Drawing Clustered Planar Graphs on Disk Arrangements

Tamara Mchedlidze$^1$ Marcel Radermacher$^1$
Ignaz Rutter$^2$ Nina Zimbel

$^1$Department of Computer Science, Karlsruhe Institute of Technology, Germany
$^2$Department of Computer Science and Mathematics, University of Passau, Germany

Abstract

Let $G = (V, E)$ be a planar graph and let $\mathcal{V}$ be a partition of $V$. We refer to the graphs induced by the vertex sets in $\mathcal{V}$ as clusters. Let $\mathcal{D}_C$ be an arrangement of pairwise disjoint disks with a bijection between the disks and the clusters. Akitaya et al. [2] give an algorithm to test whether $(G, \mathcal{V})$ can be embedded onto $\mathcal{D}_C$ with the additional constraint that edges are routed through a set of pipes between the disks. If such an embedding exists, we prove that every clustered graph and every disk arrangement without pipe-disk intersections has a planar straight-line drawing where every vertex is embedded in the disk corresponding to its cluster. This result can be seen as an extension of the result by Alam et al. [3] who solely consider biconnected clusters. Moreover, we prove that it is NP-hard to decide whether a clustered graph has such a straight-line drawing, if we permit pipe-disk intersections, even if all disks have unit size. This answers an open question of Angelini et al. [4].

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E-mail addresses: mched@iti.uka.de (Tamara Mchedlidze), radermacher@kit.edu (Marcel Radermacher), rutter@fim.uni-passau.de (Ignaz Rutter)
1 Introduction

We study whether a clustered planar graph \( C \) has a planar straight-line drawing on a prescribed set of disks where each edge is allowed to intersect the boundary of each disk at most once. More formally, a (flat) clustering of a graph \( G = (V, E) \) is a partition \( V = \{V_1, \ldots, V_k\} \) of the vertex set \( V \). We refer to the pair \( C = (G, V) \) as a clustered graph and the graphs \( G_i = (V_i, E_i) \) induced by \( V_i \) as clusters. The set of edges \( E_i \) of a cluster \( G_i \) are intra-cluster edges and the set of edges with endpoints in different clusters are inter-cluster edges. A disk arrangement \( D_C = \{D_1, \ldots, D_k\} \) of \( C \) is a set of disks in the plane together with a bijective mapping \( \mu(V_i) = D_i \) between the clusters \( V \) and the disks \( D \).

A pipe \( p_{ij} \) of two clusters \( V_i, V_j \) is the convex hull of the disks \( D_i \) and \( D_j \), i.e., the smallest convex set of points containing \( D_i \) and \( D_j \); see Figure 1. Observe that the boundary of \( p_{ij} \) is composed of two line segments \( u_{ij}, b_{ij} \) and two circular arcs. We refer to a topological planar drawing of \( G \), i.e., the drawing of each edge is a curve, as an embedding of \( G \). A \( D_C \)-framed embedding of \( G \) is an embedding of \( G \) where each vertex \( v \in V_i \) lies in the interior of the disk \( D_i \) and each edge \( uv \), with \( u \in V_i \) and \( v \in V_j \), lies entirely in the pipe of \( V_i \) and \( V_j \).

Given a cluster planar graph \( C \), a disk arrangement \( D_C \) of \( C \) and a \( D_C \)-framed embedding \( \psi \), Godau [11] proves that it is \( \mathcal{NP} \)-hard to decide whether \( G \) has a \( D_C \)-framed straight-line drawing \( \Gamma \) such that \( \psi \) is homeomorphic to \( \Gamma \). The gadgets in the proof contain disks of size 0, i.e., the positions of some vertices are fixed. Moreover, there are disks that are entirely contained in a larger disk, i.e., there exist two disks \( d_i, d_j, i \neq j \) with \( d_i \subset d_j \). Angelini et al. [4] consider the case where \( G \) is not embedded but all disks have unit size. More formally, they show that given a planar graph \( G \), it is \( \mathcal{NP} \)-hard to decide whether \( G \) has a \( D_C \)-framed straight-line drawing. For unit disks, they leave the computational complexity of the question whether a \( D_C \)-framed embedding has a corresponding \( D_C \)-framed straight-line drawing as an open question. Banyassady et al. [6] show that this problem is \( \mathcal{NP} \)-hard in case that \( G \) is the intersection graph of \( D_C \), i.e., each vertex corresponds to a disk and two vertices are joined by an edge if the intersection of the corresponding disks is not empty.

The computational complexity of the following problem has not been considered: Given a cluster planar graph \( C = (G, V) \), a set of pairwise disjoint disks \( D \) and a \( D_C \)-framed embedding \( \psi \), does \( C \) admit a \( D_C \)-framed straight-line drawing of \( C \) that is homeomorphic to \( \psi \). Thereby, we consider two \( D_C \)-framed embeddings \( \psi, \psi' \) of \( C \) to be homeomorphic if (i) \( \psi \) and \( \psi' \) have the same combinatorial embedding and the same outer face, (ii) each edge \( e \) of \( G \) crosses a line segment \( u_{ij} \) (or \( b_{ij} \)) of a pipe \( p_{ij} \) in \( \psi \) if and only if it crosses the respective line segment in \( \psi' \), (iii) and it does so in the same order. Observe that every edge in a \( D_C \)-framed straight-line drawing intersects the boundary of a pipe at most twice; see Figure 1. Thus, in the following we assume as a necessary condition that an edge in a \( D_C \)-framed embedding crosses the boundary of a pipe at most twice.
Figure 1: (a) The light-blue region shows the pipe $p_{ij}$ of the disks $D_i$ and $D_j$. An edge in a $D_C$-framed straight-line drawing intersects the boundary of a pipe at most two times. Thus, the $D_C$-framed embedding described in (b) does not correspond to $D_C$-framed straight-line drawing. The drawing in (c) is not homeomorphic to (a), since the edge in (c) intersects different parts of the boundaries of the pipes.

Related Work  Feng et al. [10] introduced the notion of clustered graphs and c-planarity. A graph $G$ together with a recursive partitioning of the vertex set is considered to be a clustered graph. An embedding of $G$ is c-planar if (i) each cluster $c$ is drawn within a connected region $R_c$, (ii) two regions $R_c, R_d$ intersect if and only if the cluster $c$ contains the cluster $d$ or vice versa, and (iii) every edge intersects the boundary of a region at most once. They prove that a c-planar embedding of a connected clustered graph can be computed in $O(n^2)$ time. It is an open question whether this result can be extended to disconnected clustered graphs. Many special cases of this problem have been considered [7].

Eades et al. [9] prove that every c-planar graph has a c-planar straight-line drawing where each cluster is drawn in a convex region. Angelini et al. [5] strengthen this result by showing that every c-planar graph has a c-planar straight-line drawing in which every cluster is drawn in an axis-parallel rectangle. The result of Akitaya et al. [2] implies that in $O(n \log n)$ time one can decide whether an abstract graph with a flat clustering has an embedding where each vertex lies in a prescribed topological disk and every edge is routed through a prescribed topological pipe. In general they ask whether a simplicial map $\varphi$ of $G$ onto a 2-manifold $M$ is a weak embedding, i.e., for every $\epsilon > 0$, $\varphi$ can be perturbed into an embedding $\psi_\epsilon$ with $||\varphi - \psi_\epsilon|| < \epsilon$.

Alam et al. [3] prove that it is $\text{NP}$-hard to decide whether an embedded clustered graph has a c-planar straight-line drawing where every cluster is contained in a prescribed (thin) rectangle and edges have to pass through the interval common for both rectangles. Further, they prove that all instances with biconnected clusters always admit a solution. Their result implies that graphs of this class have $D_C$-framed straight-line drawings.

Ribó [13] shows that every embedded clustered graph where each cluster is a set of independent vertices has a straight-line drawing such that every cluster lies in a prescribed disk. In contrast to our setting Ribó allows an edge $e$ to intersect a disk of a cluster $G_i$ that does not contain an endpoint of $e$. 
Figure 2: The cluster $G_i$ cannot be augmented with edges such that $G_i$ becomes biconnected.

**Contribution** We say that a disk arrangement $D_C$ is *pipe-disk intersection free* if each pipe $p_{ij}$ that contains an edge (i.e., $(V_i \times V_j) \cap E \neq \emptyset$) does not have an intersection with a disk $d_k$, where $k \neq i,j$. In Section 2 we prove that if the disk arrangement $D_C$ is pipe-disk intersection free and each pair of disks is disjoint, then every clustered planar graph $(G, V)$ with a $D_C$-framed embedding $\psi$ has a $D_C$-framed planar straight-line drawing homeomorphic to $\psi$. Taking the result of Akitaya et al. [2] into account, our result can be used to test whether an abstract clustered graph with connected clusters has a $D_C$-framed straight-line drawing. The example in Figure 2 shows that in general clusters cannot be augmented to be biconnected, if the embedding is fixed. Hence, our result is generalization of the result of Alam et al. [3]. In Section 3 we show that the problem is $\mathcal{NP}$-hard in the case that the disk arrangements is not pipe-disk intersection free. More specifically, we show that the problem is $\mathcal{NP}$-hard in case of arrangements of unit disks and as well as in the case of axis-aligned unit squares. This answers the aforementioned open question of Angelini et al. [4]. From now on we refer to a $D_C$-framed straight-line drawing of $G$ simply as a $D_C$-framed drawing of $G$.

## 2 Drawing on Disk Arrangements that are Pipe-Disk Intersection Free

Let $C = (G, V)$ be a clustered planar graph, let $D_C$ be a disk arrangement with pairwise disjoint disks that is pipe-disk intersection free, and let $\psi$ be a $D_C$-framed embedding of $C$. In this section we prove that $C$ has a $D_C$-framed drawing that is homeomorphic to $\psi$. We prove the statement by induction on the number of intra-cluster edges. In Lemma 1 we show that we can indeed reduce the number of intra-cluster edges by contracting intra-cluster edges. In Lemma 2 we prove that the statement is correct if the outer face of $C$ is a triangle and $C$ is connected, i.e., each cluster $G_i$ is connected. In Theorem 1 we extend this result to clustered graphs whose clusters are not connected.

A triangle $T$ in an embedded planar graph $G$ is *separating* if the interior and exterior of $T$ each contain a vertex of $G$. Let $e = uv$ be an intra-cluster edge of $G$ that is not an edge of a separating triangle. We obtain a contracted clustered graph $C/e$ of $C$ by removing $v$ from $G$ and connecting the neighbors of $v$ to $u$. We obtain a corresponding embedding $\psi/e$ from $\psi$ by routing the edges $vw \in E, w \neq u$ close to the original drawing of $uv$. 
Lemma 1 Let $C = (G, V)$ be a connected clustered planar graph, $D_C$ be a disk arrangement with pairwise disjoint disks that is pipe-disk intersection free and let $\psi$ be $D_C$-framed embedding of $C$. Let $e$ be an intra-cluster edge that is not an edge of a separating triangle. Then $C$ has a $D_C$-framed drawing that is homeomorphic to $\psi$ if $C/e$ has a $D_C$-framed drawing that is homeomorphic to $\psi/e$.

Proof: Let $e = uv$ and denote by $u_0, u_1, \ldots, u_k$ the neighbors of $u$ and denote by $v_0, v_1, \ldots, v_l$ the neighbors of $v$ in $C$ in clockwise order; see Figure 3a. Without loss of generality, we assume that $u_0 = v$ and $v_0 = u$. Since $e$ is not an edge of a separating triangle the set $I := \{u_2, \ldots, u_{k-1}\} \cap \{v_2, \ldots, v_{l-1}\}$ is empty. Denote by $u$ the vertex obtained by the contraction of $e$. Let $G_i$ be the cluster of $u$ and $v$, and let $D_i$ be the corresponding disk in $D_C$.

Consider a $D_C$-framed drawing $\Gamma/e$ of $C/e$ homeomorphic to $\psi/e$; see Figure 3b. Then there is a small disk $D_u \subset D_i$ around $u$ such that for every point $p$ in $D_u$ moving $u$ to $p$ yields a $D_C$-framed drawing that is homeomorphic to $\psi/e$.

We obtain a straight-line drawing $\Gamma$ of $C$ from $\Gamma/e$ as follows; see Figure 3c. First, we remove the edges $u_i$ from $\Gamma/e$. The edges $u_1, u_k$ partition $D_u$ into two regions $r_u, r_v$ such that the intersection of $r_v$ with $u_i$ is empty for all $i \in \{2, \ldots, k-1\}$. We place $v$ in $r_v$ and connect it to $u$ and the vertices $v_1, \ldots, v_l$. Since $r_v$ is a subset of $D_u$ and $I = \emptyset$, we have that the new drawing $\Gamma$ is planar. Since $v$ is placed in $r_v$, the edge $uv$ is in between $u_1$ and $u_k$ in the rotational order of edges around $u$. Hence, $\Gamma$ is homeomorphic to $\psi$. Finally, $\Gamma$ is a $D_C$-framed drawing since, $D_u$ is entirely contained in $D_i$ and thus are $u$ and $v$.

Lemma 2 Let $C$ be a connected clustered graph with a triangular outer face $T$, let $D_C$ be a disk arrangement with pairwise disjoint disks that is pipe-disk intersection free, and let $\psi$ be a $D_C$-framed embedding of $C$. Moreover, let $\Gamma_T$ be a $D_C$-framed drawing of $T$. Then $C$ has a $D_C$-framed drawing that is homeomorphic to $\psi$ with the outer face drawn as $\Gamma_T$.

Proof: We prove the theorem by induction on the number of intra-cluster edges.

Figure 3: (a) Since $uv$ is not an edge of a separating triangle edges $xu, xv$ do not exist. (b) Moving $u$ within disk $d_u$ preserves the embedding of $G/uv$. (c) Drawing of $G$ obtained from (b) by placing $v$ in $r_v$. 

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\includegraphics[width=\textwidth]{figure3c}
\caption{(c)}
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First, consider the case that every intra-cluster edge of $C$ is an edge on the boundary of the outer face. Note that there are at most three vertices in the interior of a single disk. Thus, $C$ is either a triangle as depicted in Figure 4a and Figure 4b, or each cluster is a single vertex. Since $D_C$ is pipe-disk intersection free, the graph in Figure 4a and Figure 4b $C$ does not contain any further vertices. Let $\Gamma$ be the drawing obtained from $\Gamma_T$ by placing every vertex that does not lie on the outer face on the center point of its corresponding disk. Since $D_C$ is a pipe-disk intersection free and $\Gamma_T$ is convex, the resulting drawing is planar and thus a $D_C$-framed drawing of $C$ that is homeomorphic to the embedding $\psi$.

Let $S$ be a separating triangle of $C$ that splits $C$ into two subgraphs $C_{in}$ and $C_{out}$ so that $C_{in} \cap C_{out} = S$ and the outer face of $C_{out}$ and $C$ coincide. Note that $C_{in}$ and $C_{out}$ are connected as otherwise $C$ itself would not be connected. Then by the induction hypothesis $C_{out}$ has the $D_C$-framed drawing $\Gamma_{out}$ with the outer face drawn as $\Gamma_{out}[S]$, where $\Gamma_{out}[S]$ is the drawing of $S$ in $\Gamma_{out}$. Then we obtain a $D_C$-framed drawing of $C$ by merging $\Gamma_{in}$ and $\Gamma_{out}$.

Consider an intra-cluster edge $e$ that does not lie on the boundary of the outer face and is not an edge of a separating triangle. Then by the induction hypothesis, $C/e$ has a $D_C$-framed drawing with the outer face drawn as $\Gamma_{in}$. It follows by Lemma 1 that $C$ has a $D_C$-framed drawing homeomorphic to $\psi$. □

**Theorem 1** Every clustered graph $C$ with a $D_C$-framed embedding $\psi$ has a $D_C$-framed drawing homeomorphic to $\psi$ if the disk arrangement $D_C$ is pairwise disjoint and pipe-disk intersection free.

**Proof:** We obtain a clustered graph $C'$ from $C$ by adding a new triangle $T$ to the graph and assigning each vertex of $T$ to a newly constructed cluster. Let $\Gamma_T$ be a drawing of $T$ that contains all disks in $D_C$ in its interior. We obtain a new disk arrangement $D_C'$ from $D_C$ by adding a sufficiently small disk for each vertex of $\Gamma_T$. The embedding $\psi$ together with $\Gamma_T$ is a $D_C'$-framed embedding $\psi'$ of $C'$.

According to Feng et al. [10] there is a simple connected clustered graph $C''$ that contains $C'$ as a subgraph whose embedding $\psi''$ is $D_C$-framed and contains $\psi'$. By Lemma 2 there is a $D_C$-framed drawing $\Gamma''$ of $C''$ homeomorphic to $\psi''$.
with the outer face drawn as $\Gamma_T$. The drawing $\Gamma''$ contains a $\mathcal{D}_C$-framed drawing of $\mathcal{C}$. □

3 Drawing on Arrangements with Pipe-Disk Intersections

In this section we study the following problem referred to as $\mathcal{D}_C$-framed Drawings with Pipe-Disk Intersections. Given a planar clustered graph $\mathcal{C} = (G, \mathcal{V})$, a disk arrangement $\mathcal{D}_C$ with pairwise disjoint disks that is not disk-pipe intersection free, and a $\mathcal{D}_C$-framed embedding $\psi$ of $\mathcal{C}$, is there a $\mathcal{D}_C$-framed drawing $\Gamma$ that is homeomorphic to $\psi$?

Note that if the disks $\mathcal{D}_C$ are allowed to overlap and $G$ is the intersection graph of $\mathcal{D}_C$, the problem is known to be $\text{NP}$-hard [6]. Thus, in the following we require that the disks do not overlap, but there can be pipe-disk intersections. By Alam at al. [3] it follows that the problem restricted to thin touching rectangles instead of disks is $\text{NP}$-hard. Their reduction heavily relies on the fact that the rectangles are thin. We strengthen this result and prove that in case that the rectangles are either axis-aligned unit squares or unit disks and are not allowed to touch the problem remains $\text{NP}$-hard.

To prove $\text{NP}$-hardness we reduce from Planar Monotone 3-SAT [8]. For each literal and clause we construct a clustered graph $\mathcal{C}$ with an arrangement of disks (squares) $\mathcal{D}_C$ of $\mathcal{C}$ such that each disk (square) contains exactly one vertex. We refer to these instances as literal and clause gadgets. In order to transport information from the literals to the clauses, we construct a copy and inverter gadget. For each gadget we first construct an arrangement of unit squares and state its important properties in this case. This is followed by the corresponding arrangement of unit disks. We emphasize the differences that have to be dealt with to preserve the properties of the gadgets when considering unit disks instead of unit squares. The design of the gadgets is inspired by Alam et al. [3], but the restriction to unit disks and squares rather than thin touching rectangles, requires a more complex construction and a careful placement of the geometric objects. The green and red regions in the figures of the gadget correspond to positive and negative drawings of the literal gadget. The green and red line segments indicate that for each truth assignment of the variables our gadgets indeed have $\mathcal{D}_C$-framed straight-line drawings. Negative versions of the literal and clause gadget are obtained by mirroring vertically. Hence, we assume that variables and clauses are positive. Each gadget covers a set of checkerboard cells. This simplifies the assembly of the gadgets in the final reduction. Note that in the following constructions all squares and disks will be of unit size. Moreover, we consider only axis-aligned squares.

3.1 Regulator

A line $l$ separates the euclidean plane in two half planes $h_a$ and $h_b$ and we denote by $\overline{h_a}$ the complement of $h_a$. These half planes are spanned by $l$. We
say that \( l \) supports \( h_a \) (\( h_b \)). Let \( B \) be an axis-aligned square that contains a vertex \( v \) in its interior and let \( h_a, h_b \) be two half planes whose supporting lines have a unique intersection point \( q \) that lies in the interior of \( B \); see Figure 5.

We describe the construction of a gadget that restricts the feasible placements of \( v \) in a \( \mathcal{D}_C \)-framed drawing by a half plane \( h \) that excludes a placement of \( v \) in \( h_a \cap h_b \) but allows for a placement in \( h_a \cap B \) or \( h_b \cap B \). Since \( q \) lies in the interior of \( B \), there is a half plane \( h \) that does not contain \( q \) and for each \( i = a, b \), \( h_i \cap h_a \cap h_b = \emptyset \).

Let \( h, h_a, h_b \) and \( B \) as described before. We construct a regulator gadget of \( v \) in \( B \) with respect to \( h_a \) and \( h_b \) as follows. Let \( l_h \) be the supporting line of \( h \). We create two axis-aligned squares \( R \) and \( O \) such that \( R, O \) and \( B \) intersect \( l_h \) in this order and \( h \) neither intersects the interior of \( R \) nor the interior of \( O \). Place a vertex \( u \) in \( R \) and route an edge \( uv \) through \( h \cup R \cup B \). In case that \( h \) instead of \( h_a \) and \( h_b \) is given, we refer to the gadget as the regulator of \( v \) with respect to a (single) half plane \( h \).

**Lemma 3** Let \( W \) be a regulator gadget of \( v \) in \( B \) with respect to \( h_a \) and \( h_b \). For every point \( p_v \in h \cap B \) there is a \( \mathcal{D}_C \)-framed drawing \( \Gamma \) such that \( v \) lies on \( p_v \). There is no \( \mathcal{D}_C \)-framed drawing of \( W \) such that \( v \) lies in \( h \cap B \).

**Proof:** By construction of \( W \), there is for every point \( p_v \in h \cap B \) a \( \mathcal{D}_C \)-framed drawing \( \Gamma \) such that \( v \) lies on \( p_v \).

The supporting line \( l_h \) of \( h \) intersects the boundary of \( R \) and does not intersect the interior of \( O \). Let \( r \) and \( o \) be points in the intersection of \( l_h \) with \( R \) and \( O \), respectively. Since \( \Gamma \) is homeomorphic to \( W \) the edge \( uv \) intersects \( l_h \) on the ray starting in \( o \) in the direction towards \( r \). Therefore, \( u \) and \( v \) lie on different sides of \( l_h \). Since \( u \in R \), it follows that \( v \in h \). \( \square \)

We refer to the intersection \( h \cap B \) as the regulated region of \( v \) in \( B \). Thus, by the construction of \( W \), the regulated region \( Q \) has a non-empty intersection.
with \( h_a \cap B \) and \( h_b \cap B \). Thus, by the lemma for each placement of \( v \) in \( Q \cap h_i \cap B, i = a,b \), there is a \( \mathcal{D}_C \)-framed drawing. On the other hand, since \( h \cap h_a \cap h_b \cap B = \emptyset \), there is no \( \mathcal{D}_C \)-framed drawing such that \( v \) lies in \( h_a \cap h_b \cap B \).

### 3.2 Literal Gadget

In this section we construct a clustered graph \( C \) with an arrangement of squares \( \mathcal{D}_C \) that models a literal \( u \). The positive literal gadget is depicted in Figure 6a. We obtain the negative literal gadget by mirroring vertically.

The center block is a unit square \( C \) with corners \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) in clockwise order. For each corner \( \alpha_i \) of \( C \) consider a line \( l_i \) that is tangent to \( C \) in \( \alpha_i \), i.e. \( l_i \cap C = \{ \alpha_i \} \). Let \( p_i \) be the intersection of the lines \( l_{i-1} \) and \( l_i \) where \( l_0 = l_4 \); refer to Figure 6a. Let \( R_1, \ldots, R_4 \) be four pairwise non-intersecting squares that are disjoint from \( C \) such that \( R_i \) contains \( p_i \) in its interior. We add a cycle \( v_1 v_2 v_3 v_4 v_1 \) to the graph such that \( v_i \in R_i \). We refer to the vertex \( v_i \) as the cycle vertex of the cycle block \( R_i \). For each \( i \), let \( \eta_i \) be a half plane that contains \( R_{i+1} \) but does not intersect \( C \). Within \( \eta_i \) we place a regulator \( W_i \) of \( v_i \) with respect to \( h_i \) and \( h_{i-1} \), where \( h_i \) is the half plane spanned by \( l_i \) that does not contain \( C \). This finishes the construction.

We now show that there exist two disjoint regions \( P_i \) and \( N_i \) in \( R_i \) that correspond to a positive and negative drawing of the literal gadget. Consider \( R_1 \) and its two adjacent squares \( R_4 \) and \( R_2 \). Let \( Q_i \) be the regulated region of \( R_i \) with respect to \( W_i \). Then the intersection \( I_1 := \overline{h_4} \cap \overline{h_1} \cap Q_1 \neq \emptyset \). We refer to \( I_1 \) as the infeasible region of \( R_1 \). The intersection \( h_1 \cap Q_1 \) is the positive region \( P_1 \) of \( R_1 \). The region \( h_4 \cap Q_1 \) is the negative region \( N_1 \) of \( R_1 \). Regions...
Figure 7: Since \( v_i \) does not lie in \( h_i \cap R_i \) (green) and \( l_i \) is tangent to \( C \), \( v_{i+1} \) lies in the \( h_{i+1} \cap R_{i+1} \) (red).

\( P_1, N_1, I_1 \) are by construction not empty. The positive, negative and infeasible region of \( R_i, i \neq 1 \) are defined analogously.

**Property 2** If \( \Gamma \) is a \( DC \)-framed drawing of a literal gadget, then no cycle vertex \( v_i \) lies in the infeasible region of \( R_i \). Moreover, either each cycle vertex \( v_i \) lies in the positive region \( P_i \) or each vertex \( v_i \) lies in the negative region \( N_i \).

**Proof:** Consider a \( DC \)-framed drawing \( \Gamma \) with an edge \( v_iv_{i+1} \) such that \( v_i \) lies in \( P_i \), i.e., \( v_i \) lies in \( h_i \cap R_i \); see Figure 7. We show that \( v_{i+1} \) lies in \( N_{i+1} \). If \( v_{i+1} \) lies in \( h_i \), then \( v_i \) and \( v_{i+1} \) lie on the same side of \( l_i \). Since \( l_i \) is tangent to \( \alpha_i \), \( v_i v_{i+1} \) intersects \( C \). It follows that \( v_{i+1} \) lies in \( h_i \) and therefore in the negative region \( N_{i+1} \).

Assume that \( v_1 \) lies in its infeasible region \( I_1 \), then \( v_2 \) lies in \( N_2 \) by the above observation. Likewise, \( v_3, v_4, v_1 \) lie in \( N_3, N_4, N_1 \), respectively. This contradicts \( N_1 \cap I_1 = \emptyset \). Similarly, we get that each vertex \( v_i, i \neq 1 \), cannot lie in the invisible region \( I_i \). Thus, each \( v_i \) either lies in \( P_i \) or in \( N_i \). Moreover, if one \( v_i \) lies in \( N_i \) the above observation yields that all of them lie in their negative region.

The green and red squares in Figure 6 indicate that there is a positive and a negative realization of the literal gadget, i.e., there is a \( DC \)-framed drawing of the literal gadget where all cycle vertices lie either in a positive or in a negative region. In order to simplify the following constructions, we fix the position of the green and red squares as depicted. We refer to these positions as the positive and negative placement of the vertices \( v_i \) and denote them by \( p^+_{X,i} \) and \( p^-_{X,i} \). To reduce the notation, we drop the index \( i \) and simply refer to \( p^+_{X} \) and \( p^-_{X} \) as the positive and negative placements of the literal \( X \). Thus, the literal gadget has the following property.

**Property 3** The positive and negative placements induce a \( DC \)-framed drawing of the literal gadget, respectively.
Figure 8: The literal gadget with unit disks. The endpoints of the blue segment in the interior of the central disk $C$ are the points $\beta_i$.

Unit Disks

The construction of the literal gadget with unit disks follows the same principle as the construction using unit squares; see Figure 8. Only instead of the four corners $\alpha_i$ we choose four points $\beta_i$ that are equally distributed along the boundary of the central disk. The position of the disk $R_i$ have to be adjusted so that it contains the intersection of the tangents of the central disks in the points $\beta_{i-1}$ and $\beta_i$.

3.3 Copy and Inverter Gadget

In this section, we describe the copy and inverter gadget; see Figure 9. The copy gadget connects two positive or two negative literal gadgets $X$ and $Y$ such that a drawing of $X$ is positive if and only if the drawing of $Y$ is positive. Correspondingly, the inverter gadget connects a positive literal gadget $X$ to a negative literal gadget $Y$ such that the drawing of $X$ is positive if and only if the drawing of $Y$ is negative. The construction of the inverter and the copy gadget are symmetric.

Let $X$ and $Y$ be two positive literal gadgets whose center blocks are aligned on the $x$-axis with a sufficiently large distance. We construct the copy gadget that connects $X$ and $Y$ as follows. Let $R_X$ and $R_Y$ be the two cycle blocks of the literal gadgets $X$ and $Y$, respectively, with minimal distance on the $x$-axis. For $A \in \{X, Y\}$, let $P_A$ and $N_A$ be the positive and negative regions of $R_A$. Since $P_A$ and $N_A$ are convex and their intersection is empty, there exists a half plane $h_A$ that contains $N_A$ but not $P_A$, and vice versa. In a reversed manner, we call $h_A$ a positive half-plane $h^+_A$ of $A$ if it contains the negative region $N_A$, otherwise it is negative and we denote it by $h^-_A$.

Consider a positive half-plane $h^+_X$ of $X$ and a negative half-plane $h^-_Y$ of $Y$; refer to Figure 9d. We create two non-intersecting squares $O^+_X$ and $O^-_Y$ that are
contained in the intersection of $h_X$ and $h_Y$ such that a corner of $O_X$ and $O_Y$ lie on the supporting line of $h_X$ and $h_Y$, respectively. Recall that we denote the complement of a half-plane $h$ by $\overline{h}$. Let $I$ be the intersection of the supporting lines of $h_X$ and $h_Y$. We place a square $B$ with a vertex $b$ in interior so that the intersection $I$ lies in the interior of $B$. Additionally, we add a regulator of $b$ with respect to $h_X$ and $h_Y$ to exclude the intersection $h_X \cap h_Y$ as feasible placement of $b$. We route the edges $bv_X$ and $bv_Y$ through $R_X \cup h_X \cup B$ and $R_Y \cup h_Y \cup B$ respectively. This construction ensures that in a $DC$-framed drawing a placement of the vertex $v_X$ in the positive region $P_X$ excludes the possibility that the vertex $v_Y$ lies in the negative region $N_Y$. In order to ensure that $v_X$ cannot lie at the same time in $N_X$ as $v_Y$ in $P_Y$, we construct a square $B'$ with respect to a negative half-plane $h_X$ of $X$ and a positive half-plane $h_Y$ of $Y$ analogously to $B$. If the distance between $X$ and $Y$ is sufficiently large, we can ensure that the intersection of $B$ and $B'$ is empty. In the construction of the inverter gadget the square $B$ is constructed with respect to $h_X$ and $h_Y$, and $B'$ with respect to $h_X$ and $h_Y$. We refer to the corresponding gadgets as copy and inverter gadget. We say that the copy and inverter gadget connect two literals.

**Property 4** Let $\Gamma$ be a $DC$-framed drawing of two positive (negative) literals gadgets $X$ and $Y$ connected by a copy gadget. Then the $DC$-framed drawing of $X$ in $\Gamma$ is positive if and only if the $DC$-framed drawing of $Y$ is positive.

**Proof:** By Property 2 the vertices $v_X$ and $v_Y$ of $X$ and $Y$ cannot lie in the infeasible regions of $X$ and $Y$, respectively. Thus, similar to the proof of Lemma 2 we can assume for the sake of contradiction that the vertex $b$ of the block $B$ lies in the intersection of $h_X$ and $h_Y$. Thus, vertex $v_X$ lies in the negative region of $R_X$ and $v_Y$ in the positive region of $R_Y$. But then vertex $b'$ of the block $B'$ lies in $h_X$ and $h_Y$. However, this is not possible due to the regulator of $b'$. \[\square\]

The same argumentation is applicable to the inverter gadget.

**Property 5** Let $\Gamma$ be a $DC$-framed drawing of a positive literal gadget $X$ and a negative literal gadget $Y$ connected by an inverter gadget. Then the $DC$-framed drawing of $X$ in $\Gamma$ is positive if and only if the $DC$-framed drawing of $Y$ is negative.

The green and red squares in Figure 9b and in Figure 10 indicate that for a positive and a negative placement of $X$ there is $DC$-framed drawing of copy and inverter gadget, respectively. Thus, the copy and inverter gadget have the following property.

**Property 6** The positive (negative) placement of two literals gadgets $X, Y$ induces a $DC$-framed straight-line drawing of a copy [inverter] gadget that connects $X$ and $Y$. 

Figure 9: (a) The copy gadget. The thick transparent green and red lines depict the half planes $h^+_X, h^+_Y$ and $h^-_X, h^-_Y$, respectively.

(b) Green and red regions depict positive and negative regions, respectively.
Figure 10: Positive and negative realizations of the inverter gadget.
Figure 11: Observation

Unit Disks
Squares have the property that there is a set of tangents through a corner point of the square. On the other hand, at each point on the boundary of a disk the tangent to the disk is unique. The following observation helps to show that this restriction does not invalidate the correctness of the unit-disk gadgets.

Observation 7 Let $A$ and $B$ be two disks and let $P$ be a non-empty subset of $A$; see Figure 11. Moreover, let $p \in P$ and $q \in B$. Let $i$ be the intersection of the segment $pq$ and the supporting line of a half plane $h$ that contains $q$ and such that $h \cap P = \emptyset$. Let $C$ be a disk such that $pq$ is tangent to $C$ in the point $i$. Let $Q$ be the set of points in $B$ so that for each $q' \in Q$ there is a point $p' \in P$ such that the segment $p'q'$ does not intersect $C$. Then $Q$ is a strict subset of $h \cap B$.

Recall that, for $A = X, Y$, let $p_A^+$ and $p_A^-$ be the positive and negative placements of $X$ and $Y$. Denote by $h_A^+$ and $h_A^-$ the positive and negative half-planes, respectively, of the disk $D_A$; see Figure 12. Moreover, let $q_A^+$ and $q_A^-$ be points in $h_X^+ \cap B$ and $h_Y^+ \cap B$. Let $O_X^+ (O_Y^-)$ be a disk such that $p_X^+q_X^+ (p_Y^-q_Y^-)$ is tangent to $O_X^+ (O_Y^-)$ in intersection of $p_X^+q_X^+ (p_Y^-q_Y^-)$ with the supporting line of $h_X^+ (h_Y^-)$. The disks $O_X^-$ and $O_Y^+$ are positioned accordingly. The regulators of $B$ and $B'$ and Observation 7 ensure $X$ has a positive $D_C$-framed drawing if and only if $Y$ has a positive $D_C$-framed drawing.

3.4 Clause Gadget
We construct a clause gadget with respect to three positive literal gadgets $X, Y, Z$ arranged as depicted in Figure 13. The negative clause gadget, i.e., a clause with three negative literal gadgets, is obtained by mirroring vertically.

We construct the clause gadget in two steps. First, we place a transition block $T_A$ close to each literal gadget $A \in \{X, Y, Z\}$. In the second step, we connect the transition block to a vertex $k$ in a clause block $K$ such that for every placement of $k$ in $K$ at least one drawing of the literal gadgets has to be positive.
Figure 12: (a) Disk copy gadget. (b) Disk inverter gadget.
Consider the literal gadget $X$ and let $R_X$ be the rightmost cycle block of $X$. Let $h_X$ be a negative half-plane of $R_X$, i.e., $h_X$ contains the positive region but not the negative region; refer to Figure 14. We now place a transition block $T_X$ such that the intersection $T_X \cap h_X$ has small area. Recall that $p_X^+$ and $p_X^-$ denote the positive and negative placements of $X$, respectively. Let $q_X^-$ be a point in $T_X \cap h_X$. Note that, in the following $l^-$ and $l^+$ denote lines and not the half-planes left or right of a line. Let $i$ be the intersection point of the supporting line $l_X^-$ of $h_X^-$ and the line segment $p_X^- q_X^-$. We place a square $Q_X$ such that $l_X^-$ is tangent to $Q_X$ at point $i$. We place a transition vertex $t_X$ in the interior of $T_X$ and route the edge $v_X t_X$ through $h_X^+ \cup T_X \cup R_X$, where $v_X \in R_X$.

Observe that $q_X^-$ allows for a negative drawing of $X$; see Figure 14. Let $l_X^+$ be a line that is tangent to $Q_X$ and that contains $p_X^+$. Then each point on $l_X^+$ that lies in the interior of $T_X$ allows for a positive drawing of $X$. Let $q_X^+$ be the point on $l_X^+$ that maximizes the distance to $q_X^-$. We refer to $q_X^-$ and $q_X^+$ as
Figure 14: $D_C$-framed drawings of the transition block of literal $X$

Figure 15: (a) Initial placement of $q_∅$ and the corresponding half planes $h^−_A$. (b) Setting after perturbing $h^−_A$. The green segments indicate that each $q^+_A, A = X, Y, Z$ can be connected with a line segment to each intersection $i_{X,Y}, i_{X,Z}, i_{Y,Z}$.

Let $K$ be the clause block as depicted in Figure 13. Further, let $q_∅$ be a point in the interior of $K$. Let $f^−_A$, for $A \in \{X, Y, Z\}$, be half planes such that the supporting lines of all three half planes intersect at $q_∅$ and such that $f^−_A$ does not contain the negative region $N_A$ of the transition block $T_A$; see Figure 15. Recall that $q^+_A$ denotes the negative placement of $t_A$ in the transition block $T_A$. Let $n_X$ and $n_Y$ be two lines whose intersection lies in the interior of $f^+_X \cap f^+_Y \cap K$ and that contain $q^+_X$ and $q^+_Y$, respectively. Moreover, denote by $n_Z$ a line that
contains $q_Z$ with a non-empty intersection with $f_Z^- \cap K$. We position a square $O_A$ that is tangent to $n_A$ at point $n_A \cap l_A^-$, where $l_A^-$ is the supporting line of $f_A^-$ and such that the intersection of the interior of $O_A$ and $f_A^-$ is empty. By construction of $O_A$ all three literals gadgets $X,Y,Z$ have negative $D_C$-framed drawings if and only if $k$ lies on $q_0$. Slightly perturbing the positions of the squares $O_A$ ensures that the intersection $f_X^- \cap f_Y^- \cap f_Z^-$ is empty. Denote by $i_{B,C}$, for $B,C \in \{X,Y,Z\}$ with $B \neq C$, the intersection of $n_B$ and $n_C$. To ensure that there are the necessary positive and negative drawings, the perturbation operation has to ensure that the intersection of the line through $q_X^+$ and $i_{X,Y}$ with $n_B$ and $f_X^-$ has the pattern as depicted in Figure 16 and correspondingly for the literals $Y$ and $Z$. Thus, the clause gadget has the following property.

Property 8 There is no $D_C$-framed drawing of the clause gadget such that the $D_C$-framed drawing of each literal gadget is negative. For all remaining combinations of positive and negative drawings of the literal gadgets $X,Y$ and $Z$ there is a $D_C$-framed drawing of the clause gadget.

Unit Disks

We utilize Observation 7 twice to ensure the correctness of the clause gadget with unit disks. First, recall that the square $Q_X$ in Figure 14 is positioned such that $Q_X$ is tangent to the supporting line of $h_X^-$ and the line $l^-$ that contains $p_X^-$ and $q_X^-$, in point $i$. Replacing $Q_X$ by a disk $Q'_X$ that such that the disk is tangent to $l^-$ in point $i$ ensures that $q_X^-$ corresponds to a negative drawing of $X$. Moreover, by Observation 7 the set of points that possibly allow for a negative drawing is a subset of $h_X^- \cap Q'_X$. The disks $Q'_Y, Q'_Z$ are constructed analogously.

Second, recall the construction of the square $O_A$ for $A = X,Y,Z$. The disk $O'_A$ that corresponds to the square $O_A$ is placed such that the line $n_A$ is tangent to $O'_A$ in the intersection of $n_A$ with the supporting line of the half place $f_A^-$. Figure 18 shows the final clause gadget with unit disks.
3.5 Reduction

A 3-SAT instance \((U, C)\) on a set \(U\) of \(n\) boolean variables and \(m\) clauses \(C\) is monotone if each clause either contains only positive or only negative literals. It is planar if the bipartite graph \(G_{U,C} = (U \cup C, \{uc \mid u \in c \text{ or } \overline{u} \in c \text{ with } u \in U \text{ and } c \in C\})\) is planar. A rectilinear representation of a monotone planar 3-SAT instance is a drawing of \(G_{U,C}\) where each vertex is represented as an axis-aligned rectangle and the edges are vertical line segments touching their endpoints; see Figure 19a. Further, all vertices corresponding to variables lie on a common line \(l\), the positive and negative clauses are separated by \(l\).

The problem Monotone Planar 3-SAT asks whether a monotone planar 3-SAT instance with a given rectilinear representation is satisfiable. De Berg and Khosravi [8] proved that Monotone Planar 3-SAT is \(\mathcal{NP}\)-complete. We use this problem to show that the DC-framed Drawings with Pipe-Disk
Figure 18: Clause Construction
The problem of intersecting with axis-aligned unit squares and unit disks is \( \mathcal{NP} \)-hard even when the clustered graph has maximum vertex degree 5 and its obstacle number is 2.

**Proof:** Let \((U, C)\) be a planar monotone 3-SAT instance with a rectilinear representation \(\Pi\). Let \(l\) be a horizontal or vertical line that intersects \(\Pi\). The line \(l\) splits \(\Pi\) into two drawings \(\Pi_L\) and \(\Pi_R\) that are left and right of \(l\), respectively. For a positive factor \(x\), we obtain from \(\Pi\) a new rectilinear representation by moving \(\Pi_R\) \(x\) units to the right. We fill the resulting gap between \(\Pi_L\) and \(\Pi_R\) with infinitely many copies of \(l \cap \Pi\). This operation of stretching the drawing at line \(l\) allows us to do the following necessary modifications.

In the following we modify \(\Pi\) to fit on a checkerboard of \(O(|C|)\) rows and columns where each column has width \(d\) and every row has height \(d\). A row or column is odd if its index is an odd number, otherwise it is even. The pair \((i, j)\) refers to the cell in column \(i\) and row \(j\). We align all vertices corresponding to variables in the rectilinear representation in row 0 so that the leftmost variable vertex is in column 1; refer to Figure 19(b). The width of each rectangle \(r_u\) of variable \(u\) is increased to cover \(2 \cdot n_u - 1\) columns, where \(n_u\) is the number of occurrences of \(u\) and \(\overline{u}\) in \(C\). To ensure that each \(r_u\) starts in an odd column, we increase the distance between two consecutive variables so that the number of columns between the variables is odd and is at least three. Since we are able to add an arbitrary number of columns between two consecutive variables, we can assume without loss of generality that no two edges of the rectilinear representation share a column and that their columns are odd. We adapt the
rectangle of a clause so that it covers five rows and at least six columns, and so that its left and right sides are aligned with the leftmost and rightmost incoming edges, respectively. Note that the positive clauses lie in rows with positive indices and the negative clauses in rows with negative indices. Each operation adds at most a constant number of columns and rows per vertex and per edge to the layout. Thus, the width and height of the final layout is in $O(|C|)$. Further, it can be computed in time polynomial in $|C|$.

In the following we construct a planar embedded graph $C$ and an arrangement of squares $D_C$ of $C$. We use the modified rectilinear layout to locally replace the variable by a sequence of positive and negative literals connected by either a copy or an inverter gadget. Clauses are replaced with the clause gadget and then connected with a sequence of literals and copy gadget to the respective literal in the variable.

Observe that the literal gadget is constructed so that all its squares fit in a larger square $S$. The copy and inverter gadget together with two literals is constructed so that they fit in rectangle three times the size of $S$. The clause gadget fits in a rectangle of width six times the size of the square $S$ and its height is five times the height of $S$.

We assume that the size of the square $S$ and the size of the squares of the checkerboard coincide. Let $r = 0$ be the row that contains the variable vertices. Every column contains at most one edge of the rectilinear representation. Thus, we place a positive literal gadget in cell $(i, r)$ if the edge in column $i$ connects a variable $u$ to a positive clause. Otherwise, we place a negative literal gadget in cell $(i, r)$. Since every edge of the rectilinear representation lies in an odd column, we can connect two literals of the same variable by either a copy or inverter gadget depending on whether both literals are positive or negative, or one is positive and the other negative.

We substitute an edge $e$ of the rectilinear representation that connects a variable to a positive clause as follows. Let $i$ be the column of $e$. If the cell $(i, r_e)$ is covered by $e$ and $r_e$ is odd, we place a positive literal gadget in cell $(i, r_e)$. The copy gadget can be rotated in order to connect a literal gadget in cell $(i, r_e)$ to a literal gadget in a cell $(i, r_e + 2)$.

Let $R_c$ be the rectangle that corresponds to the positive clause $c$ in the modified rectilinear representation. We insert a clause gadget in $R_c$ and justify it on the right of it so that the literal gadget $Z$ lies in an odd column. Note that by the construction of clause gadget this fixes the position of the corresponding literal gadgets $X$ and $Y$. Finally, the literal gadget $X$, $Y$ and $Z$ can be connected to their variables $x$, $y$ and $z$ as depicted in Figure 19b. A negative clause is obtained by vertically mirroring the construction of a positive clause.

We now argue that the embedding of the graph $C$ is planar and that the pairwise intersections of squares in the arrangement $D_C$ are empty. Observe that, every gadget is entirely embedded in the modified rectilinear representation. Recall that the rectilinear representation is planar and all gadget are placed in disjoint cells. Therefore, the pairwise intersection of squares in $D_C$ is empty. Moreover, each literal gadget is planar embedded in a single cell, each clause is embedded in a rectangle that covers five rows and six columns, and finally
each copy and inverter gadget together with its two literal gadget is embedded in either a single row and 3 columns or in 3 rows and a single column. Thus, since the modified rectilinear representation is planar and the pairwise intersections of squares in $D_C$ are empty, the graph $C$ has a planar embedding. Finally, the maximal vertex degree of the literal gadget is three, the maximal degree a clause gadget is four. Connecting two literal gadgets by copy or inverter gadget increases the maximum vertex degree of $C$ to five. Further, the obstacle number of the clause gadget is one and the obstacle number of the literal, copy and the inverter gadget is two.

It is left to show that the layout can be computed in polynomial time. As already argued the modified rectilinear representation $\Pi$ of the monotone planar 3-SAT instance can be computed polynomial time. Moreover, the height and width of $\Pi$ is linear in $|C|$. Thus, we inserted a number of gadgets linear in $|C|$. Further, the coordinates of each gadget are independent of the instance $(U, C)$, thus overall the representation of the final arrangement $D_C$ is polynomial in $|U|$ and $|C|$. Placing a single gadget requires polynomial time, thus overall the clustered graph $C$ and the arrangement $D_C$ of squares can be computed in polynomial time.

**Correctness** Assume that $(U, C)$ is satisfiable. Depending on whether a variable $u$ is true or false, we place all cycle vertices on a positive placement of a positive literal gadget and on the negative placement of negative literal gadget of the variable. Correspondingly, if $u$ is false, we place the vertices on the negative and positive placements, respectively. By Property 3, the placement induces a $D_C$-framed drawing of all literal gadgets. Property 6 ensures that the copy and the inverter gadgets have a $D_C$-framed drawing. Since at least one variable of each clause is true, there is a $D_C$-framed drawing of each clause gadget by Property 8.

Now consider the clustered graph $C$ has a $D_C$-framed drawing. Let $X$ and $Y$ be two positive literal gadgets or two negative literal gadgets connected with a copy gadget. By Property 4, a drawing of $X$ is positive if and only if the drawing of $Y$ is positive. Property 5 ensures that the drawing of a positive literal gadget $X$ is positive if and only if the drawing of the negative literal gadget $Y$ is negative, in case that both are joined with an inverter gadget. Further, Property 2 states that each cycle vertex lies either in a positive or negative region. Thus, the truth value of a variable $u$ can be consistently determined by any drawing of a positive or negative literal gadget of $u$. By Property 8, the clause gadget has no $D_C$-framed drawing of the clause gadget such that all literal gadgets have a negative drawing. Thus, the truth assignment indeed satisfies $C$. □

### 4 Conclusion

We proved that every clustered planar graph with a pipe-disk intersection free disk arrangement $D_C$ and with a $D_C$-framed embedding $\psi$ has a $D_C$-framed
straight-line drawing homeomorphic to $\psi$. In case of arrangements of unit disks and unit squares with pipe-disk intersections the problem becomes $\mathcal{NP}$-hard. This answers an open question of Angelini et al. [4]. We are not aware whether the problem is known to be in $\mathcal{NP}$. Due to the geometric nature of the problem, we ask whether techniques developed by Abrahamsen et al. [1] can be used to prove $\exists \mathcal{R}$-hardness. The cycles in the literal and copy gadget are crucial for our reduction. Thus, we ask whether the problem becomes tractable for restricted graph classes, e.g., trees, outerplanar graphs, or planar graphs that have maximum vertex degree 4.
References


