\|x\|^2 - 2 \operatorname{Re} [\bar{\alpha} \langle x, y \rangle] + |\alpha|^2 \|y\|^2] \\ &\quad - |\alpha|^2 [\|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2] \\ &= \|x\|^4 - 2 \|x\|^2 \operatorname{Re} [\bar{\alpha} \langle x, y \rangle] + |\alpha|^2 |\langle x, y \rangle|^2. \end{aligned}$$

Since

$$(2.22) \quad \operatorname{Re} [\bar{\alpha} \langle x, y \rangle] \leq |\alpha| |\langle x, y \rangle|,$$



A Potpourri of Schwarz Related Inequalities in Inner Product Spaces (II)

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 12 of 27

hence

$$(2.23) \quad \begin{aligned} I(\alpha) &\geq \|x\|^4 - 2\|x\|^2 |\alpha| |\langle x, y \rangle| + |\alpha|^2 |\langle x, y \rangle|^2 \\ &= (\|x\|^2 - |\alpha| |\langle x, y \rangle|)^2 \geq 0. \end{aligned}$$

Conversely, if (2.21) holds true, then $I(\alpha) = 0$, showing that the equality case holds in (2.20).

Now, if the equality case holds in (2.20), then we must have equality in (2.22) and in (2.23) implying that

$$\operatorname{Re}[\langle x, \alpha y \rangle] = |\alpha| |\langle x, y \rangle| \quad \text{and} \quad |\alpha| |\langle x, y \rangle| = \|x\|^2$$

which imply (2.21). □

The following result may be stated.

Corollary 2.7. *Let $(H; \langle \cdot, \cdot \rangle)$ be as above and $x, y \in H$, $\alpha \in \mathbb{K} \setminus \{0\}$ and $r > 0$ such that $|\alpha| \geq r$. If*

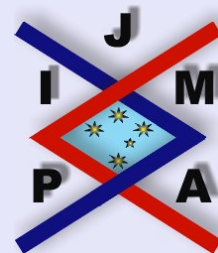
$$(2.24) \quad \|x - \alpha y\| \leq r \|y\|,$$

then

$$(2.25) \quad \frac{\sqrt{|\alpha|^2 - r^2}}{|\alpha|} \leq \frac{|\langle x, y \rangle|}{\|x\| \|y\|} (\leq 1).$$

Proof. From (2.24) and (2.20) we have

$$\|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \leq \frac{r^2}{|\alpha|^2} \|x\|^2 \|y\|^2,$$



**A Potpourri of Schwarz Related
Inequalities in Inner Product
Spaces (II)**

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 13 of 27

that is,

$$\frac{(|\alpha|^2 - r^2)}{|\alpha|^2} \|x\|^2 \|y\|^2 \leq |\langle x, y \rangle|^2,$$

which is clearly equivalent to (2.25). □

Remark 4. Since for $\Gamma, \gamma \in \mathbb{K}$ the following statements are equivalent

(i) $\operatorname{Re} \langle \Gamma y - x, x - \gamma y \rangle \geq 0,$

(ii) $\|x - \frac{\gamma + \Gamma}{2} \cdot y\| \leq \frac{1}{2} |\Gamma - \gamma| \|y\|,$

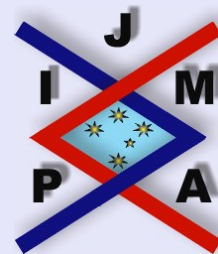
hence by Corollary 2.7 we deduce

$$(2.26) \quad \frac{2 [\operatorname{Re} (\Gamma \bar{\gamma})]^{\frac{1}{2}}}{|\Gamma + \gamma|} \leq \frac{|\langle x, y \rangle|}{\|x\| \|y\|},$$

provided $\operatorname{Re} (\Gamma \bar{\gamma}) > 0$, an inequality that has been obtained in a different way in [3].

Corollary 2.8. If $x, y \in H$, $\alpha \in \mathbb{K} \setminus \{0\}$ and $\rho > 0$ such that $\|x - \alpha y\| \leq \rho$, then

$$(2.27) \quad (0 \leq) \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \leq \frac{\rho^2}{|\alpha|^2} \|x\|^2.$$



A Potpourri of Schwarz Related Inequalities in Inner Product Spaces (II)

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 14 of 27

3. Other Inequalities

The following result holds.

Proposition 3.1. Let $x, y \in H \setminus \{0\}$ and $\varepsilon \in (0, \frac{1}{2}]$. If

$$(3.1) \quad (0 \leq) 1 - \varepsilon - \sqrt{1 - 2\varepsilon} \leq \frac{\|x\|}{\|y\|} \leq 1 - \varepsilon + \sqrt{1 - 2\varepsilon},$$

then

$$(3.2) \quad (0 \leq) \|x\| \|y\| - \operatorname{Re} \langle x, y \rangle \leq \varepsilon \|x - y\|^2.$$

Proof. If $x = y$, then (3.2) is trivial.

Suppose $x \neq y$. Utilising the inequality (2.5) from [6], we can state that

$$\frac{\|x\| \|y\| - \operatorname{Re} \langle x, y \rangle}{\|x - y\|^2} \leq \frac{2 \|x\| \|y\|}{(\|x\| + \|y\|)^2}$$

for any $x, y \in H \setminus \{0\}$, $x \neq y$.

Now, if we assume that

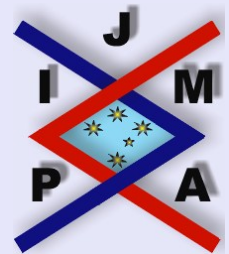
$$\frac{2 \|x\| \|y\|}{(\|x\| + \|y\|)^2} \leq \varepsilon,$$

then, after some manipulation, we get that

$$\varepsilon \|x\|^2 + 2(\varepsilon - 1) \|x\| \|y\| + \varepsilon \|y\|^2 \geq 0,$$

which, for $\varepsilon \in (0, \frac{1}{2}]$ and $y \neq 0$, is clearly equivalent to (3.1).

The proof is complete. □



A Potpourri of Schwarz Related
Inequalities in Inner Product
Spaces (II)

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 15 of 27

The following result may be stated:

Proposition 3.2. *Let $(H; \langle \cdot, \cdot \rangle)$ be a real or complex inner product space. Then for any $x, y \in H$ and $\varphi \in \mathbb{R}$ one has:*

$$\begin{aligned} \|x\| \|y\| &= [\cos 2\varphi \cdot \operatorname{Re} \langle x, y \rangle + \sin 2\varphi \cdot \operatorname{Im} \langle x, y \rangle] \\ &\leq \frac{1}{2} [|\cos \varphi| \|x - y\| + |\sin \varphi| \|x + y\|]^2. \end{aligned}$$

Proof. For $\varphi \in \mathbb{R}$, consider the complex number $\alpha = \cos \varphi - i \sin \varphi$. Then $\alpha^2 = \cos 2\varphi - i \sin 2\varphi$, $|\alpha| = 1$ and by the inequality (1.6) we deduce the desired result. \square

From a different perspective, we may consider the following results as well:

Theorem 3.3. *Let $(H; \langle \cdot, \cdot \rangle)$ be a real or complex inner product space, $\alpha \in \mathbb{K}$ with $|\alpha - 1| = 1$. Then for any $e \in H$ with $\|e\| = 1$ and $x, y \in H$, we have*

$$(3.3) \quad |\langle x, y \rangle - \alpha \langle x, e \rangle \langle e, y \rangle| \leq \|x\| \|y\|.$$

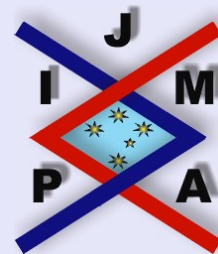
The equality holds in (3.3) if and only if there exists a $\lambda \in \mathbb{K}$ such that

$$(3.4) \quad \alpha \langle x, e \rangle e = x + \lambda y.$$

Proof. It is known that for $u, v \in H$, we have equality in the Schwarz inequality

$$(3.5) \quad |\langle u, v \rangle| \leq \|u\| \|v\|$$

if and only if there exists a $\lambda \in \mathbb{K}$ such that $u = \lambda v$.



A Potpourri of Schwarz Related
Inequalities in Inner Product
Spaces (II)

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 16 of 27

If we apply (3.5) for $u = \alpha \langle x, e \rangle e - x$, $v = y$, we get

$$(3.6) \quad |\langle \alpha \langle x, e \rangle e - x, y \rangle| \leq \| \alpha \langle x, e \rangle e - x \| \| y \|$$

with equality iff there exists a $\lambda \in \mathbb{K}$ such that

$$\alpha \langle x, e \rangle e = x + \lambda y.$$

Since

$$\begin{aligned} \| \alpha \langle x, e \rangle e - x \|^2 &= |\alpha|^2 |\langle x, e \rangle|^2 - 2 \operatorname{Re} [\alpha] |\langle x, e \rangle|^2 + \| x \|^2 \\ &= (|\alpha|^2 - 2 \operatorname{Re} [\alpha]) |\langle x, e \rangle|^2 + \| x \|^2 \\ &= (|\alpha - 1|^2 - 1) |\langle x, e \rangle|^2 + \| x \|^2 \\ &= \| x \|^2 \end{aligned}$$

and

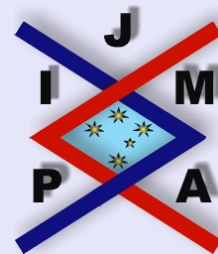
$$\langle \alpha \langle x, e \rangle e - x, y \rangle = \alpha \langle x, e \rangle \langle e, y \rangle - \langle x, y \rangle$$

hence by (3.6) we deduce the desired inequality (3.3). □

Remark 5. If $\alpha = 0$ in (3.3), then it reduces to the Schwarz inequality.

Remark 6. If $\alpha \neq 0$, then (3.3) is equivalent to

$$(3.7) \quad \left| \langle x, e \rangle \langle e, y \rangle - \frac{1}{\alpha} \langle x, y \rangle \right| \leq \frac{1}{|\alpha|} \| x \| \| y \|.$$



**A Potpourri of Schwarz Related
Inequalities in Inner Product
Spaces (II)**

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 17 of 27

Utilising the continuity property of modulus for complex numbers, i.e., $|z - w| \geq ||z| - |w||$ we then obtain

$$\left| |\langle x, e \rangle \langle e, y \rangle| - \frac{1}{|\alpha|} |\langle x, y \rangle| \right| \leq \frac{1}{|\alpha|} \|x\| \|y\|,$$

which implies that

$$(3.8) \quad |\langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{|\alpha|} [|\langle x, y \rangle| + \|x\| \|y\|].$$

For $e = \frac{z}{\|z\|}$, $z \neq 0$, we get from (3.8) that

$$(3.9) \quad |\langle x, z \rangle \langle z, y \rangle| \leq \frac{1}{|\alpha|} [|\langle x, y \rangle| + \|x\| \|y\|] \|z\|^2$$

for any $\alpha \in \mathbb{K} \setminus \{0\}$ with $|\alpha - 1| = 1$ and $x, y, z \in H$.

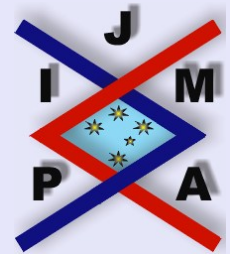
For $\alpha = 2$, we get from (3.9) the *Buzano inequality* [1]

$$(3.10) \quad |\langle x, z \rangle \langle z, y \rangle| \leq \frac{1}{2} [|\langle x, y \rangle| + \|x\| \|y\|] \|z\|^2$$

for any $x, y, z \in H$.

Remark 7. In the case of real spaces the condition $|\alpha - 1| = 1$ is equivalent to either $\alpha = 0$ or $\alpha = 2$. For $\alpha = 2$ we deduce from (3.7) that

$$(3.11) \quad \frac{1}{2} [|\langle x, y \rangle| - \|x\| \|y\|] \leq \langle x, e \rangle \langle e, y \rangle \leq \frac{1}{2} [|\langle x, y \rangle| + \|x\| \|y\|]$$



A Potpourri of Schwarz Related Inequalities in Inner Product Spaces (II)

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 18 of 27

for any $x, y \in H$ and $e \in H$ with $\|e\| = 1$, which implies Richard's inequality [8]:

$$(3.12) \quad \frac{1}{2} [\langle x, y \rangle - \|x\| \|y\|] \|z\|^2 \leq \langle x, z \rangle \langle z, y \rangle \leq \frac{1}{2} [\langle x, y \rangle + \|x\| \|y\|] \|z\|^2,$$

for any $x, y, z \in H$.

The following result concerning a generalisation for orthonormal families of the inequality (3.3) may be stated.

Theorem 3.4. Let $(H; \langle \cdot, \cdot \rangle)$ be a real or complex inner product space, $\{e_i\}_{i \in F}$ a finite orthonormal family, i.e., $\langle e_i, e_j \rangle = \delta_{ij}$ for $i, j \in F$, where δ_{ij} is Kronecker's delta and $\alpha_i \in \mathbb{K}$, $i \in F$ such that $|\alpha_i - 1| = 1$ for each $i \in F$. Then

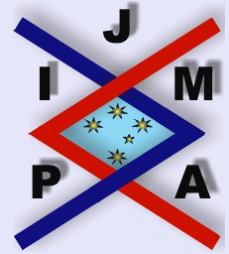
$$(3.13) \quad \left| \langle x, y \rangle - \sum_{i \in F} \alpha_i \langle x, e_i \rangle \langle e_i, y \rangle \right| \leq \|x\| \|y\|.$$

The equality holds in (3.13) if and only if there exists a constant $\lambda \in \mathbb{K}$ such that

$$(3.14) \quad \sum_{i \in F} \alpha_i \langle x, e_i \rangle e_i = x + \lambda y.$$

Proof. As above, by Schwarz's inequality, we have

$$(3.15) \quad \left| \left\langle \sum_{i \in F} \alpha_i \langle x, e_i \rangle e_i - x, y \right\rangle \right| \leq \left\| \sum_{i \in F} \alpha_i \langle x, e_i \rangle e_i - x \right\| \|y\|$$



A Potpourri of Schwarz Related Inequalities in Inner Product Spaces (II)

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 19 of 27

with equality if and only if there exists a $\lambda \in \mathbb{K}$ such that $\sum_{i \in F} \alpha_i \langle x, e_i \rangle e_i = x + \lambda y$.

Since

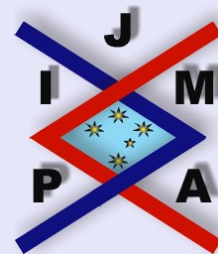
$$\begin{aligned} & \left\| \sum_{i \in F} \alpha_i \langle x, e_i \rangle e_i - x \right\|^2 \\ &= \|x\|^2 - 2 \operatorname{Re} \left\langle x, \sum_{i \in F} \alpha_i \langle x, e_i \rangle e_i \right\rangle + \left\| \sum_{i \in F} \alpha_i \langle x, e_i \rangle e_i \right\|^2 \\ &= \|x\|^2 - 2 \sum_{i \in F} \bar{\alpha}_i \langle x, e_i \rangle \overline{\langle x, e_i \rangle} + \sum_{i \in F} |\alpha_i|^2 |\langle x, e_i \rangle|^2 \\ &= \|x\|^2 + \sum_{i \in F} |\langle x, e_i \rangle|^2 (|\alpha_i|^2 - 2 \operatorname{Re} \alpha_i) \\ &= \|x\|^2 + \sum_{i \in F} |\langle x, e_i \rangle|^2 (|\alpha_i - 1|^2 - 1) \\ &= \|x\|^2, \end{aligned}$$

hence the inequality (3.13) is obtained. \square

Remark 8. If the space is real, then the nontrivial case one can get from (3.13) is for all $\alpha_i = 2$, obtaining the inequality

$$(3.16) \quad \frac{1}{2} [\langle x, y \rangle - \|x\| \|y\|] \leq \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \leq \frac{1}{2} [\langle x, y \rangle + \|x\| \|y\|]$$

that has been obtained in [5].



A Potpourri of Schwarz Related
Inequalities in Inner Product
Spaces (II)

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

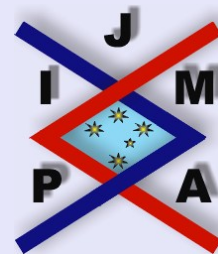
Page 20 of 27

Corollary 3.5. *With the above assumptions, we have*

$$(3.17) \quad \left| \sum_{i \in F} \alpha_i \langle x, e_i \rangle \langle e_i, y \rangle \right| \leq |\langle x, y \rangle| + \left| \langle x, y \rangle - \sum_{i \in F} \alpha_i \langle x, e_i \rangle \langle e_i, y \rangle \right|$$

$$\leq |\langle x, y \rangle| + \|x\| \|y\|, \quad x, y \in H,$$

where $|\alpha_i - 1| = 1$ for each $i \in F$ and $\{e_i\}_{i \in F}$ is an orthonormal family in H .



**A Potpourri of Schwarz Related
Inequalities in Inner Product
Spaces (II)**

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 21 of 27

4. Applications for the Triangle Inequality

In 1966, Diaz and Metcalf [2] proved the following reverse of the triangle inequality:

$$(4.1) \quad \left\| \sum_{i=1}^n x_i \right\| \geq r \sum_{i=1}^n \|x_i\|,$$

provided the vectors x_i in the inner product space $(H; \langle \cdot, \cdot \rangle)$ over the real or complex number field \mathbb{K} are nonzero and

$$(4.2) \quad 0 \leq r \leq \frac{\operatorname{Re} \langle x_i, a \rangle}{\|x_i\|} \quad \text{for each } i \in \{1, \dots, n\},$$

where $a \in H$, $\|a\| = 1$. The equality holds in (4.2) if and only if

$$(4.3) \quad \sum_{i=1}^n x_i = r \left(\sum_{i=1}^n \|x_i\| \right) a.$$

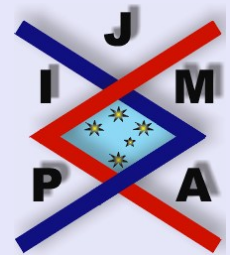
The following result may be stated:

Proposition 4.1. *Let $e \in H$ with $\|e\| = 1$, $\varepsilon \in (0, \frac{1}{2}]$ and $x_i \in H$, $i \in \{1, \dots, n\}$ with the property that*

$$(4.4) \quad (0 \leq) 1 - \varepsilon - \sqrt{1 - 2\varepsilon} \leq \|x_i\| \leq 1 - \varepsilon + \sqrt{1 - 2\varepsilon}$$

for each $i \in \{1, \dots, n\}$. Then

$$(4.5) \quad \sum_{i=1}^n \|x_i\| \leq \left\| \sum_{i=1}^n x_i \right\| + \varepsilon \sum_{i=1}^n \|x_i - e\|^2.$$



A Potpourri of Schwarz Related
Inequalities in Inner Product
Spaces (II)

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 22 of 27

Proof. Utilising Proposition 3.1 for $x = x_i$ and $y = e$, we can state that

$$\|x_i\| - \operatorname{Re} \langle x_i, e \rangle \leq \varepsilon \|x_i - e\|^2$$

for each $i \in \{1, \dots, n\}$. Summing over i from 1 to n , we deduce that

$$(4.6) \quad \sum_{i=1}^n \|x_i\| \leq \operatorname{Re} \left\langle \sum_{i=1}^n x_i, e \right\rangle + \varepsilon \sum_{i=1}^n \|x_i - e\|^2.$$

By Schwarz's inequality in $(H; \langle \cdot, \cdot \rangle)$, we also have

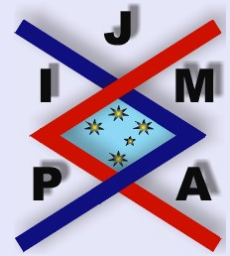
$$(4.7) \quad \operatorname{Re} \left\langle \sum_{i=1}^n x_i, e \right\rangle \leq \left| \operatorname{Re} \left\langle \sum_{i=1}^n x_i, e \right\rangle \right| \leq \left| \left\langle \sum_{i=1}^n x_i, e \right\rangle \right| \\ \leq \left\| \sum_{i=1}^n x_i \right\| \|e\| = \left\| \sum_{i=1}^n x_i \right\|.$$

Therefore, by (4.6) and (4.7) we deduce (4.5). □

In the same spirit, we can prove the following result as well:

Proposition 4.2. *Let $(H; \langle \cdot, \cdot \rangle)$ be a real or complex inner product space and $e \in H$ with $\|e\| = 1$. Then for any $\varphi \in \mathbb{R}$ one has the inequality:*

$$(4.8) \quad \sum_{i=1}^n \|x_i\| \leq \left\| \sum_{i=1}^n x_i \right\| + \frac{1}{2} \sum_{i=1}^n [|\cos \varphi| \|x_i - e\| + |\sin \varphi| \|x_i + e\|]^2.$$



A Potpourri of Schwarz Related
Inequalities in Inner Product
Spaces (II)

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 23 of 27

Proof. Applying Proposition 3.2 for $x = x_i$ and $y = e$, we have:

$$(4.9) \quad \|x_i\| \leq \cos 2\varphi \cdot \operatorname{Re} \langle x_i, e \rangle + \sin 2\varphi \cdot \operatorname{Im} \langle x_i, e \rangle \\ + \frac{1}{2} [|\cos \varphi| \|x_i - e\| + |\sin \varphi| \|x_i + e\|]^2$$

for any $i \in \{1, \dots, n\}$.

Summing in (4.5) over i from 1 to n , we have:

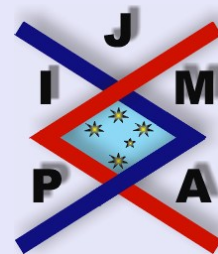
$$(4.10) \quad \sum_{i=1}^n \|x_i\| \leq \cos 2\varphi \cdot \operatorname{Re} \left\langle \sum_{i=1}^n x_i, e \right\rangle + \sin 2\varphi \cdot \operatorname{Im} \left\langle \sum_{i=1}^n x_i, e \right\rangle \\ + \frac{1}{2} \sum_{i=1}^n [|\cos \varphi| \|x_i - e\| + |\sin \varphi| \|x_i + e\|]^2.$$

Now, by the elementary inequality for the real numbers m, p, M and P ,

$$mM + pP \leq (m^2 + p^2)^{\frac{1}{2}} (M^2 + P^2)^{\frac{1}{2}},$$

we have

$$(4.11) \quad \cos 2\varphi \cdot \operatorname{Re} \left\langle \sum_{i=1}^n x_i, e \right\rangle + \sin 2\varphi \cdot \operatorname{Im} \left\langle \sum_{i=1}^n x_i, e \right\rangle \\ \leq (\cos^2 2\varphi + \sin^2 2\varphi)^{\frac{1}{2}} \\ \times \left(\left[\operatorname{Re} \left\langle \sum_{i=1}^n x_i, e \right\rangle \right]^2 + \left[\operatorname{Im} \left\langle \sum_{i=1}^n x_i, e \right\rangle \right]^2 \right)^{\frac{1}{2}}$$



**A Potpourri of Schwarz Related
Inequalities in Inner Product
Spaces (II)**

S.S. Dragomir

Title Page

Contents



Go Back

Close

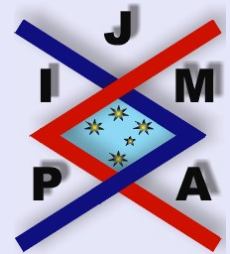
Quit

Page 24 of 27

$$= \left| \left\langle \sum_{i=1}^n x_i, e \right\rangle \right| \leq \left\| \sum_{i=1}^n x_i \right\| \|e\| = \left\| \sum_{i=1}^n x_i \right\|,$$

where for the last inequality we used Schwarz's inequality in $(H; \langle \cdot, \cdot \rangle)$.

Finally, by (4.10) and (4.11) we deduce the desired result (4.8). □



**A Potpourri of Schwarz Related
Inequalities in Inner Product
Spaces (II)**

S.S. Dragomir

Title Page

Contents



Go Back

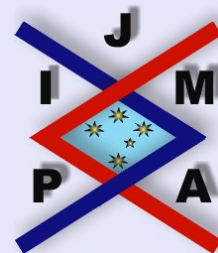
Close

Quit

Page 25 of 27

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A Potpourri of Schwarz Related Inequalities in Inner Product Spaces (II)

S.S. Dragomir

Title Page

Contents



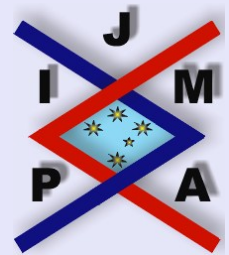
Go Back

Close

Quit

Page 26 of 27

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**A Potpourri of Schwarz Related
Inequalities in Inner Product
Spaces (II)**

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 27 of 27