

# Journal of Inequalities in Pure and Applied Mathematics

ON THE SEQUENCE  $(p_n^2 - p_{n-1}p_{n+1})_{n \geq 2}$

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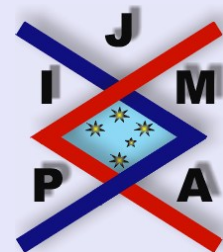
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Abstract

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## Abstract

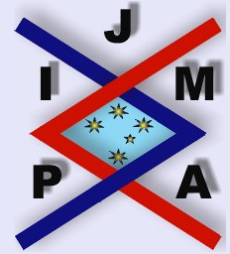
Let  $p_n$  be the  $n$ -th prime number and  $x_n = p_n^2 - p_{n-1}p_{n+1}$ . In this paper, we study sequences containing the terms of the sequence  $(x_n)_{n \geq 1}$ . The main result asserts that the series  $\sum_{n=1}^{\infty} x_n/p_n^2$  is convergent, without being absolutely convergent.

*2000 Mathematics Subject Classification:* 11A25, 11N05, 11N36

*Key words:* Prime Numbers, Sequences, Series, Asymptotic Behaviour.

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# 1. Introduction

We shall use the following notation:

$p_n$  the  $n$ -th prime number

$$x_n = p_n^2 - p_{n-1}p_{n+1} \text{ for } n \geq 2,$$

$$d_n = p_{n+1} - p_n \text{ for } n \geq 1,$$

$$q_n = \frac{p_{n+1}}{p_n} \text{ for } n \geq 1,$$

$f(x) \asymp g(x)$  if there exist  $c_1, c_2, M > 0$  such that

$$c_1 f(x) < g(x) < c_2 f(x) \text{ for every } x > M.$$

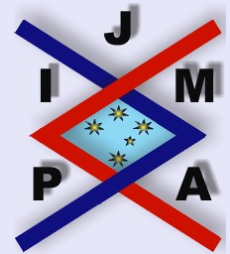
It will be our aim here to study the sequence  $(x_n)_{n \geq 2}$  defined above.

It was proved in [1] that the sequence  $(d_n)_{n \geq 1}$  is not monotone. A similar result holds for the sequence  $(q_n)_{n \geq 1}$  as well. This means that the sequence  $(x_n)_{n \geq 2}$  has infinitely many positive terms, and infinitely many negative terms, hence it is not monotone.

In [1], the so-called method of the triple sieve (due to Vigo Brun) was used to prove that

$$(1.1) \quad \sum_{p_n \leq x} \left| \log \frac{q_n}{q_{n-1}} \right| \asymp \log x.$$

This result plays an essential role in the following paragraph of the present paper.



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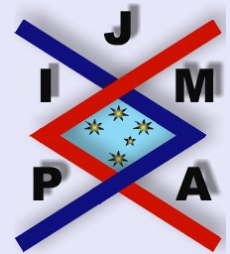
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Another useful result is proved in [4]:

(1.2) the series  $\sum_{n=1}^{\infty} \left(\frac{d_n}{p_n}\right)^n$  is convergent.



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## 2. The Series $\sum_{n=2}^{\infty} \frac{x_n}{p_n^2}$

**Theorem 2.1.** *The series  $\sum_{n=2}^{\infty} \frac{x_n}{p_n^2}$  is convergent, but it is not absolutely convergent.*

In order to prove this fact, we need the following lemmas.

**Lemma 2.2.** *For  $x \geq -\frac{1}{2}$ , we have*

$$x^2 + |x| \geq |\log(1+x)| \geq |x| - \frac{x^2}{2}.$$

*Proof.* The inequalities are well known for  $x > 0$ . When  $x \in [-\frac{1}{2}, 0]$ , they take on the form

$$x^2 - x \geq -\log(1+x) \geq -x - \frac{x^2}{2}.$$

Let  $f, g: [-\frac{1}{2}, 0] \rightarrow \mathbb{R}$  be defined by  $f(x) = \log(1+x) - x - \frac{x^2}{2}$  and  $g(x) = \log(1+x) - x + x^2$ , respectively. We have  $f'(x) = -\frac{x(x+2)}{1+x} \geq 0$ , and  $g'(x) = \frac{x(2x+1)}{1+x} \leq 0$ . Since  $f$  is increasing and  $f(0) = 0$ , we get  $f(x) \leq 0$ . On the other hand, we have  $g'(x) < 0$  and  $g(0) = 0$ , so that  $g(x) \geq 0$ .  $\square$

**Lemma 2.3.** *The series  $\sum_{n=2}^{\infty} \frac{d_n - d_{n-1}}{p_n}$  is convergent.*

*Proof.* Denote  $S_n = \sum_{k=2}^n \frac{d_k - d_{k-1}}{p_k}$ , so that

$$S_n = \frac{d_n}{p_n} + \sum_{k=2}^n \frac{d_k^2}{p_k p_{k-1}} - \frac{1}{2}.$$



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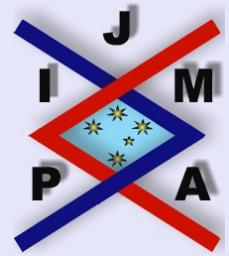


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Since  $\lim_{n \rightarrow \infty} \frac{p_{n+1}}{p_n} = 1$ , it suffices to prove that the series  $\sum_{k=2}^{\infty} \frac{d_{k-1}^2}{p_k p_{k-1}}$  is convergent. Since  $\frac{d_{k-1}^2}{p_k p_{k-1}} \sim \left(\frac{d_{k-1}}{p_{k-1}}\right)^2$  and the terms of the series are positive, it follows that the series  $\sum_{k=2}^{\infty} \frac{d_{k-1}^2}{p_k p_{k-1}}$  and  $\sum_{k=2}^{\infty} \left(\frac{d_{k-1}}{p_{k-1}}\right)^2$  are simultaneously convergent or not. Now just use (1.2) and the proof ends.  $\square$

**Lemma 2.4.** *The series  $\sum_{n=2}^{\infty} \frac{x_n^2}{p_n^4}$  is convergent.*

*Proof.* Since  $x_n = d_n d_{n-1} + p_n(d_{n-1} - d_n)$ , it follows that

$$(2.1) \quad \frac{x_n}{p_n^2} = \frac{d_n d_{n-1}}{p_n^2} + \frac{d_{n-1} - d_n}{p_n}$$

hence

$$(2.2) \quad \frac{x_n^2}{p_n^4} \leq 2 \left( \frac{d_n^2 d_{n-1}^2}{p_n^4} + \frac{(d_{n-1} - d_n)^2}{p_n^2} \right).$$

Since the series  $\sum_{n=1}^{\infty} \frac{d_n^2}{p_n^2}$  is convergent and  $\frac{d_{n-1}^2}{p_{n-1}^2} \sim \frac{d_{n-1}^2}{p_n^2}$ , it follows that the series  $\sum_{n=2}^{\infty} \frac{d_{n-1}^2}{p_n^2}$  is convergent as well. This implies that the series  $\sum_{n=2}^{\infty} \frac{\max(d_n^2, d_{n-1}^2)}{p_n^2}$  is also convergent. Since

$$\frac{d_n^2 d_{n-1}^2}{p_n^4} < \frac{\max(d_n^2, d_{n-1}^2)}{p_n^2} \quad \text{and} \quad \frac{(d_{n-1} - d_n)^2}{p_n^2} < \frac{\max(d_n^2, d_{n-1}^2)}{p_n^2},$$

we deduce by (2.2) that the series  $\sum_{n=2}^{\infty} \frac{x_n^2}{p_n^4}$  is convergent.  $\square$

**Lemma 2.5.** For  $x > 0$  we have

$$\sum_{p_n \leq x} \left| \frac{q_n - q_{n-1}}{q_{n-1}} \right| \asymp \log x.$$

*Proof.* In view of Lemma 2.2, we have

$$\left( \frac{q_n - q_{n-1}}{q_{n-1}} \right)^2 + \left| \frac{q_n - q_{n-1}}{q_{n-1}} \right| \geq \left| \log \frac{q_n}{q_{n-1}} \right| \geq \left| \frac{q_n - q_{n-1}}{q_{n-1}} \right| - \frac{1}{2} \left( \frac{q_n - q_{n-1}}{q_{n-1}} \right)^2.$$

Since

$$\frac{q_n - q_{n-1}}{q_{n-1}} = \frac{\frac{p_{n+1}}{p_n} - \frac{p_n}{p_{n-1}}}{\frac{p_n}{p_{n-1}}} = -\frac{x_n}{p_n^2},$$

we have

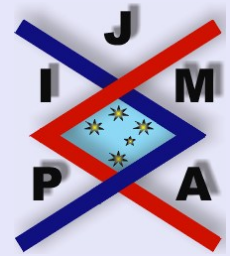
$$(2.3) \quad \frac{1}{2} \cdot \frac{x_n^2}{p_n^4} + \left| \log \frac{q_n}{q_{n-1}} \right| > \left| \frac{q_n - q_{n-1}}{q_{n-1}} \right| \geq \left| \log \frac{q_n}{q_{n-1}} \right| - \frac{x_n^2}{p_n^4}.$$

Now the desired conclusion follows by (1.1) and Lemma 2.4.  $\square$

*Proof of Theorem 2.1.* By the relation (2.1) we have

$$S_n = \sum_{k=2}^n \frac{x_k}{p_k^2} = \sum_{k=2}^n \frac{d_k d_{k-1}}{p_k^2} + \sum_{k=2}^n \frac{d_{k-1} - d_k}{p_k}.$$

Since  $d_k d_{k-1} \leq \max(d_k^2, d_{k-1}^2)$ , and since the series  $\sum_{n=2}^{\infty} \frac{\max(d_n^2, d_{n-1}^2)}{p_n^2}$  is convergent, see the proof of Lemma 2.4, it follows that the series  $\sum_{n=2}^{\infty} \frac{d_n d_{n-1}}{p_n^2}$  is



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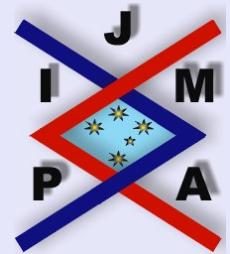
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convergent too. Consequently the sequence  $(S'_n)_{n \geq 1}$ , defined by  $S'_n = \sum_{k=2}^n \frac{d_k d_{k-1}}{p_k^2}$  is convergent. Lemma 2.3 implies that the sequence  $(S''_n)_{n \geq 1}$ , defined by  $S''_n = \sum_{k=2}^n \frac{d_{k-1} - d_k}{p_k}$  is convergent as well. It then follows that the sequence  $(S_n)_{n \geq 2}$  is convergent, that is, the series  $\sum_{n=2}^{\infty} \frac{x_n}{p_n^2}$  is convergent.

On the other hand, Lemma 2.5 and the relation  $\left| \frac{q_n - q_{n-1}}{q_{n-1}} \right| = \frac{|x_n|}{p_n^2}$  imply that

$$(2.4) \quad \sum_{p_n \leq x} \frac{|x_n|}{p_n^2} \asymp \log x,$$

hence the series  $\sum_{n=2}^{\infty} \frac{x_n}{p_n^2}$  is not absolutely convergent. □



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### 3. The Series $\sum_{n=2}^{\infty} \frac{|x_n|}{p_n^2 \log^\alpha n}$

Since the series  $\sum_{n=2}^{\infty} \frac{|x_n|}{p_n^2}$  is divergent, it is natural to study what “correction” does it need to become convergent. In this connection, we prove the following fact.

**Theorem 3.1.** *The series  $\sum_{n=2}^{\infty} \frac{|x_n|}{p_n^2 \log^\alpha n}$  is convergent if and only if  $\alpha > 1$ .*

*Proof.* We are going to put to use a technique from [3].

To begin with, we recall an inequality due to Abel: Let  $a_k, b_k \in \mathbb{R}$ ,  $k \in \overline{1, n}$  such that, if  $S_i = \sum_{k=1}^i b_k$ , then  $S_i \geq 0$  for  $i \in \overline{1, n}$ . Then  $\sum_{i=1}^n a_i b_i = S_1(a_1 - a_2) + S_2(a_2 - a_3) + \dots + S_{n-1}(a_{n-1} - a_n) + S_n a_n$ , which implies the inequalities

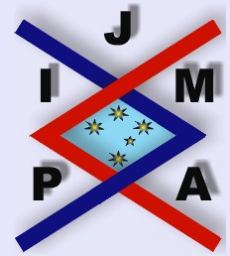
$$(3.1) \quad \sum_{i=1}^n a_i b_i \geq a_n S_n \quad \text{provided } a_1 \geq \dots \geq a_n,$$

and

$$(3.2) \quad \sum_{i=1}^n a_i b_i \leq a_1 S_n \quad \text{when } a_1 \leq \dots \leq a_n.$$

It follows by (2.4) that there exist  $c_1$  and  $c_2$  such that  $0 < c_1 < c_2$  and

$$(3.3) \quad c_1 \log x < \sum_{p_n \leq x} \frac{|x_n|}{p_n^2} < c_2 \log x \quad \text{for all } x \geq 2.$$



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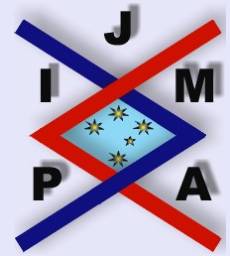
For  $\alpha > 0$  and  $n \geq 1$ , we denote  $a_1 = 1, b_1 = 0$  and for  $i \geq 2$   $a_i = \frac{1}{\log^\alpha i}$  and  $b_i = \frac{c'}{i} \cdot \frac{|x_i|}{p_i^2}$ , where  $c' > 0$  is chosen such that  $S_1, S_2, \dots, S_n \geq 0$ . Such a choice is possible because  $\sum_{2 \leq i \leq x} \frac{1}{i} \sim \log x$  and (3.3) holds.

It now follows by (3.1) that  $\sum_{i=2}^n \frac{1}{\log^\alpha i} \left( \frac{c'}{i} - \frac{|x_i|}{p_i^2} \right) \geq 0$ , that is,  $\sum_{i=2}^n \frac{|x_i|^2}{p_i^2 \log^\alpha i} < c' \sum_{i=2}^n \frac{1}{i \log^\alpha i}$ . Since the series  $\sum_{i=2}^\infty \frac{1}{i \log^\alpha i}$  is convergent for  $\alpha > 1$ , we deduce that the series  $\sum_{n=2}^\infty \frac{|x_n|}{p_n^2 \log^\alpha n}$  is convergent as well.

One can similarly show that there exists  $c'' > 0$  such that

$$\sum_{i=2}^n \frac{|x_i|^2}{p_i^2 \log^\alpha i} > c'' \sum_{i=2}^n \frac{1}{i \log^\alpha i}.$$

Since the series  $\sum_{i=2}^\infty \frac{1}{i \log^\alpha i}$  is divergent for  $\alpha \leq 1$ , it follows that in this case the series  $\sum_{n=2}^\infty \frac{|x_n|}{p_n^2 \log^\alpha n}$  is in turn divergent.  $\square$



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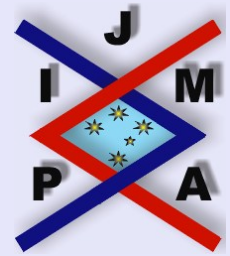
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