**Journal of Inequalities in Pure and Applied Mathematics**

# ON THE SEQUENCE $(p_n^2 - p_{n-1}p_{n+1})_{n\geq 2}$

LAURENŢIU PANAITOPOL
Faculty of Mathematics
14 Academiei St.
RO-70109 Bucharest, Romania.
*EMail*: pan@al.math.unibuc.ro


ISSN (electronic): 1443-5756
090-01

volume 3, issue 4, article 53,
2002.

*Received 17 December, 2001;*
*accepted 24 May, 2002.*

*Communicated by:* L. Toth

## Abstract

Let $p_n$ be the $n$-th prime number and $x_n = p_n^2 - p_{n-1}p_{n+1}$. In this paper, we study sequences containing the terms of the sequence $(x_n)_{n\geq 1}$. The main result asserts that the series $\sum_{n=1}^{\infty} x_n/p_n^2$ is convergent, without being absolutely convergent.

*2000 Mathematics Subject Classification:* 11A25, 11N05, 11N36
*Key words:* Prime Numbers, Sequences, Series, Asymptotic Behaviour.

# Contents



Title Page
Contents
Go Back
Close
Quit

# 1. Introduction

We shall use the following notation:

$$\begin{aligned} &p_n \text{ the } n\text{-th prime number}\\ &x_n = p_n^2 - p_{n-1}p_{n+1} \text{ for } n \geq 2,\\ &d_n = p_{n+1} - p_n \text{ for } n \geq 1,\\ &q_n = \frac{p_{n+1}}{p_n} \text{ for } n \geq 1,\\ &f(x) \asymp g(x) \text{ if there exist } c_1, c_2, M > 0 \text{ such that}\\ &\quad c_1 f(x) < g(x) < c_2 f(x) \text{ for every } x > M. \end{aligned}$$

It will be our aim here to study the sequence $(x_n)_{n\geq 2}$ defined above.

It was proved in [1] that the sequence $(d_n)_{n\geq 1}$ is not nonotone. A similar result holds for the sequence $(q_n)_{n\geq 1}$ as well. This means that the sequence $(x_n)_{n\geq 2}$ has infinitely many positive terms, and infinitely many negative terms, hence it is not monotone.

In [1], the so-called method of the triple sieve (due to Vigo Brun) was used to prove that

$$\sum_{p_n \leq x} \left| \log \frac{q_n}{q_{n-1}} \right| \asymp \log x. \tag{1.1}$$

This result plays an essential role in the following paragraph of the present paper.

Another useful result is proved in [4]:

$$\text{the series } \sum_{n=1}^{\infty} \left( \frac{d_n}{p_n} \right)^n \text{ is convergent.} \tag{1.2}$$

## 2. The Series $\sum_{n=2}^{\infty} \frac{x_n}{p_n^2}$

**Theorem 2.1.** *The series* $\sum_{n=2}^{\infty} \frac{x_n}{p_n^2}$ *is convergent, but it is not absolutely convergent.*

In order to prove this fact, we need the following lemmas.

**Lemma 2.2.** *For* $x \geq -\frac{1}{2}$, *we have*

$$x^2 + |x| \geq |\log(1+x)| \geq |x| - \frac{x^2}{2}.$$

*Proof.* The inequalities are well known for $x > 0$. When $x \in \left[-\frac{1}{2}, 0\right]$, they take on the form

$$x^2 - x \geq -\log(1+x) \geq -x - \frac{x^2}{2}.$$

Let $f, g \colon \left[-\frac{1}{2}, 0\right] \to \mathbb{R}$ be defined by $f(x) = \log(1+x) - x - \frac{x^2}{2}$ and $g(x) = \log(1+x) - x + x^2$, respectively. We have $f'(x) = -\frac{x(x+2)}{1+x} \geq 0$, and $g'(x) = \frac{x(2x+1)}{1+x} \leq 0$. Since $f$ is increasing and $f(0) = 0$, we get $f(x) \leq 0$. On the other hand, we have $g'(x) < 0$ and $g(0) = 0$, so that $g(x) \geq 0$. □

**Lemma 2.3.** *The series* $\sum_{n=2}^{\infty} \frac{d_n - d_{n-1}}{p_n}$ *is convergent.*

*Proof.* Denote $S_n = \sum_{k=2}^{n} \frac{d_k - d_{k-1}}{p_k}$, so that

$$S_n = \frac{d_n}{p_n} + \sum_{k=2}^{n} \frac{d_{k-1}^2}{p_k p_{k-1}} - \frac{1}{2}.$$

Since $\lim_{n\to\infty}\frac{p_{n+1}}{p_n}=1$, it suffices to prove that the series $\sum_{k=2}^{\infty}\frac{d_{k-1}^2}{p_kp_{k-1}}$ is convergent. Since $\frac{d_{k-1}^2}{p_kp_{k-1}}\sim\left(\frac{d_{k-1}}{p_{k-1}}\right)^2$ and the terms of the series are positive, it follows that the series $\sum_{k=2}^{\infty}\frac{d_{k-1}^2}{p_kp_{k-1}}$ and $\sum_{k=2}^{\infty}\left(\frac{d_{k-1}}{p_{k-1}}\right)^2$ are simultaneously convergent or not. Now just use (1.2) and the proof ends. □

**Lemma 2.4.** *The series $\sum_{n=2}^{\infty}\frac{x_n^2}{p_n^4}$ is convergent.*

*Proof.* Since $x_n=d_nd_{n-1}+p_n(d_{n-1}-d_n)$, it follows that

$$\frac{x_n}{p_n^2}=\frac{d_nd_{n-1}}{p_n^2}+\frac{d_{n-1}-d_n}{p_n} \tag{2.1}$$

hence

$$\frac{x_n^2}{p_n^4}\le 2\left(\frac{d_n^2d_{n-1}^2}{p_n^4}+\frac{(d_{n-1}-d_n)^2}{p_n^2}\right). \tag{2.2}$$

Since the series $\sum_{n=1}^{\infty}\frac{d_n^2}{p_n^2}$ is convergent and $\frac{d_{n-1}^2}{p_{n-1}^2}\sim\frac{d_{n-1}^2}{p_n^2}$, it follows that the series $\sum_{n=2}^{\infty}\frac{d_{n-1}^2}{p_n^2}$ is convergent as well. This implies that the series $\sum_{n=2}^{\infty}\frac{\max(d_n^2,d_{n-1}^2)}{p_n^2}$ is also convergent. Since

$$\frac{d_n^2d_{n-1}^2}{p_n^4}<\frac{\max(d_n^2,d_{n-1}^2)}{p_n^2}\quad\text{and}\quad\frac{(d_{n-1}-d_n)^2}{p_n^2}<\frac{\max(d_n^2,d_{n-1}^2)}{p_n^2},$$

we deduce by (2.2) that the series $\sum_{n=2}^{\infty}\frac{x_n^2}{p_n^4}$ is convergent. □

**Lemma 2.5.** *For $x > 0$ we have*

$$\sum_{p_n \le x} \left| \frac{q_n - q_{n-1}}{q_{n-1}} \right| \asymp \log x.$$

*Proof.* In view of Lemma 2.2, we have

$$\left( \frac{q_n - q_{n-1}}{q_{n-1}} \right)^2 + \left| \frac{q_n - q_{n-1}}{q_{n-1}} \right| \ge \left| \log \frac{q_n}{q_{n-1}} \right| \ge \left| \frac{q_n - q_{n-1}}{q_{n-1}} \right| - \frac{1}{2} \left( \frac{q_n - q_{n-1}}{q_{n-1}} \right)^2 .$$

Since

$$\frac{q_n - q_{n-1}}{q_{n-1}} = \frac{\frac{p_n+1}{p_n} - \frac{p_n}{p_{n-1}}}{\frac{p_n}{p_{n-1}}} = -\frac{x_n}{p_n^2},$$

we have

$$\frac{1}{2} \cdot \frac{x_n^2}{p_n^4} + \left| \log \frac{q_n}{q_{n-1}} \right| > \left| \frac{q_n - q_{n-1}}{q_{n-1}} \right| \ge \left| \log \frac{q_n}{q_{n-1}} \right| - \frac{x_n^2}{p_n^4}. \tag{2.3}$$

Now the desired conclusion follows by (1.1) and Lemma 2.4. □

*Proof of Theorem 2.1.* By the relation (2.1) we have

$$S_n = \sum_{k=2}^{n} \frac{x_k}{p_k^2} = \sum_{k=2}^{n} \frac{d_k d_{k-1}}{p_k^2} + \sum_{k=2}^{n} \frac{d_{k-1} - d_k}{p_k}.$$

Since $d_k d_{k-1} \le \max(d_k^2, d_{k-1}^2)$, and since the series $\sum_{n=2}^{\infty} \frac{\max(d_n^2, d_{n-1}^2)}{p_n^2}$ is convergent, see the proof of Lemma 2.4, it follows that the series $\sum_{n=2}^{\infty} \frac{d_n d_{n-1}}{p_n^2}$ is convergent too. Consequently the sequence $(S'_n)_{n\geq 1}$, defined by $S'_n = \sum_{k=2}^{n} \frac{d_k d_{k-1}}{p_k^2}$ is convergent. Lemma 2.3 implies that the sequence $(S''_n)_{n\geq 1}$, defined by $S''_n = \sum_{k=2}^{n} \frac{d_{k-1}-d_k}{p_k}$ is convergent as well. It then follows that the sequence $(S_n)_{n\geq 2}$ is convergent, that is, the series $\sum_{n=2}^{\infty} \frac{x_n}{p_n^2}$ is convergent.

On the other hand, Lemma 2.5 and the relation $\left|\frac{q_n - q_{n-1}}{q_{n-1}}\right| = \frac{|x_n|}{p_n^2}$ imply that

$$\sum_{p_n \leq x} \frac{|x_n|}{p_n^2} \asymp \log x, \tag{2.4}$$

hence the series $\sum_{n=2}^{\infty} \frac{x_n}{p_n^2}$ is not absolutely convergent. □

## 3. The Series $\sum_{n=2}^{\infty} \frac{|x_n|}{p_n^2 \log^\alpha n}$

Since the series $\sum_{n=2}^{\infty} \frac{|x_n|}{p_n^2}$ is divergent, it is natural to study what "correction" does it need to become convergent. In this connection, we prove the following fact.

**Theorem 3.1.** *The series* $\sum_{n=2}^{\infty} \frac{|x_n|}{p_n^2 \log^\alpha n}$ *is convergent if and only if* $\alpha > 1$.

*Proof.* We are going to put to use a technique from [3].

To begin with, we recall an inequality due to Abel: Let $a_k, b_k \in \mathbb{R}$, $k \in \overline{1,n}$ such that, if $S_i = \sum_{k=1}^{i} b_k$, then $S_i \geq 0$ for $i \in \overline{1,n}$. Then $\sum_{i=1}^{n} a_i b_i = S_1(a_1 - a_2) + S_2(a_2 - a_3) + \cdots + S_{n-1}(a_{n-1} - a_n) + S_n a_n$, which implies the inequalities

$$\sum_{i=1}^{n} a_i b_i \geq a_n S_n \text{ provided } a_1 \geq \cdots \geq a_n, \tag{3.1}$$

and

$$\sum_{i=1}^{n} a_i b_i \leq a_n S_n \text{ when } a_1 \leq \cdots \leq a_n. \tag{3.2}$$

It follows by (2.4) that there exist $c_1$ and $c_2$ such that $0 < c_1 < c_2$ and

$$c_1 \log x < \sum_{p_n \leq x} \frac{|x_n|}{p_n^2} < c_2 \log x \text{ for all } x \geq 2. \tag{3.3}$$

For $\alpha > 0$ and $n \geq 1$, we denote $a_1 = 1, b_1 = 0$ and for $i \geq 2$ $a_i = \frac{1}{\log^\alpha i}$ and $b_i = \frac{c'}{i} \cdot \frac{|x_i|}{p_i^2}$, where $c' > 0$ is chosen such that $S_1, S_2, \dots, S_n \geq 0$. Such a choice is possible because $\sum_{2 \leq i \leq x} \frac{1}{i} \sim \log x$ and (3.3) holds.

It now follows by (3.1) that $\sum_{i=2}^n \frac{1}{\log^\alpha i}\left(\frac{c'}{i} - \frac{|x_i|}{p_i^2}\right) \geq 0$, that is, $\sum_{i=2}^n \frac{|x_i|^2}{p_i^2 \log^\alpha i} < c' \sum_{i=2}^n \frac{1}{i \log^\alpha i}$. Since the series $\sum_{i=2}^\infty \frac{1}{i \log^\alpha i}$ is convergent for $\alpha > 1$, we deduce that the series $\sum_{n=2}^\infty \frac{|x_n|}{p_n^2 \log^\alpha n}$ is convergent as well.

One can similarly show that there exists $c'' > 0$ such that

$$\sum_{i=2}^n \frac{|x_i|^2}{p_i^2 \log^\alpha i} > c'' \sum_{i=2}^n \frac{1}{i \log^\alpha i}.$$

Since the series $\sum_{i=2}^\infty \frac{1}{i \log^\alpha i}$ is divergent for $\alpha \leq 1$, it follows that in this case the series $\sum_{n=2}^\infty \frac{|x_n|}{p_n^2 \log^\alpha n}$ is in turn divergent. □

# References


[1] P. ERDŐS AND A. RÉNYI, Some problems and results on consecutive primes, *Simon Stevin*, **27** (1950), 115–125.

[2] P. ERDŐS AND P. TURÁN, On some new question on the distribution of prime numbers, *Bull. Amer. Math. Soc.*, **54** (1948), 371–378.

[3] L. PANAITOPOL, Properties of the series of differences of prime numbers, *Publications du Centre de Recherches en Mathématiques Pures. Univ. Neuchâtel, Sér. I, Fasc.*, **31** (2000), 21–28.

[4] P. VLAMOS, Inequalities involving the sequence of the differences of the prime numbers, *Publications du Centre de Recherches en Mathématiques Pures. Univ. Neuchâtel, Sér. I, Fasc.*, **32** (2001), 32–40.