

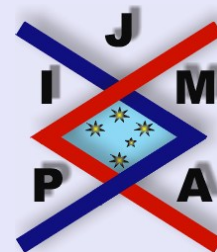
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ON SOME CLASSES OF ANALYTIC FUNCTIONS

KHALIDA INAYAT NOOR AND M.A. SALIM

Department of Mathematics and Computer Science
College of Science
United Arab Emirates University
P.O. Box 17551, Al-Ain
United Arab Emirates.

EMail: KhalidaN@uaeu.ac.ae



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Abstract

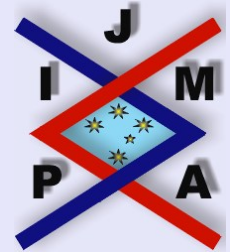
We define some classes of analytic functions related with the class of functions with bounded boundary rotation and study these classes with reference to certain integral operators.

2000 Mathematics Subject Classification: 30C45, 30C50.

Key words: Close-to-convex functions, Univalent functions, Bounded boundary rotation, Integral operator.

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1. Introduction

Let \mathcal{A} denote the class of functions f of the form $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ which are analytic in the unit disk $E = \{z : |z| < 1\}$. Let C, S^*, K and S be the subclasses of \mathcal{A} which are respectively convex, starlike, close-to-convex and univalent in E . It is known that $C \subset S^* \subset K \subset S$. In [1], Kaplan showed that $f \in K$ if, and only if, for $z \in E$, $0 \leq \theta_1 < \theta_2 \leq 2\pi$, $0 < r < 1$,

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \left\{ 1 + \frac{r e^{i\theta} f''(r e^{i\theta})}{f'(r e^{i\theta})} \right\} d\theta > -\pi, \quad z = r e^{i\theta}.$$

Let $V_k (k \geq 2)$ be the class of locally univalent functions $f \in \mathcal{A}$ that map E conformally onto a domain whose boundary rotation is at most $k\pi$. It is well known that $V_2 \equiv C$ and $V_k \subset K$ for $2 \leq k \leq 4$.

Definition 1.1. Let $f \in \mathcal{A}$ and $f'(z) \neq 0$. Then $f \in T_k(\lambda)$, $k \geq 2$, $0 \leq \lambda < 1$ if there exists a function $g \in V_k$ such that, for $z \in E$

$$\operatorname{Re} \left\{ \frac{f'(z)}{g'(z)} \right\} > \lambda.$$

The class $T_k(0) = T_k$ was considered in [2, 3] and $T_2(0) = K$, the class of close-to-convex functions.

Definition 1.2. Let $f \in \mathcal{A}$ and $\frac{f(z)f'(z)}{z} \neq 0$, $z \in E$. Then $f \in T_k(a, \gamma, \lambda)$, $\operatorname{Re} a \geq 0$, $0 \leq \gamma \leq 1$ if, and only if, there exists a function $g \in T_k(\lambda)$ such that

$$(1.1) \quad z f'(z) + a f(z) = (a + 1) z (g'(z))^\gamma, \quad z \in E.$$



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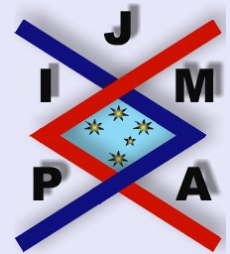
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We note that $T_k(0, 1, \lambda) = T_k(\lambda)$ and $T_2(0, 1, \lambda) = K(\lambda) \subset K$, and it follows that $f \in T_k(a, \gamma, \lambda)$ if, and only if, there exists $F \in T_k(\infty, \gamma, \lambda)$ such that

$$f(z) = \frac{a+1}{z^a} \int_0^z t^{a-1} F(t) dt.$$



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2. Preliminary Results

Lemma 2.1 ([2]). Let $f \in \mathcal{A}$. Then, for $0 \leq \theta_1 < \theta_2 \leq 2\pi$, $z = re^{i\theta}$, $0 < r < 1$, $f \in T_k$ if and only if

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \left\{ \frac{zf'(z)}{f'(z)} \right\} d\theta > -\frac{k}{2}\pi.$$

Lemma 2.2. Let $q(z)$ be analytic in E and of the form $q(z) = 1 + b_1z + \dots$ for $|z| = r \in (0, 1)$. Then, for $a, c_1, \theta_1, \theta_2$ with $a \geq 1$, $\operatorname{Re}(c_1) \geq 0$, $0 \leq \theta_1 < \theta_2 \leq 2\pi$,

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \left\{ q(z) + \frac{azq'(z)}{c_1a + q(z)} \right\} d\theta > -\beta_1\pi; \quad (0 < \beta_1 \leq 1)$$

implies

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} q(z) d\theta > -\beta_1\pi, \quad z = re^{i\theta}.$$

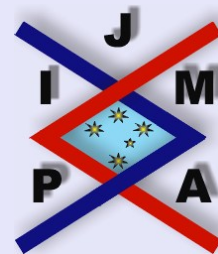
This result is a direct consequence of the one proved in [4] for $\beta_1 = 1$.

From (1.1) and Lemma 2.1, we can easily have the following:

Lemma 2.3. A function $f \in T_k(\infty, \gamma, \lambda)$ if and only if, it may be represented as $f(z) = p(z) \cdot zG'(z)$, where $G \in V_k$ and $\operatorname{Re} p(z) > \lambda$, $z \in E$.

Proof. Since $f \in T_k(\infty, \gamma, \lambda)$, we have

$$\begin{aligned} f(z) &= z(g'(z))^\gamma, \quad g \in T_k(\lambda) \\ &= z[G_1'(z)p_1(z)]^\gamma, \quad G_1 \in V_k, \quad \operatorname{Re} p_1(z) > \lambda \\ &= zG'(z).p(z), \end{aligned}$$



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where $G'(z) = (G'_1(z))^\gamma \in V_k$ and $p(z) = (p_1(z))^\gamma$, $\operatorname{Re} p(z) > \lambda$, since $0 \leq \gamma \leq 1$.

The converse case follows along similar lines. □

Using Lemma 2.1 and Lemma 2.3, we have:

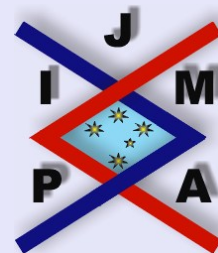
Lemma 2.4.

(i) Let $f \in T_k(0, \gamma, \lambda)$. Then, with $z = re^{i\theta}$, $0 \leq \theta_1 < \theta - 2 \leq 2\pi$,

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \left\{ \frac{(zf'(z))'}{f'(z)} \right\} d\theta > -\frac{k\gamma}{2}, \quad \text{see also [3].}$$

(ii) Let $f \in T_k(\infty, \gamma, \lambda)$. Then, for $z = re^{i\theta}$ and $0 \leq \theta_1 < \theta_2 \leq 2\pi$,

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} d\theta > -\frac{k\gamma}{2}.$$



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3. Main Results

Theorem 3.1. For $0 < \alpha < \frac{1}{1-\lambda+\lambda\beta}$, $0 < \beta < \frac{\lambda}{1-\lambda}$, $0 \leq \lambda < \frac{1}{2}$ and $f, g \in T_k(\infty, \gamma, \lambda)$, $z \in E$, let

$$(3.1) \quad F(z) = \left[\left(\beta + \frac{1}{\alpha} \right) z^{1-\frac{1}{\alpha}} \int_0^z t^{\frac{1}{\alpha}-2} (f(t))^\beta g(t) dt \right]^{\frac{1}{1+\beta}}.$$

Then F_1 , with $F = zF_1'$ and $0 < \gamma < 1$, $k \leq \frac{2}{\gamma}$, is close-to-convex and hence univalent in E .

Proof. We can write (3.1) as

$$(3.2) \quad (F(z))^{\beta+1} = \left(\beta + \frac{1}{\alpha} \right) z^{1-\frac{1}{\alpha}} \int_0^z t^{\frac{1}{\alpha}-2} (f(t))^\beta g(t) dt.$$

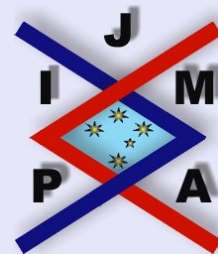
Let

$$(3.3) \quad \frac{zF'(z)}{F(z)} = \frac{(zF_1'(z))'}{F_1'(z)} = (1-\lambda)H(z) + \lambda,$$

where $H(z)$ is analytic in E and $H(z) = 1 + c_1z + c_2z^2 + \dots$.

We differentiate (3.2) logarithmically to obtain

$$(\beta + 1) \frac{zF'(z)}{F(z)} = \left(1 - \frac{1}{\alpha} \right) + \frac{z^{\frac{1}{\alpha}-1} (f(z))^\beta g(z)}{\int_0^z t^{\frac{1}{\alpha}-2} (f(t))^\beta g(t) dt}.$$



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Using (3.2) and differentiating again, we have after some simplifications,

$$(1 - \lambda)zH' \frac{\int_0^z t^{\frac{1}{\alpha}-2} (f(t))^\beta g(t) dt}{z^{\frac{1}{\alpha}-1} (f(z))^\beta g(z)} + (1 - \lambda)H(z) \\ = \frac{\beta}{1 + \beta} \cdot \frac{zf'(z)}{f(z)} + \frac{1}{\beta + 1} \cdot \frac{zg'(z)}{g(z)} - \lambda.$$

Now

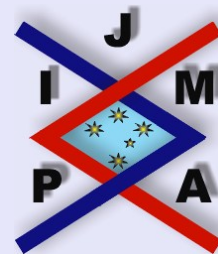
$$\frac{z^{\frac{1}{\alpha}-1} (f(z))^\beta g(z)}{\int_0^z t^{\frac{1}{\alpha}-2} (f(t))^\beta g(t) dt} = \left(\frac{1}{\alpha} - 1\right) + (1 + \beta) \frac{zF'(z)}{F(z)}.$$

Hence

$$-\lambda + \frac{\beta}{1 + \beta} \cdot \frac{zf'(z)}{f(z)} + \frac{1}{\beta + 1} \cdot \frac{zg'(z)}{g(z)} \\ = (1 - \lambda)H(z) + \frac{(1 - \lambda)zH'(z)}{(1 - \lambda)(1 + \beta)H(z) + \left(\frac{1}{\alpha} - 1\right) + \lambda(1 + \beta)}$$

and we have

$$(3.4) \quad \frac{1}{1 - \lambda} \left[\frac{\beta}{1 + \beta} \left(\frac{zf'(z)}{f(z)} - \lambda \right) + \frac{1}{1 + \beta} \left(\frac{zg'(z)}{g(z)} - \lambda \right) \right] \\ = H(z) + \frac{\frac{1}{(1+\beta)(1-\lambda)} zH'(z)}{H(z) + \left[\frac{(\frac{1}{\alpha}-1)}{(1+\beta)(1-\lambda)} + \frac{\lambda}{1-\lambda} \right]}.$$



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Since $f, g \in T_k(\infty, \gamma, \lambda)$, so with $z = re^{i\theta}$, $0 \leq \theta_1 < \theta_2 \leq 2\pi$,

$$\frac{\beta}{1+\beta} \int_{\theta_1}^{\theta_2} \operatorname{Re} \left\{ \frac{1}{1-\lambda} \left(\frac{zf'(z)}{f(z)} - \lambda \right) \right\} d\theta + \frac{1}{1+\beta} \int_{\theta_1}^{\theta_2} \operatorname{Re} \left\{ \frac{1}{1-\lambda} \left(\frac{zg'(z)}{g(z)} - \lambda \right) \right\} d\theta > \frac{-k\gamma}{2} \pi,$$

and, therefore,

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \left[H(z) + \frac{\frac{1}{(1+\beta)(1-\lambda)} z H'(z)}{H(z) + \left\{ \frac{(\frac{1}{\alpha}-1)}{(1+\beta)(1-\lambda)} + \frac{\lambda}{1-\lambda} \right\}} \right] d\theta > \frac{-k\gamma}{2} \pi.$$

Now using Lemma 2.2 with $a = \frac{1}{(1+\beta)(1-\lambda)} \geq 1$, $c_1 = \left\{ \left(\frac{1}{\alpha} - 1 \right) + (1 + \beta)\lambda \right\} \geq 0$, we obtain the required result. \square

Theorem 3.2. Let $f, g \in T_k(\infty, \gamma, \lambda)$, α, c, δ and ν be positively real, $0 < \alpha \leq \frac{1}{1-\lambda}$, $c > \alpha(1-\lambda)$, $(\nu + \delta) = \alpha$. Then the function F defined by

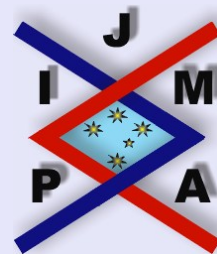
$$(3.5) \quad [F(z)]^\alpha = cz^{\alpha-c} \int_0^z t^{(c-\delta-\nu)-1} (f(t))^\delta (g(t))^\nu dt$$

belongs to $T_k(\infty, \gamma, \lambda)$ for $k \leq \frac{2}{\gamma}$, $0 < \gamma < 1$.

Proof. First we show that there exists an analytic function F satisfying (3.5).

Let

$$\begin{aligned} G(z) &= z^{-(\nu+\delta)} (f(z))^\delta (g(z))^\nu \\ &= 1 + d_1 z + d_2 z^2 + \dots \end{aligned}$$



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and choose the branches which equal 1 when $z = 0$. For

$$K(z) = z^{(c-\nu-\delta)-1} (f(z))^\delta (g(z))^\nu = z^{c-1} G(z),$$

we have

$$L(z) = \frac{c}{z^c} \int_0^z K(t) dt = 1 + \frac{c}{1+c} d_1 z + \dots$$

Hence L is well-defined and analytic in E .

Now let

$$F(z) = [z^\alpha L(z)]^{\frac{1}{\alpha}} = z [L(z)]^{\frac{1}{\alpha}},$$

where we choose the branch of $[L(z)]^{\frac{1}{\alpha}}$ which equals 1 when $z = 0$. Thus F is analytic in E and satisfies (3.5).

Set

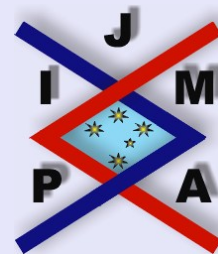
$$(3.6) \quad \frac{zF'(z)}{F(z)} = (1 - \lambda)h(z) + \lambda,$$

and let

$$\begin{aligned} \frac{zf'(z)}{f(z)} &= (1 - \lambda)h_1(z) + \lambda \\ \frac{zg'(z)}{g(z)} &= (1 - \lambda)h_2(z) + \lambda. \end{aligned}$$

Now, from (3.5), we have

$$(3.7) \quad z^{(c-\alpha)} [F(z)]^\alpha \left[(c - \alpha) + \alpha \frac{zF'(z)}{F(z)} \right] = c \left[z^{(c-\delta-\nu)-1} (f(z))^\delta (g(z))^\nu \right].$$



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We differentiate (3.7) logarithmically and use (3.6) to obtain

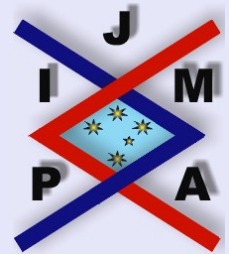
$$\begin{aligned} & \alpha(1-\lambda) \left[h(z) + \frac{zh'(z)}{(c-\alpha) + \alpha\{\lambda + (1-\lambda)h(z)\}} \right] + (\delta + \nu - \alpha) \\ &= \delta \frac{zf'(z)}{f(z)} + \nu \frac{zg'(z)}{g(z)} - \alpha\lambda \\ &= \delta \left[\frac{zf'(z)}{f(z)} - \lambda \right] + \nu \left[\frac{zg'(z)}{g(z)} - \lambda \right]. \end{aligned}$$

This gives us

$$\begin{aligned} & \left[h(z) + \frac{zh'(z)}{(c-\alpha) + \alpha\{\lambda + (1-\lambda)h(z)\}} \right] \\ &= \frac{\delta}{\alpha(1-\lambda)} \left[\frac{zf'(z)}{f(z)} - \lambda \right] + \frac{\nu}{\alpha(1-\lambda)} \left[\frac{zg'(z)}{g(z)} - \lambda \right]. \end{aligned}$$

Since $f, g \in T_k(\infty, \gamma, \lambda)$, we have, for $0 \leq \theta_1 < \theta_2 \leq 2\pi$, $z = re^{i\theta}$,

$$\begin{aligned} & \int_{\theta_1}^{\theta_2} \operatorname{Re} \left[h(z) + \frac{zh'(z)}{(c-\alpha) + \alpha\{\lambda + (1-\lambda)h(z)\}} \right] d\theta \\ &= \left[\frac{\delta}{\alpha} \int_{\theta_1}^{\theta_2} \operatorname{Re} h_1(z) d\theta + \frac{\nu}{\alpha} \int_{\theta_1}^{\theta_2} \operatorname{Re} h_2(z) d\theta \right] \\ &> \frac{\delta}{\alpha} \left(-\frac{\gamma k}{2} \pi \right) + \frac{\nu}{\alpha} \left(-\frac{\gamma k}{2} \pi \right) \\ &= \frac{\delta + \nu}{\alpha} \left(-\frac{\gamma k}{2} \pi \right) = -\frac{\gamma k}{2} \pi, \end{aligned}$$



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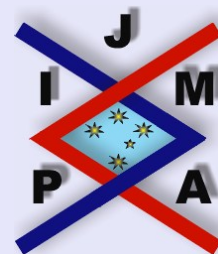
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where we have used Lemma 2.4.

Now using Lemma 2.2 with $a = \frac{1}{\alpha(1-\lambda)} > 1$, for $\alpha < \frac{1}{1-\lambda}$ and

$$c_1 = c - \alpha + \alpha\lambda = c - \alpha(1 - \lambda) \geq 0,$$

we obtain the required result. □



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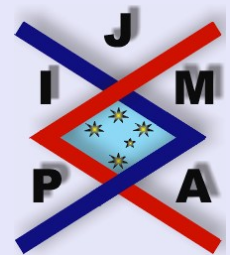
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