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CONTINUITY PROPERTIES OF CONVEX-TYPE SET-VALUED MAPS

KAZIMIERZ NIKODEM

Department of Mathematics,
University of Bielsko–Biała Willowa 2,
PL-43-309 Bielsko–Biała, Poland.
E-Mail: knik@ath.bielsko.pl

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It is well known that convex functions defined on an infinite-dimensional space need not be continuous and midconvex (Jensen convex) functions, they may be discontinuous even if they are defined on an open interval in \mathbb{R} . However, their continuity follows from other regularity assumptions, such as continuity at a point, upper boundedness on a set with non-empty interior, measurability, lower semicontinuity, closedness of the epigraph, etc. (cf. e.g. [26], [12]). The aim of this note is to collect similar results for convex set-valued maps. Such maps arise naturally from, e.g., the constraints of convex optimization problems and play an important role in various questions of convex analysis and economic theory (cf. [4], [5], [13], [27], [28], [29] for more information). Conditions implying their continuity can be found, among others, in [3], [6], [7], [8], [9], [16], [17], [18], [19], [20], [22], [23], [24], [25], [27], [30], [31].

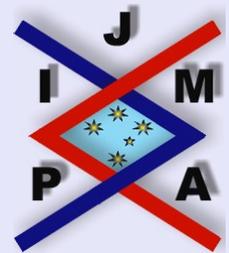
Let X and Y be topological vector spaces (real and Hausdorff in the whole paper), D be a convex subset of X and K be a convex cone in Y (i.e. $K + K \subset K$ and $tK \subset K$ for all $t \geq 0$). Denote by $n(Y)$, $b(Y)$, $c(Y)$ and $cc(Y)$ the families of all non-empty, non-empty bounded, non-empty compact and non-empty compact convex subsets of Y , respectively.

A set-valued map (s.v. map for short) $F : D \rightarrow n(Y)$ is said to be K -convex if

$$(1) \quad tF(x_1) + (1 - t)F(x_2) \subset F(tx_1 + (1 - t)x_2) + K$$

for all $x_1, x_2 \in D$ and $t \in [0, 1]$; F is called K -midconvex (or K -Jensen convex) if

$$(2) \quad \frac{F(x_1) + F(x_2)}{2} \subset F\left(\frac{x_1 + x_2}{2}\right) + K,$$



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for all $x_1, x_2 \in D$. Equivalently, F is K -convex (K -midconvex) if its *epigraph*, i.e. the set

$$\text{epi}F = \{(x, y) \in D \times Y : y \in F(x) + K\},$$

is a convex (midconvex) subset of $X \times Y$.

Note that F is K -convex (K -midconvex) with $K = \{0\}$ iff its *graph*, i.e. the set

$$\text{gr}F = \{(x, y) \in D \times Y : y \in F(x)\},$$

is a convex (midconvex) subset of $X \times Y$.

If F is single-valued and Y is endowed with the relation \leq_K of partial order defined by $x \leq_K y : \iff y - x \in K$, then condition (1) reduces to the following one

$$F(tx_1 + (1 - t)x_2) \leq_K tF(x_1) + (1 - t)F(x_2).$$

In particular if $Y = \mathbb{R}$ and $K = [0, \infty)$, we obtain the standard definition of convex functions.

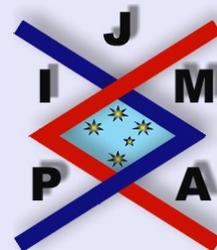
We say that a set-valued map $F : D \rightarrow n(Y)$ is K -continuous at a point $x_0 \in D$ if for every neighbourhood W of zero in Y there exists a neighbourhood U of zero in X such that

$$(3) \quad F(x_0) \subset F(x) + W + K$$

and

$$(4) \quad F(x) \subset F(x_0) + W + K$$

for every $x \in (x_0 + U) \cap D$. Only when condition (3) (condition (4)) is fulfilled, we say that F is K -lower semicontinuous (K -upper semicontinuous) at x_0 . The



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K -continuity in the case where $K = \{0\}$ means the continuity with respect to the Hausdorff topology on $n(Y)$. If K is a normal cone (i.e. if there exists a base \mathcal{W} of neighbourhoods of zero in Y such that $W = (W - K) \cap (W + K)$ for every $W \in \mathcal{W}$) and F is a single-valued function, then K -continuity means continuity. Note also that in the case where F is a real-valued function and $K = [0, \infty)$ then conditions (3) and (4) define the classical upper and lower semicontinuity of F at x_0 , respectively.

We start with the following result showing that for K -midconvex s.v. maps K -lower semicontinuity at a point implies K -continuity on the whole domain.

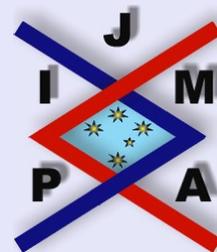
Theorem 1. ([17, Thm. 3.3]; cf. also [6]). *Let X and Y be topological vector spaces, D be a convex open subset of X , and K be a convex cone in Y . Assume that $F : D \rightarrow b(Y)$ and $G : D \rightarrow n(Y)$ are s.v. maps such that $G(x) \subset F(x) + K$, for all $x \in D$. If F is K -midconvex and G is K -lower semicontinuous at a point of D , then F is K -continuous on D .*

As an immediate consequence of this theorem (under the same assumptions on X , Y , D and K) we get the following corollaries. Recall that a function $f : D \rightarrow Y$ is a selection of $F : D \rightarrow n(Y)$ if $f(x) \in F(x)$ for all $x \in D$.

Corollary 2. *If a s.v. map $F : D \rightarrow b(Y)$ is K -midconvex and K -lower semicontinuous at a point of D , then it is K -continuous on D .*

Corollary 3. *If a s.v. map $F : D \rightarrow b(Y)$ is K -midconvex and has a selection continuous at a point of D , then it is K -continuous on D .*

In the centre of many results giving conditions under which midconvex (or convex) functions are continuous there are two basic theorems. The first one is



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the theorem of Bernstein and Doetsch, stating that midconvex functions bounded above on a set with non-empty interior are continuous, and the second one is the theorem of Sierpiński, stating that measurable midconvex functions are continuous (cf. [26], [12]). The next two theorems are far-reaching generalizations of those results for K -midconvex s.v. maps.

We say that an s.v. map F is K -upper bounded on a set A if there exists a bounded set $B \subset Y$ such that $F(x) \cap (B - K) \neq \emptyset$, for all $x \in A$.

Theorem 4. ([17, Thm. 3.4]). *Let X and Y be topological vector spaces, D – an open convex subset of X and K – a convex cone in Y . If an s.v. map $F : D \rightarrow b(Y)$ is K -midconvex and K -upper bounded on a subset of D with non-empty interior, then F is K -continuous on D .*

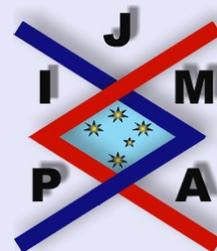
Remark 1. *In the case where $X = \mathbb{R}^n$, it is sufficient to assume that the set A is of positive Lebesgue measure. Indeed, if F is K -upper bounded on A , then, by the K -midconvexity, it is also K -upper bounded on the set $(A + A)/2$, which, by the classical Steinhaus theorem, has non-empty interior.*

Recall that a set-valued map $F : \mathbb{R}^n \supset D \rightarrow n(Y)$ is Lebesgue measurable if for every open set $W \subset Y$ the set

$$F^+(W) = \{t \in D : F(x) \subset W\}$$

is Lebesgue measurable.

Theorem 5. ([17, Thm. 3.8]; cf. also [30]). *Let D be a convex open subset of \mathbb{R}^n , Y be a topological vector space, and K be a convex cone in Y . Assume that $F : D \rightarrow b(Y)$ and $G : D \rightarrow b(Y)$ are s.v. maps such that $G(x) \subset F(x) + K$, for all $x \in D$. If F is K -midconvex and G is Lebesgue measurable, then F is K -continuous on D .*



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Under the same assumptions on D , Y and K we have the following corollaries.

Corollary 6. *If a s.v. map $F : D \rightarrow b(Y)$ is K -midconvex and Lebesgue measurable, then it is K -continuous on D .*

Corollary 7. *If a s.v. map $F : D \rightarrow b(Y)$ is K -midconvex and has a Lebesgue measurable selection, then it is K -continuous on D .*

The next result generalizes the well known result stating that convex functions defined on an open subset of a finite-dimensional space are continuous.

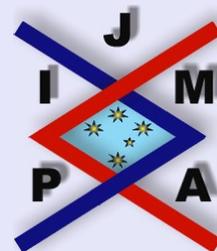
Theorem 8. ([17, Thm. 3.7]; cf. also [24]). *Let D be a convex open subset of \mathbb{R}^n , Y be a topological vector space, and K be a convex cone in Y . If a s.v. map $F : D \rightarrow b(Y)$ is K -convex, then it is K -continuous on D .*

Now we present a generalization of the classical closed graph theorem.

Theorem 9. ([18, Thm. 1]). *Let X be a Baire topological vector space, D be a convex open subset of X , Y be a locally convex topological vector space and K be a convex cone in Y . Assume that there exist compact sets $B_n \subset Y$, $n \in \mathbb{N}$, such that*

$$(5) \quad \bigcup_{n \in \mathbb{N}} (B_n - K) = Y.$$

If a s.v. map $F : D \rightarrow b(Y)$ is K -midconvex and its epigraph is closed in $D \times Y$, then it is K -continuous on D .



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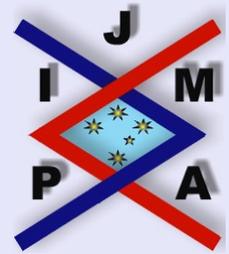
Remark 2. *The assumption (5) is trivially satisfied if Y is a locally compact space (and K is an arbitrary convex cone in Y). It is also fulfilled if there exists an order unit in Y , i.e. such an element $e \in Y$ that for every $y \in Y$ we can find an $n \in \mathbb{N}$ with $y \in ne - K$ (we put then $B_n = \{ne\}$). In particular, if $\text{int } K \neq \emptyset$, then every element of $\text{int } K$ is an order unit in Y . The above result extends the closed graph theorem proved by Ger [10] for midconvex operators and crosses with the closed graph theorems due to Borwein [6], Ricceri [25] and Robinson-Ursescu [27], [31] (cf. also [2]).*

The next result generalizes the known theorem stating that lower semicontinuous convex functions are continuous. Given a convex cone K in a topological vector space Y we denote by K^* the set of all continuous linear functionals on Y which are nonnegative on K , i.e.

$$K^* = \{y^* \in Y^* : y^*(y) \geq 0, \text{ for every } y \in K\}.$$

Theorem 10. ([19, Thm. 1]). *Let X be a Baire topological vector space, D – a convex open subset of X , Y – a locally convex topological vector space and K – a convex cone in Y . Moreover, assume that there exist bounded sets $B_n \subset Y$, $n \in \mathbb{N}$, such that condition (5) holds. If a s.v. map $F : D \rightarrow cc(Y)$ is K –midconvex and for every $y^* \in K^*$ the functional $x \mapsto f_{y^*}(x) = \inf y^*(F(x))$, $x \in D$, is lower semicontinuous on D , then F is K –continuous on D .*

It is easy to check that if a s.v. map $F : D \rightarrow b(Y)$ is K –upper semicontinuous at a point, then for every $y^* \in K^*$ the functional f_{y^*} defined above is lower semicontinuous at this point. Therefore, as a consequence of the above theorem, we get the following result.



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Corollary 11. *Let X, D, Y and K be such as in Theorem 10. If a K -midconvex s.v. map $F : D \rightarrow cc(Y)$ is K -upper semicontinuous on D , then it is K -continuous on D .*

Now we will present the Mazur's criterion for continuity of K -midconvex s.v. maps. It is related to the following question posed by S. Mazur [15]: In a Banach space E there is given an additive functional f such that, for every continuous function $x : [0, 1] \rightarrow E$, the superposition $f \circ x$ is Lebesgue measurable. Is f continuous?

The answer to that question, in the affirmative, was given by I. Labuda and R.D. Mauldin [14]. R. Ger [11] showed that the same remains true in the case where f is a midconvex functional defined on an open convex subset D of E . More precisely, he proved that each midconvex functional $f : D \rightarrow E$ such that for every continuous function $x : [0, 1] \rightarrow D$, the superposition $f \circ x$ admits a Lebesgue measurable majorant, is continuous. The next theorem is a set-valued generalization of this result.

Theorem 12. ([20, Thm. 1]). *Let E be a real Banach space, D – an open convex subset of E , Y – a locally convex topological vector space and K – a convex cone in Y . Moreover, assume that there exist bounded sets $B_n \subset Y$, $n \in \mathbb{N}$, such that condition (5) holds. If a set-valued map $F : D \rightarrow cc(Y)$ is K -midconvex and for every continuous function $x : [0, 1] \rightarrow D$ there exists a Lebesgue measurable set-valued map $G : [0, 1] \rightarrow c(Y)$ such that*

$$G(t) \subset F(x(t)) + K, \quad t \in [0, 1],$$

then F is K -continuous on D .



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As an immediate consequence of the above theorem (under the same assumptions on E, D, Y and K) we obtain the following corollaries.

Corollary 13. *If a set-valued map $F : D \rightarrow cc(Y)$ is K -midconvex and for every continuous function $x : [0, 1] \rightarrow D$ the superposition $F \circ x$ is Lebesgue measurable, then F is K -continuous on D .*

Corollary 14. *If a set-valued map $F : D \rightarrow cc(Y)$ is K -midconvex and for every continuous function $x : [0, 1] \rightarrow D$ the superposition $F \circ x$ has a Lebesgue measurable selection, then F is K -continuous on D .*

Now assume that $\lambda : D^2 \rightarrow (0, 1)$ is a fixed function. We say that a set-valued map $F : D \rightarrow n(Y)$ is (K, λ) -convex if

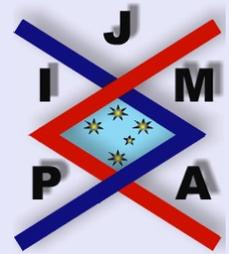
$$(6) \quad \lambda(x, y)F(x) + (1 - \lambda(x, y))F(y) \subset F(\lambda(x, y)x + (1 - \lambda(x, y))y) + K$$

for all $x, y \in D$. Clearly, K -convex set-valued maps are (K, λ) -convex with every function λ ; K -midconvex set-valued maps are (K, λ) -convex with the constant function $\lambda = 1/2$. For real-valued functions and $K = [0, \infty)$ condition (6) reduces to

$$F(\lambda(x, y)x + (1 - \lambda(x, y))y) \leq \lambda(x, y)F(x) + (1 - \lambda(x, y))F(y), \quad x, y \in D.$$

Such functions were introduced and discussed by Zs. Páles in [21], who obtained a Bernstein–Doetsch-type theorem for them. The next result is a set-valued generalization of this theorem.

Theorem 15. (*[1, Thm. 1]*). *Let $D \subset \mathbb{R}^n$ be an open convex set, $\lambda : D^2 \rightarrow (0, 1)$ be a function continuous in each variable, Y be a locally convex space and K be a closed convex cone in Y . If a s.v. map $F : D \rightarrow c(Y)$ is (K, λ) -convex and locally K -upper bounded at a point of D , then it is K -convex.*



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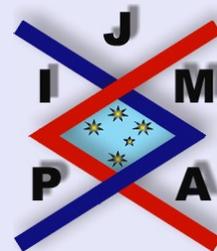
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Finally we present a Sierpiński-type theorem for (K, λ) -convex s.v. maps.

Theorem 16. ([1, Thm. 2]). *Let Y , K , and D be such as in Theorem 15 and $\lambda : D^2 \rightarrow (0, 1)$ be a continuously differentiable function. If a s.v. map $F : D \rightarrow c(Y)$ is (K, λ) -convex and Lebesgue measurable, then it is also K -convex.*



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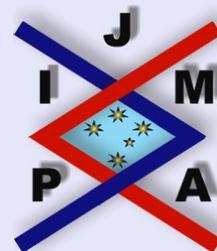
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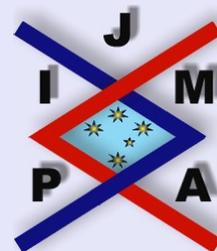
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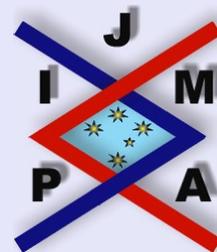
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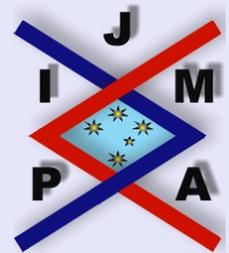
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