SEQUENCES ON SETS OF FOUR NUMBERS

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Abstract

The following problem has been open since 1985: Does there exist an infinite word w over a finite set of non-negative integers such that w does not contain any two consecutive blocks with the same length and the same sum? This problem was considered independently by Brown and Freedman (in 1987), Pirillo and Varricchio (in 1994), and Halbeisen and Hungerbüher (in 2000). We show that the answer is "no" for all 4-element sets $\{a, b, c, d\}$ where $a < b < c < d$ are real numbers satisfying the Sidon equation $a + d = b + c$. For any finite subset T of $\mathbb{R}$, we define $g(T)$ to be the maximum length of a word over T which does not contain any two consecutive blocks with the same length and the same sum. (We allow $g(T) = \infty$.) In general, very little is known about g. Here we find the exact values of $g(T)$ for all 4-element sets of real numbers $T = \{a, b, c, d\},\ a + d = b + c$. We also show that $g(T) \geq 50$ for all 4-element sets of real numbers, with equality if and only if T is an arithmetic progression.

1. Introduction

Paul Erdős [7] asked whether there exists an infinite sequence w (often called an infinite *word*–we will use the terms "word" and "sequence" interchangeably) on a finite number of symbols in which no two consecutive blocks are permutations (anagrams) of one another, that is, w has no factorization of the form $w = ABCD$, where A, B, C are finite words (A may be empty, but B must be non-empty), B is a permutation of C, and D is an infinite word. Usually a word of the form BB (where B is not empty) is called simply a *square*, and a word BC, where B is a permutation of C, is called an *abelian square*.

For example, the words $B = aabab$ and $C = abbaa$ are permutations of one another, so the word $BC = aababaabbaa$ is an abelian square. The word BC contains as factors the squares $aa, bb, abab, baba$, and the abelian squares $abababba$ and $bababb$. We also say that the words B and C have the same *composition* if BC is an abelian square.

An infinite word which contains no abelian square (and hence answers the question of Erdős) was constructed by Evdomikov [8], using 25 symbols, in 1968. P. A. B. Pleasants constructed an infinite word with no abelian square on 5 symbols in 1970, and the ultimate result, an infinite word with no abelian square on 4 symbols, was constructed by V. Keränen [11] in 1992. (It is easy to show, by looking at several cases, that there does not exist such a word on 3 symbols. A survey of this problem up to 1971 appears in [4].)

Definition 1. Let $B = a_1a_2 \ldots a_n$, where $a_1, a_2, \cdots a_n \in \mathbb{R}$. Then we write

$$|B| = n \text{ and } \sum B = \sum_{i=1}^{n} a_i.$$

We call $|B|$ the *length* of B, and $\sum B$ the *sum* of B. If $\sum B = \sum C$ and $|B| = |C|$, we say that BC is an *additive square*.

The question of the existence of an infinite word w on a finite set of positive integers which contains no additive square (clearly a stronger requirement than containing no abelian square) was raised in [3]. See also [1], [2], [5], [9], [10], [12].

J. Cassaigne, J. D. Currie, L. Schaeffer, J. Shallit [6] constructed an infinite word on $\{0, 1, 3, 4\}$ which contains no additive *cube*, i. e., which contains no factor ABC where $|A| = |B| = |C|$ and $\sum A = \sum B = \sum C$.

Definition 2. A sequence of numbers, finite or infinite, is called *good* if it contains no additive square.

Definition 3. Let T be a finite set of real numbers. Then $g(T)$ denotes the maximum length of all good sequences on T. If there is no maximum, we write $g(T) = \infty$. (If $T = \{a, b, c, d\}$, then for convenience we write $g(a, b, c, d)$ instead of $g(\{a, b, c, d\})$.

As an example, the word 31304511 on the numbers $0, 1, 3, 4, 5$ contains the additive square BC where $B = 304, C = 511$. (This word also contains the additive squares 1304 and 11.) We will see (as part of Theorem 1 below) that $g(0, 1, 2, 3) = 50$ and $g(0, 1, 5, 6) = 60$.

Considering the real numbers as a vector space over the rational numbers, we note that a sequence on four independent real numbers, $\{\alpha, \beta, \gamma, \delta\}$ is good if and only if there do not exist two consecutive blocks of equal composition, that is, an abelian square. This follows from the uniqueness of sums of the form $x\alpha + y\beta + z\gamma + w\delta$ where x, y, z, w are integers. Using Keränen's infinite sequence on four symbols without abelian squares, and hence without additive squares, we get, in the case where $\alpha, \beta, \gamma, \delta$ are independent over the rationals, $g(\alpha, \beta, \gamma, \delta) = \infty$.

Similarly, for any fixed positive integer N, the sums $x + yN + zN^2 + wN^3$, where x, y, z, w are integers, are unique for $0 \leq x, y, z, w < N$. So, again using Keränen's remarkable result, we see that $g(1, N, N^2, N^3) \geq N - 1$. (Construct a sequence on the numbers $\{1, N, N^2, N^3\}$ of length $N - 1$ with no abelian square. Since each of the four numbers occurs less than N times, there is no additive square.) Thus, Keränen's Theorem implies that

$$g(1, N, N^2, N^3) \rightarrow \infty \text{ as } N \rightarrow \infty.$$

We do not know whether or not all the $g(1, N, N^2, N^3)$ are finite. But in any case, using a standard combinatorial method, one sees that

$$g(1, N, N^2, N^3) \rightarrow \infty \text{ as } N \rightarrow \infty$$

implies Keränen's Theorem. Thus it would be nice to have an independent proof of this fact.

In this paper we will only consider sets T consisting of four distinct real numbers (except for the last corollaries). For ease of discussion we will always assume that the elements of T are listed in their natural order. Call the set of all such 4-tuples $\mathcal{A}$. We will determine the value of g for all members of a special subset, $\mathcal{B}$, of $\mathcal{A}$, namely,

$$\mathcal{B} = \{\{a, b, c, d\} : \quad a + d = b + c\}.$$

We will also show that $g(T) \geq 50$ for all $T \in \mathcal{A}$ and determine exactly when equality holds. So far, these seem to be the only results, with any degree of generality, related to the calculation of $g(T)$.

1.1. Affine Transformations

Consider an affine transformation $x \rightarrow \mu x + \sigma$ (where μ, σ are real numbers, $\mu \neq 0$). Let w be a word on $\{a, b, c, d\}$ and v the word obtained from w by replacing each number x in w with $\mu x + \sigma$. It is easy to see that w is an additive square if and only if v is an additive square. Similarly, w is good if and only if v is good. It follows that g is invariant under an affine transformation. That is,

$$g(a, b, c, d) = g(\mu a + \sigma, \mu b + \sigma, \mu c + \sigma, \mu c + \sigma), \; (\mu, \sigma \text{ real}, \; \mu \neq 0).$$

The sets $\mathcal{A}$ and $\mathcal{B}$ are each closed under affine transformations and, in particular, closed under positive affine transformations (i.e., where $\mu > 0$). We use the positive affine transformations to partition the sets $\mathcal{A}$ and $\mathcal{B}$ (respectively) into equivalence classes where two 4-tuples are equivalent if one is transformed into the other by some positive affine transformation. It follows that every equivalence class in $\mathcal{B}$ has a unique representative of the form $\{0, 1, 1 + \epsilon, 2 + \epsilon\}, \; \epsilon > 0$. It is interesting that there is a canonical 1-1 correspondence between the equivalence classes in $\mathcal{B}$ and the interval $(0, \infty)$.

Similarly, every equivalence class in $\mathcal{A}$ has a unique representative of the form $\{0, 1, 1+\epsilon, 1+\epsilon+\delta\}$, $\epsilon, \delta > 0$. Hence, there is a canonical 1-1 correspondence between the equivalence classes in $\mathcal{A}$ and the open first quadrant of the plane.

2. Main Results

Recall that a *good sequence* (on a set of reals) is a sequence which does not contain an additive square. When a, b, c, d are real numbers, $g(a, b, c, d)$ denotes the maximum possible number of terms in a good sequence on the set $\{a, b, c, d\}$. Recall also that $\mathcal{B}$ is the set of all 4-element sets $\{a, b, c, d\}$ of real numbers, where $a < b < c < d$ and $a + d = b + c$.

Definition 4. The *reversal* of a sequence $w = x_1 x_2 \cdots x_n$ is the sequence

$$w^r = x_n x_{n-1} \cdots x_1.$$

(Clearly if a sequence w is good then so is w^r.)

Theorem 1. *Other than in the exceptional cases listed below in Table 1, if $\{a, b, c, d\}$ is any set of real numbers with $a < b < c < d$ and $a+d = b+c$, then $g(a, b, c, d) = 60$, and there are exactly 8 distinct good sequences on $\{a, b, c, d\}$ of length 60. These 8 sequences are described in the Appendix.*

Exceptional cases: *If $\{a, b, c, d\}$ is equivalent to any of the 4-tuples listed in Table 1 below, then the value of $g(a, b, c, d)$, and the number of distinct maximum length good sequences on $\{a, b, c, d\}$, are given in the table. Each line contains an exceptional 4-tuple $\{a, b, c, d\}$, the value of $g(a, b, c, d)$, and the number of distinct maximum length good sequences on $\{a, b, c, d\}$.*

{0,1,2,3}	50	16
{0,1,3,4}	55	4
{0,1,4,5}	55	4
{0,1,5,6}	60	4
{0,2,3,5}	55	4
{0,2,5,7}	60	4
{0,3,4,7}	58	4
{0,3,5,8}	60	4

Table 1

Descriptions of all the maximum length good sequences in these exceptional cases are given in the Appendix.

Proof. We rely heavily on the output from computer programs. Consider any 4-tuple in $\mathcal{B}$. It is clear from the previous section that this 4-tuple is equivalent to a 4-tuple of the form $\{0, 1, \alpha, 1 + \alpha\}, \alpha > 1$. Consider a sequence S on these four numbers and two consecutive equal length blocks, E and F, in S. Let r, s, t, u be the counts for $0, 1, \alpha, 1 + \alpha$ respectively, in block E and r', s', t', u' be the same for block F. Hence, length(E)=length(F) implies

$$r + s + t + u = r' + s' + t' + u'.$$

Now S will not be good if, in these blocks,

$$s + t\alpha + u(1 + \alpha) = s' + t'\alpha + u'(1 + \alpha). \tag{1}$$

This equation is equivalent to

$$(t - t' + u - u')\alpha = s' - s + u' - u. \tag{2}$$

There are two ways that (2) can hold:

$$t - t' + u - u' = s' - s + u' - u = 0, \text{ or} \tag{2a}$$

$$t - t' + u - u' \text{ and } s' - s + u' - u \text{ are both non-zero, and} \tag{2b}$$

$$\alpha = \frac{s' - s + u' - u}{t - t' + u - u'}. \tag{3}$$

We now construct a computer program, PROG 1, that finds all the longest sequences on $\{0, 1, \alpha, 1+\alpha\}, \alpha > 1$, which do not have two consecutive equal length blocks where $t - t' + u - u' = s' - s + u' - u = 0$.

This program produces the eight length 60 sequences mentioned in the first part of the statement of the theorem. The program replaces $\{0, 1, \alpha, 1 + \alpha\}$ with any equivalent $\{a, b, c, d\}$ respectively in the final output. See the Appendix for a description of these sequences.

Now, it is clear that these eight sequences are good for some but not for all 4-tuples in $\mathcal{B}$. For example, if $\alpha = \pi$ (or any irrational number), then (2) above cannot occur because PROG 1 guarantees that the case (2a) is avoided, and the case (2b) is avoided since the right-hand side of (3) is rational. Hence, $g(0, 1, \pi, 1 + \pi) = 60$ and all eight sequences in the statement of the theorem are good for any 4-tuple equivalent to $\{0, 1, \pi, 1 + \pi\}$.

However, consider $\{0, 1, 2, 3\}$. Here $\alpha = 2$. It turns out that, in each of the eight sequences, there occur consecutive equal length blocks such that $(s'-s+u'-u)/(t-t' + u - u') = 2$. Hence (3) holds and therefore (2) and (1) hold. So, none of the eight sequences is good for $\{0, 1, 2, 3\}$ and we must have $g(0, 1, 2, 3) < 60$.

In general, we have to find, for each of the eight length 60 sequences, S, found by PROG 1, all the ordered pairs (X, Y) where $X = s' - s + u' - u, Y = t - t' + u - u'$

and $X/Y > 1$, the calculation of X and Y being performed for all the consecutive blocks E, F of equal length in S (there are $30^2 = 900$ such pairs of intervals). Each such found pair, (X, Y), will produce an equivalence class of 4-tuples in $\mathcal{B}$ for which at least one of the eight sequences is not good. For example, since $X = 5$ and $Y = 3$ occurs, the 4-tuple $\{0, 1, 5/3, 8/3\}$, equivalently, $\{0, 3, 5, 8\}$, represents an equivalence class in $\mathcal{B}$ for which at least one of the eight sequences in Table 1 is not good.

So we write another Perl program, PROG 2, that calculates all of the (X, Y) just described. (PROG 2 must be run eight times, once for each of the eight sequences.) Somewhat surprisingly, this produces only eight distinct equivalence classes in $\mathcal{B}$ represented by the eight exceptions in Table 1 of the theorem.

The last step is to use a third program, PROG 3, run on the eight exceptional 4-tuples, to find all of the longest good sequences. This provides the values for g and the number of maximum length good sequences which appear in Table 1. This completes the proof of the theorem. □

Theorem 2. *For any 4-tuple* $T = \{a, b, c, d\}$ *in* $\mathcal{A}$, $g(T) \geq 50$*, and equality holds only when* T *is a 4-term arithmetic progression.*

Proof. T is equivalent to a 4-tuple of the form $\{0, 1, \alpha, \beta\}, 1 < \alpha < \beta$. Let S be the 51-term sequence:

$(1, 0, \alpha, 0, \beta, \alpha, 0, 1, \beta, \alpha, \beta, 0, \alpha, 0, 1, 0, \alpha, 0, \beta, \alpha, 0, 1, \beta, \alpha, 0, \beta, \alpha, \beta, 1, \beta, \alpha, \beta,$
$0, \alpha, 0, 1, 0, \alpha, 0, \beta, \alpha, 0, 1, \beta, \alpha, \beta, 0, \alpha, 0, 1, \alpha)$.

(The first 50 terms of S are from one of the 16 maximum length good sequences on $\{0, 1, 2, 3\}$, replacing $\{0, 1, 2, 3\}$ by $\{0, 1, \alpha, \beta\}$). This sequence will turn out to be good for all reals $\alpha, \beta, (1 < \alpha < \beta)$ except when $\beta = \alpha + 1$. But, if $\beta = \alpha + 1$, then $\{0, 1, \alpha, \beta\} = \{0, 1, \alpha, \alpha + 1\}$ is in $\mathcal{B}$ and Theorem 1 takes care of the values for $g(T)$ in this case, including the fact that $g(T) = 50$ only if $\alpha = 2$, i.e., T is a 4-term arithmetic progression.

To show that S is good for all other T, we use the notation established in the proof of Theorem 1 and observe that S will not be good for $\{0, 1, \alpha, \beta\}$ if there are two consecutive equal length blocks where

$$s + t\alpha + u\beta = s' + t'\alpha + u'\beta$$

or

$$(u' - u)\beta = (t - t')\alpha + (s - s'). \tag{4}$$

We write a program (PROG 4, similar to PROG 2) which, using the above sequence S, finds all the triples $(u' - u), (t - t'), (s - s')$ (one triple for each pair of consecutive blocks E, F of equal length in S.) and then we examine the output. There are 650 adjacent block pairs to examine, but after eliminating duplications, PROG 4 produces the following 51 triples:

(0 0 1) (0 -1 0) (0 1 0) (1 0 0) (-1 -1 0) (0 0 -1) (1 0 1) (1 1 0) (-1 0 0) (-1 0 -1) (0 -1 1) (-1 1 -1) (1 1 -1) (1 -1 1) (0 1 -1) (-1 -1 1) (2 0 1) (-2 0 0) (1 0 -1) (-2 0 -1) (-1 0 1) (-2 -1 -1) (-2 -1 0) (-2 1 -1) (2 0 0) (-2 -1 1) (-1 1 0) (-3 0 0) (2 1 0) (1 -1 0) (-2 1 0) (-3 -1 0) (-4 0 0) (3 0 0) (2 0 -1) (3 0 1) (2 1 -1) (3 1 -1) (3 -1 1) (2 -1 1) (-3 -1 1) (-3 1 -1) (3 0 -1) (4 0 1) (-2 0 1) (-3 0 -1) (-1 -1 -1) (2 -1 0) (3 1 0) (-3 -1 -1) (0 -1 -1).

Note that (0 0 0) does not occur. Examining these triples further, we see that the only one that would allow the existence of $\alpha, \beta, (1 < \alpha < \beta)$ such that equation (4) holds is (-1 -1 -1), which produces $\beta = \alpha + 1$. All the other triples (like (1 1 -1)) lead to equation (4) ($\beta = \alpha - 1$ in this case) which cannot hold and allow $1 < \alpha < \beta$ at the same time. □

Corollary 1. *If U is any finite set of real numbers with at least 5 elements, then $g(U) > 50$.*

Proof. If $T \subset U$, then $g(U) \geq g(T)$. Since U has 5 or more elements there must be a 4-element subset, T, which does not form an arithmetic progression. By Theorem 2, $g(T) \geq 51$. □

Lemma 1. *If $1 \leq k \leq 4$, the maximum length of a good palindrome on k numbers is $2^k - 1$.*

Proof. For $k = 1, 2$, this is trivial. For $k = 3$, it's not hard to show by hand that *any* good sequence A on 3 numbers (not just a palindrome) has $|A| \leq 7$. (There are 18 good sequences of length 7 (six of these are palindromes) on 3 distinct numbers, namely $abacaba, abacbab, abcbabc$, where a, b, c is any permutation of the three numbers.)

Now assume that w is a good palindrome on 4 numbers with $|w| \geq 16$. We may assume without loss of generality that w has odd length, say $w = AxA^r$, where $|x| = 1, |A| \geq 8$, and A^r denotes the reversal of A. Since $|A| > 7$, all four numbers must occur in A, in particular $A = BxC$. But then $w = BxCxC^rxB^r$, and w contains the additive square (in fact, the abelian square) $xCxC^r$, contradicting the assumption that w is good. Finally, $w = 01020104010201$ is a good palindrome on four numbers, of length 15. □

Remark For $k = 5$, there are arbitrarily long good palindromes on five numbers. For let N be given. Then as previously remarked, there is a good word of length $N - 1$ on the numbers $\{1, N, N^2, N^3\}$. Denote this word by w. Then $w(\sum w)w^r$ is a good palindrome on 5 numbers, with length $2N - 1$.

Corollary 2. *For each k, $2 \leq k \leq 4$, if T is a set of k numbers, and there is no infinite good sequence on T, then the number of maximum length good sequences on T numbers is even.*

Proof. For $k = 2, 3$, this can be checked by hand. If $|T| = 4$, then by Theorem 1 and Corollary 1, $g(T) \geq 50 > 2^4 - 1$, hence by Lemma 1 there are no palindromes among the set of maximum length good sequences. Since the reversal of a good sequence is good, the Corollary follows. □

Comment. While Theorem 1 provides a precise answer, Theorem 2 and Corollary 1 leave considerable room for improvement.

Programs. Programs 1 through 4, written in Perl, are available from either author. They require a Perl processor such as TextWrangler.

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Appendix

Here we describe all the maximum length good sequences mentioned in Theorem 1. It will be convenient to use the notation in the following definition.

Definition 5. If $w = x_1x_2\dots x_n$ is a sequence on the 4-tuple $\{a,b,c,d\}$ in $\mathcal{B}$ ($a < b < c < d$ and $a+d = b+c$), we set $w^* = x_1^*x_2^*\dots x_n^*$, where

$$x_i^* = a + d - x_i, \quad 1 \le i \le n.$$

Since $x \to x*$ is an affine transformation it is clear that w is a good sequence if and only if $w*$ is a good sequence (see Section 1.1).

First we describe the 8 maximum length (length 60) sequences for all the 4-tuples $\{a,b,c,d\}$ in $\mathcal{B}$, which are not equivalent to any of the 4-tuples in Table 1. Let $S(x,y,z,w)$ be the sequence

$S(x,y,z,w)$ = x y z x w y w z y z x z y w y z y w x w z x z y z x w x z x w y w x w y z y w y x w x z y z x z y w y z y x z x w y z w.

Then the eight good sequences of length 60 on $\{a,b,c,d\}$ are $S(a,d,b,c)$, $S(a,d,c,b)$, $S(d,a,b,c)$, $S(d,a,c,b)$, and their reversals, which are $S(b,c,a,d)$, $S(b,c,d,a)$, $S(c,b,a,d)$, $S(c,b,d,a)$. Alternatively, these are

$$R,\ R^r,\ R^*,\ R^{r*},\ T,\ T^r,\ T^*,\ T^{r*},$$

where

$R = S(a,d,b,c)$ = a d b a c d c b d b a b d c d b d c a c b a b d b a c a b a c d c a c d b d c d a c a b d b a b d c d b d a b a c d b c,

$T = S(a,d,c,b)$ = a d c a b d b c d c a c d b d c d b a b c a c d c a b a c a b d b a b d c d b d a b a c d c a c d b d c d a c a b d c b.

In the three cases in Table 1 where $g(a,b,c,d) = 60$ (that is, when $\{a,b,c,d\}$ is equivalent to one of $\{0,1,5,6\}$ or $\{0,2,5,7\}$ or $\{0,3,5,8\}$) the 4 maximum length good sequences are $S(a,d,c,b)$, $S(d,a,b,c)$, and their reversals $S(b,c,d,a)$, and $S(c,b,a,d)$. Alternatively, these are T, T^r, T^*, T^{r*}, where T is as above.

In the case where $\{a,b,c,d\}$ is equivalent to $\{0,1,2,3\}$, the 16 length 50 good sequences on $\{0,1,2,3\}$ are

$$A, B, C, D, A^r, B^r, C^r, D^r, A^*, B^*, C^*, D^*, A^{r*}, B^{r*}, C^{r*}, D^{r*},$$

where

$A =$ b a c a d c a b d c d a c a b a c a d c a b d c a d c d b d c d a c a b a c a d c a b d c d a c a b,

$B =$ b a c a d c d b a c d a c a b a c a d c d b d c d a c d b a c d a c a b a c a d c d b a c a d c a b,

$C =$ b d c a d c d b a c a d c d b d c d a c d b a c d a c a b a c a d c d b d c d a c d b a c a d c d b,

$D =$ b d c d a c a b d c a d c d b d c d a c a b a c a d c a b d c a d c d b d c d a c a b d c a d c d b.

(It is curious that the first 45 terms of A^r and B are identical, as are the first 45 terms of C^r and D.)

For the cases where $\{a,b,c,d\}$ is equivalent to one of $\{0,1,3,4\}, \{0,1,4,5\}$, $\{0,2,3,5\}$, the four good sequences of length 55 are E, E^r, E^*, E^{r*}, where

$E =$ b a b d a c a b d b a b d c d b d a b a c a b a d b a c d b a d b d c d b d a b a c a b d b a b d c d a b d b.

Finally, for the case $\{a,b,c,d\}$ equivalent to $\{0,3,4,7\}$, the four good sequences of length 58 are F, F^*, F^r, F^{r*}, where

$F =$ a b a c a d c a b d c a d c d b d c d a c a b a c a d c a b d c b d b a b d b c d c a c d c b d c a b d c b d b a b.

It is a bit of a curiosity that (since $\{0,1,5,6\}$ is an exceptional 4-tuple) the sequences $S(0,6,5,1)$, $S(6,0,1,5)$, $S(1,5,6,0)$, $S(5,1,0,6)$ are good, whereas the sequences $S(0,6,1,5)$, $S(6,0,5,1)$, $S(1,5,0,6)$, $S(5,1,6,0)$ are not good. Indeed, S(0,6,1,5) =

$$061056\ \underline{5161016561650510161}\ \underline{0501056505616560501}\ 6101656160105615,$$

where the smallest additive square has been underlined. Each of the two segments has length 19 and sum 57. (This is not an abelian square, since the first segment has 3 zeros and the second segment has 6 zeros.)