PARTITIONS WITH DESIGNATED SUMMANDS IN WHICH ALL PARTS ARE ODD

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Abstract

In 2002, Andrews, Lewis and Lovejoy studied the partition function $PDO(n)$, the number of partitions of n with designated summands in which all parts are odd and found several identities and congruences satisfied by the function. In this paper, we find further identities and congruences satisfied by $PDO(n)$.

1. Introduction

In [1], Andrews, Lewis and Lovejoy introduced and studied a new class of partitions, partitions with designated summands. Partitions with designated summands are constructed by taking ordinary partitions and tagging exactly one of each part size. For example, there are 10 partitions of 4 with designated summands, namely,

$$4', \quad 3'+1', \quad 2'+2, \quad 2+2', \quad 2'+1'+1, \quad 2'+1+1',$$

$$1'+1+1+1, \quad 1+1'+1+1, \quad 1+1+1'+1, \quad 1+1+1+1'.$$

The total number of partitions of n with designated summands is denoted by $PD(n)$. Hence, $PD(4) = 10$. Further studies on $PD(n)$ were carried out by Chen, Ji, Jin, and Shen [7].

In the same paper [1], Andrews, Lewis and Lovejoy also studied $PDO(n)$, the number of partitions of n with designated summands in which all parts are odd. From the above example, $PDO(4) = 5$. Note that $(PDO(n))$ is sequence A102186 in the On-line Encyclopedia of Integer Sequences available at "https://oeis.org/A102186." The generating function found by Andrews, Lewis and Lovejoy for $PDO(n)$ is

$$\sum_{n=0}^{\infty} PDO(n)q^n = \frac{(q^4;q^4)_\infty (q^6;q^6)_\infty^2}{(q;q)_\infty (q^3;q^3)_\infty (q^{12};q^{12})_\infty}, \tag{1}$$

where, here and the sequel, for $|q| < 1$ and positive integers n, we use the standard notation

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \quad \text{and} \quad (a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n).$$

By using q-series and modular forms, they found (1) as well as the following identities.

Theorem 1.1 *[1, Theorem 21 and Theorem 22] We have*

$$\sum_{n=0}^{\infty} PDO(2n)q^n = \frac{(q^4;q^4)_\infty^2 (q^6;q^6)_\infty^4}{(q;q)_\infty^2 (q^3;q^3)_\infty^2 (q^{12};q^{12})_\infty^2}, \tag{2}$$

$$\sum_{n=0}^{\infty} PDO(2n+1)q^n = \frac{(q^2;q^2)_\infty^6 (q^{12};q^{12})_\infty^2}{(q;q)_\infty^4 (q^4;q^4)_\infty^2 (q^6;q^6)_\infty^2}, \tag{3}$$

$$\sum_{n=0}^{\infty} PDO(3n)q^n = \frac{(q^2;q^2)_\infty^2 (q^6;q^6)_\infty^4}{(q;q)_\infty^4 (q^{12};q^{12})_\infty^2}, \tag{4}$$

$$\sum_{n=0}^{\infty} PDO(3n+1)q^n = \frac{(q^3;q^3)_\infty^3 (q^2;q^2)_\infty^4 (q^{12};q^{12})_\infty}{(q;q)_\infty^5 (q^4;q^4)_\infty (q^6;q^6)_\infty^2}, \tag{5}$$

$$\sum_{n=0}^{\infty} PDO(3n+2)q^n = 2\frac{(q^2;q^2)_\infty^3 (q^6;q^6)_\infty (q^{12};q^{12})_\infty}{(q;q)_\infty^4 (q^4;q^4)_\infty}. \tag{6}$$

They also deduce the following congruences.

Corollary 1.2 *[1, Corollary 19] We have*

$$PDO(9n+6) \equiv 0 \pmod 3, \tag{7}$$

$$PDO(12n+6) \equiv 0 \pmod 3, \tag{8}$$

$$PDO(12n+10) \equiv 0 \pmod 3, \tag{9}$$

$$PDO(24n) \equiv 0 \pmod 3,$$

$$PDO(24n+16) \equiv 0 \pmod 3.$$

The aim of this paper is to find proofs of (2)–(6) and the following new identities by using certain dissections of theta functions.

Theorem 1.3 *We have*

$$\sum_{n=0}^{\infty} PDO(4n)q^n = \frac{(q^2;q^2)_\infty^2 (q^4;q^4)^5 (q^{12};q^{12})_\infty^5}{(q;q)_\infty^5 (q^3;q^3)_\infty (q^6;q^6)_\infty^2 (q^8;q^8)_\infty^2 (q^{24};q^{24})_\infty^2} + 4q^2 \frac{(q^2;q^2)_\infty^4 (q^8;q^8)_\infty^4 (q^{24};q^{24})_\infty^2}{(q;q)_\infty^5 (q^3;q^3)_\infty (q^4;q^4)_\infty (q^6;q^6)_\infty^2 (q^{12};q^{12})_\infty}, \quad (10)$$

$$\sum_{n=0}^{\infty} PDO(4n+1)q^n = \frac{(q^2;q^2)_\infty^{12} (q^6;q^6)_\infty^2}{(q;q)_\infty^8 (q^3;q^3)_\infty^2 (q^4;q^4)_\infty^4}, \quad (11)$$

$$\sum_{n=0}^{\infty} PDO(4n+2)q^n = 2 \frac{(q^2;q^2)_\infty^6 (q^6;q^6)_\infty^2}{(q;q)_\infty^6 (q^3;q^3)_\infty^2}, \quad (12)$$

$$\sum_{n=0}^{\infty} PDO(4n+3)q^n = 4 \frac{(q^4;q^4)_\infty^4 (q^6;q^6)_\infty^2}{(q:q)_\infty^4 (q^3;q^3)_\infty^2}, \quad (13)$$

$$\sum_{n=0}^{\infty} PDO(6n)q^n = \frac{(q^2;q^2)_\infty^{14} (q^3;q^3)_\infty^4}{(q;q)_\infty^{12} (q^4;q^4)_\infty^4 (q^6;q^6)_\infty^2}, \quad (14)$$

$$\sum_{n=0}^{\infty} PDO(6n+2)q^n = 2 \frac{(q^2;q^2)_\infty^{13} (q^3;q^3)_\infty (q^6;q^6)_\infty}{(q;q)_\infty^{11} (q^4;q^4)_\infty^4}, \quad (15)$$

$$\sum_{n=0}^{\infty} PDO(6n+3)q^n = 4 \frac{(q^2;q^2)_\infty^2 (q^3;q^3)_\infty^4 (q^4;q^4)_\infty^4}{(q;q)_\infty^8 (q^6;q^6)_\infty^2}, \quad (16)$$

$$\sum_{n=0}^{\infty} PDO(6n+5)q^n = 8 \frac{(q^2;q^2)_\infty (q^3;q^3)_\infty (q^4;q^4)_\infty^4 (q^6;q^6)_\infty}{(q;q)_\infty^7}, \quad (17)$$

$$\sum_{n=0}^{\infty} PDO(9n+3)q^n = 4 \left\{ \frac{(q^2;q^2)_\infty^{11} (q^3;q^3)_\infty^9}{(q;q)_\infty^{15} (q^4;q^4)_\infty^2 (q^6;q^6)_\infty^3} + 4q \frac{(q^2;q^2)_\infty^8 (q^6;q^6)_\infty^6}{(q;q)_\infty^{12} (q^4;q^4)_\infty^2} \right\}, \quad (18)$$

$$\sum_{n=0}^{\infty} PDO(9n+6)q^n = 12 \frac{(q^2;q^2)_\infty^{10} (q^3;q^3)_\infty^6}{(q;q)_\infty^{14} (q^4;q^4)_\infty^2}, \quad (19)$$

$$\sum_{n=0}^{\infty} PDO(12n)q^n = \frac{(q;q)_\infty^{14}}{(q^2;q^2)_\infty^4 (q^3;q^3)_\infty^2} \left\{ \frac{(q^2;q^2)_\infty^{24} (q^3;q^3)_\infty^{12}}{(q;q)_\infty^{36} (q^6;q^6)_\infty^8} + 54q \frac{(q^2;q^2)_\infty^{16} (q^3;q^3)_\infty^8}{(q;q)_\infty^{32}} + 81q^2 \frac{(q^2;q^2)_\infty^8 (q^3;q^3)_\infty^4 (q^6;q^6)_\infty^8}{(q;q)_\infty^{28}} \right\}, \quad (20)$$

$$\sum_{n=0}^{\infty} PDO(12n+2)q^n = 2\left\{\frac{(q^2;q^2)_\infty^{12}(q^3;q^3)_\infty^{16}}{(q;q)_\infty^{20}(q^6;q^6)_\infty^{8}} + 44q\,\frac{(q^2;q^2)_\infty^{9}(q^3;q^3)_\infty^{7}(q^6;q^6)_\infty}{(q;q)_\infty^{17}}\right\}, \tag{21}$$

$$\begin{aligned}\sum_{n=0}^{\infty} PDO(12n+3)q^n = 4\Bigg\{&\frac{(q^2;q^2)_\infty^{12}(q^4;q^4)_\infty^{8}(q^6;q^6)_\infty^{4}}{(q;q)_\infty^{18}(q^8;q^8)_\infty^{4}(q^{12};q^{12})_\infty^{2}}\\ &+ 16q\,\frac{(q^2;q^2)_\infty^{11}(q^3;q^3)_\infty(q^4;q^4)_\infty^{4}(q^6;q^6)_\infty}{(q;q)_\infty^{17}}\\ &+ 4q\,\frac{(q^2;q^2)_\infty^{16}(q^6;q^6)_\infty^{4}(q^8;q^8)_\infty^{4}}{(q;q)_\infty^{18}(q^4;q^4)_\infty^{4}(q^{12};q^{12})_\infty^{2}}\\ &+ 4q\,\frac{(q^2;q^2)_\infty^{6}(q^3;q^3)_\infty^{2}(q^4;q^4)_\infty^{12}(q^{12};q^{12})_\infty^{2}}{(q;q)_\infty^{16}(q^6;q^6)_\infty^{2}(q^8;q^8)_\infty^{4}}\\ &+ 16q^2\,\frac{(q^2;q^2)_\infty^{10}(q^3;q^3)_\infty^{2}(q^8;q^8)_\infty^{4}(q^{12};q^{12})_\infty^{2}}{(q;q)_\infty^{16}(q^6;q^6)_\infty^{2}}\Bigg\},\end{aligned} \tag{22}$$

$$\sum_{n=0}^{\infty} PDO(12n+6)q^n = 12\left\{\frac{(q^2;q^2)_\infty^{11}(q^3;q^3)_\infty^{13}}{(q;q)_\infty^{19}(q^6;q^6)_\infty^{5}} + 10q\,\frac{(q^2;q^2)_\infty^{8}(q^3;q^3)_\infty^{4}(q^6;q^6)_\infty^{4}}{(q;q)_\infty^{16}}\right\}, \tag{23}$$

$$\begin{aligned}\sum_{n=0}^{\infty} PDO(12n+9)q^n = 16\Bigg\{&\frac{(q^2;q^2)_\infty^{14}(q^4;q^4)_\infty^{2}(q^6;q^6)_\infty^{4}}{(q;q)_\infty^{18}(q^{12};q^{12})_\infty^{2}}\\ &+ \frac{(q^2;q^2)_\infty^{9}(q^3;q^3)_\infty(q^4;q^4)_\infty^{10}(q^6;q^6)_\infty}{(q;q)_\infty^{17}(q^8;q^8)_\infty^{4}}\\ &+ 4q\,\frac{(q^2;q^2)_\infty^{13}(q^3;q^3)_\infty(q^6;q^6)_\infty(q^8;q^8)_\infty^{4}}{(q;q)_\infty^{17}(q^4;q^4)_\infty^{2}}\\ &+ 4q\,\frac{(q^2;q^2)_\infty^{8}(q^4;q^4)_\infty^{6}(q^3;q^3)_\infty^{2}(q^{12};q^{12})_\infty^{2}}{(q;q)_\infty^{16}(q^6;q^6)_\infty^{2}}\Bigg\},\end{aligned} \tag{24}$$

$$\sum_{n=0}^{\infty} PDO(12n+10)q^n = 6\left\{7\,\frac{(q^2;q^2)_\infty^{10}(q^3;q^3)_\infty^{10}}{(q;q)_\infty^{18}(q^6;q^6)_\infty^{2}} + 16q\,\frac{(q^2;q^2)_\infty^{7}(q^3;q^3)_\infty(q^6;q^6)_\infty^{7}}{(q;q)_\infty^{15}}\right\}. \tag{25}$$

From the previous theorem, we easily deduce the following congruences.

Corollary 1.4 *We have*

$$PDO(4n+2) \equiv 0 \pmod 2,$$

$$
\begin{aligned}
PDO(4n+3) &\equiv 0 \pmod{4},\\
PDO(6n+2) &\equiv 0 \pmod{2},\\
PDO(6n+3) &\equiv 0 \pmod{4},\\
PDO(6n+5) &\equiv 0 \pmod{8},\\
PDO(9n+3) &\equiv 0 \pmod{4},\\
PDO(9n+6) &\equiv 0 \pmod{12}, \qquad (26)\\
PDO(12n+6) &\equiv 0 \pmod{12}, \qquad (27)\\
PDO(12n+9) &\equiv 0 \pmod{16},\\
PDO(12n+10) &\equiv 0 \pmod{6}. \qquad (28)
\end{aligned}
$$

Note that, congruences (26), (27) and (28) are improved versions of (7), (8) and (9), respectively.

We also find the following congruences.

Theorem 1.5 *For all nonnegative integers n, we have*

$$
\begin{aligned}
PDO(8n+6) &\equiv 0 \pmod{4}, \qquad (29)\\
PDO(8n+7) &\equiv 0 \pmod{8}, \qquad (30)\\
PDO(18n+15) &\equiv 0 \pmod{24}, \qquad (31)\\
PDO(9n+9) &\equiv 0 \pmod{4}, \qquad (32)\\
PDO(24n+9) &\equiv 0 \pmod{8}, \qquad (33)\\
PDO(24n+15) &\equiv 0 \pmod{8}, \qquad (34)
\end{aligned}
$$

and

$$PDO(24n+21) \equiv 0 \pmod{8}. \qquad (35)$$

In the next section, we give some definitions, preliminary results and dissections of some theta functions. In the last section, we prove Theorems 1.1, 1.3, 1.5.

2. Definitions, Preliminary Results, and Proof of Some Dissections

Ramanujan's general theta function $f(a,b)$ is defined as

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1.$$

The well-known Jacobi's triple product identity takes the form

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \qquad (36)$$

Two special cases of $f(a,b)$ are defined, for $|q|<1$, by [3, p. 36, Entry 22]

$$\varphi(q) := f(q,q) = \sum_{k=-\infty}^{\infty} q^{k^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2 (q^4;q^4)_\infty^2} \tag{37}$$

and

$$\psi(q) := f(q,q^3) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}, \tag{38}$$

where the penultimate product representations in (37) and (38) arise from (36).

After Ramanujan, we also define

$$\chi(-q) = (q;q^2)_\infty = \frac{(q;q)_\infty}{(q^2;q^2)_\infty}. \tag{39}$$

Next, we recall from [5] that

$$c(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n} = 3\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}. \tag{40}$$

Now we state a lemma.

Lemma 2.1 *We have*

$$\psi(q) = f(q^3,q^6) + q\psi(q^9), \tag{41}$$

$$f(q,q^2) = \frac{\varphi(-q^3)}{\chi(-q)}, \tag{42}$$

$$\varphi^2(q) = \varphi^2(q^2) + 4q\psi^2(q^4). \tag{43}$$

Proof. See [3, p. 49, Corollary(ii)] and [3, p. 350, Eq. (2.3)] for the proofs of (41) and (42), respectively. Adding identities (v) and (vi) of [3, Entry 25, p. 40], we can easily derive (43). □

In the remaining lemmas of this section, we state and prove certain 2- and 3-dissections.

Lemma 2.2 *We have*

$$\frac{1}{(q;q)_\infty (q^3;q^3)_\infty} = \frac{(q^8;q^8)_\infty^2 (q^{12};q^{12})_\infty^5}{(q^2;q^2)_\infty^2 (q^4;q^4)(q^6;q^6)_\infty^4 (q^{24};q^{24})_\infty^2} + q\,\frac{(q^4;q^4)_\infty^5 (q^{24};q^{24})_\infty^2}{(q^2;q^2)_\infty^4 (q^6;q^6)_\infty^2 (q^8;q^8)_\infty^2 (q^{12};q^{12})}, \tag{44}$$

$$\frac{1}{(q;q)_\infty^2(q^3;q^3)_\infty^2} = \frac{(q^8;q^8)_\infty^5(q^{24};q^{24})_\infty^5}{(q^2;q^2)_\infty^5(q^6;q^6)_\infty^5(q^{16};q^{16})_\infty^2(q^{48};q^{48})_\infty^2} + 2q\ \frac{(q^4;q^4)_\infty^4(q^{12};q^{12})_\infty^4}{(q^2;q^2)_\infty^6(q^6;q^6)_\infty^6} + 4q^4\ \frac{(q^4;q^4)_\infty^2(q^{12};q^{12})_\infty^2(q^{16};q^{16})_\infty^4(q^{48};q^{48})_\infty^2}{(q^2;q^2)_\infty^5(q^6;q^6)_\infty^5(q^8;q^8)_\infty(q^{24};q^{24})_\infty}. \tag{45}$$

Proof. From [6, Corollary 8], we find

$$\psi(q)\psi(q^3) = \psi(q^4)\varphi(q^6) + q\psi(q^{12})\varphi(q^2). \tag{46}$$

Again, we have from [6, Corollary 4]

$$\varphi(q)\varphi(q^3) = \varphi(q^4)\varphi(q^{12}) + 2q\ \psi(q^2)\psi(q^6) + 4q^4\ \psi(q^8)\psi(q^{24}). \tag{47}$$

Employing (37) and (38) in (46) and (47), we easily derive (44) and (45), respectively. □

Lemma 2.3 *We have*

$$\frac{1}{(q;q)_\infty^4} = \frac{(q^4;q^4)_\infty^{14}}{(q^2;q^2)_\infty^{14}(q^8;q^8)_\infty^4} + 4q\ \frac{(q^4;q^4)_\infty^2(q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^{10}}. \tag{48}$$

Proof. Employing (37) and (38) in (43), we readily arrive at (48). □

Proofs of the results in the next lemma can be found in [2].

Lemma 2.4 *We have*

$$\frac{(q^3;q^3)_\infty}{(q;q)_\infty^3} = \frac{(q^4;q^4)_\infty^6(q^6;q^6)_\infty^3}{(q^2;q^2)_\infty^9(q^{12};q^{12})_\infty^2} + 3q\ \frac{(q^6;q^6)_\infty(q^{12};q^{12})_\infty^2(q^4;q^4)_\infty^2}{(q^2;q^2)^7}, \tag{49}$$

$$\frac{(q^3;q^3)_\infty^2}{(q;q)_\infty^4} = \frac{(q^4;q^4)_\infty^4(q^6;q^6)_\infty(q^8;q^8)_\infty^4(q^{12};q^{12})_\infty^2}{(q^2;q^2)_\infty^{10}(q^{16};q^{16})_\infty^2(q^{24};q^{24})_\infty} + 2q\ \frac{(q^4;q^4)_\infty^6(q^6;q^6)_\infty(q^{12};q^{12})_\infty^2(q^{16};q^{16})_\infty^2}{(q^2;q^2)_\infty^{10}(q^8;q^8)_\infty^2(q^{24};q^{24})_\infty} + 2q\ \frac{(q^4;q^4)_\infty(q^6;q^6)_\infty^2(q^8;q^8)_\infty^6(q^{24};q^{24})_\infty}{(q^2;q^2)_\infty^9(q^{12};q^{12})_\infty(q^{16};q^{16})^2} + 4q^2\ \frac{(q^4;q^4)_\infty^3(q^6;q^6)_\infty^2(q^{16};q^{16})_\infty^2(q^{24};q^{24})_\infty}{(q^2;q^2)_\infty^9(q^{12};q^{12})_\infty}, \tag{50}$$

$$\frac{1}{\varphi(-q)} = \frac{\varphi^3(-q^9)}{\varphi^4(-q^3)} + 2q\frac{\varphi^3(-q^9)w(q^3)}{\varphi^4(-q^3)} + 4q^2w^2(q^3)\frac{\varphi^3(-q^9)}{\varphi^4(-q^3)}, \tag{51}$$

where

$$w(q) = \frac{(q;q)_\infty (q^6;q^6)_\infty^3}{(q^2;q^2)_\infty (q^3;q^3)_\infty^3}. \tag{52}$$

Lemma 2.5 *We have*

$$\begin{aligned}
&\frac{(q^2;q^2)_\infty^6}{(q;q)_\infty^6} \\
&= \frac{(q^6;q^6)_\infty^{10}(q^9;q^9)_\infty^{16}}{(q^3;q^3)_\infty^{18}(q^{18};q^{18})_\infty^{8}} + 6q\,\frac{(q^6;q^6)_\infty^{9}(q^9;q^9)_\infty^{13}}{(q^3;q^3)_\infty^{17}(q^{18};q^{18})_\infty^{5}} + 21q^2\,\frac{(q^6;q^6)_\infty^{8}(q^9;q^9)_\infty^{10}}{(q^3;q^3)_\infty^{16}(q^{18};q^{18})_\infty^{2}} \\
&\quad + 44q^3\,\frac{(q^6;q^6)_\infty^{7}(q^9;q^9)_\infty^{7}(q^{18};q^{18})_\infty}{(q^3;q^3)_\infty^{15}} + 60q^4\,\frac{(q^6;q^6)_\infty^{6}(q^9;q^9)_\infty^{4}(q^{18};q^{18})_\infty^{4}}{(q^3;q^3)_\infty^{14}} \\
&\quad + 48q^5\,\frac{(q^6;q^6)_\infty^{5}(q^9;q^9)_\infty(q^{18};q^{18})_\infty^{7}}{(q^3;q^3)_\infty^{13}} + 16q^6\,\frac{(q^6;q^6)_\infty^{4}(q^{18};q^{18})_\infty^{10}}{(q^9;q^9)_\infty^{2}(q^3;q^3)_\infty^{12}}.
\end{aligned} \tag{53}$$

Proof. Squaring both sides of (41) and then employing (42), we have

$$\psi^2(q) = \frac{\varphi^2(-q^9)}{\chi^2(-q^3)} + q^2\psi^2(q^9) + 2q\frac{\varphi(-q^9)\psi(q^9)}{\chi(-q^3)}. \tag{54}$$

Again, squaring both sides of (51), we find that

$$\frac{1}{\varphi^2(-q)} = \frac{\varphi^6(-q^9)}{\varphi^8(-q^3)}\{1 + 4qw(q^3) + 12q^2w^2(q^3) + 16q^3w^3(q^3) + 16q^4w^4(q^3)\}. \tag{55}$$

Now, replacing q by $-q$ in (37), we have

$$\varphi(-q) = \frac{(q;q)_\infty^2}{(q^2;q^2)}. \tag{56}$$

Multiplying (54) and (55) and then employing (38), (39), (52) and (56), we easily arrive at (53) to complete the proof. □

Lemma 2.6 *We have*

$$\begin{aligned}
\frac{(q^4;q^4)_\infty}{(q;q)_\infty} &= \frac{(q^{12};q^{12})_\infty (q^{18};q^{18})_\infty^4}{(q^3;q^3)_\infty^3 (q^{36};q^{36})_\infty^2} + q\,\frac{(q^6;q^6)_\infty^2 (q^9;q^9)_\infty^3 (q^{36};q^{36})_\infty}{(q^3;q^3)_\infty^4 (q^{18};q^{18})_\infty^2} \\
&\quad + 2q^2\,\frac{(q^6;q^6)_\infty (q^{18};q^{18})_\infty (q^{36};q^{36})_\infty}{(q^3;q^3)_\infty^3}.
\end{aligned} \tag{57}$$

Proof. From [4], we have

$$\frac{c(q)}{c(q^4)} = 1 + \frac{\psi^2(q^2)}{q\psi^2(q^6)}. \tag{58}$$

Employing (40) in (58), we find that

$$\frac{(q^4;q^4)_\infty}{(q;q)_\infty} = q\frac{(q^{12};q^{12})_\infty^3}{(q^3;q^3)^3}\left\{1 + \frac{\psi^2(q^2)}{q\psi^2(q^6)}\right\}. \tag{59}$$

Next, replacing q by q^2 in (54), we have

$$\psi^2(q^2) = \frac{\varphi^2(-q^{18})}{\chi^2(-q^6)} + q^4\psi^2(q^{18}) + 2q^2\frac{\varphi(-q^{18})\psi(q^{18})}{\chi(-q^6)}. \tag{60}$$

Using (60) in (59), we obtain

$$\begin{aligned}
&\frac{(q^4;q^4)_\infty}{(q;q)_\infty} \\
&= q\frac{(q^{12};q^{12})_\infty^3}{(q^3;q^3)_\infty^3}\left\{1 + \frac{1}{q\psi^2(q^6)}\left(\frac{\varphi^2(-q^{18})}{\chi^2(-q^6)} + q^4\psi^2(q^{18}) + 2q^2\frac{\varphi(-q^{18})\psi(q^{18})}{\chi(-q^6)}\right)\right\} \\
&= q\frac{(q^{12};q^{12})_\infty^3}{(q^3;q^3)_\infty^3}\left\{1 + q^3\frac{\psi^2(q^{18})}{\psi^2(q^6)}\right\} \\
&\quad + q\frac{(q^{12};q^{12})_\infty^3}{(q^3;q^3)_\infty^3}\left\{\frac{\varphi^2(-q^{18})}{q\psi^2(q^6)\chi^2(-q^6)} + 2q\,\frac{\varphi(-q^{18})\psi(q^{18})}{\psi^2(q^6)\chi(-q^6)}\right\}.
\end{aligned} \tag{61}$$

Employing (38), (39) and (56) in (61), we find that

$$\begin{aligned}
\frac{(q^4;q^4)_\infty}{(q;q)_\infty} &= q\frac{(q^{12};q^{12})_\infty^3}{(q^3;q^3)_\infty^3}\left\{1 + q^3\frac{\psi^2(q^{18})}{\psi^2(q^6)}\right\} + \frac{(q^{12};q^{12})_\infty(q^{18};q^{18})_\infty^4}{(q^3;q^3)_\infty^3(q^{36};q^{36})_\infty^2} \\
&\quad + 2q^2\,\frac{(q^{18};q^{18})_\infty(q^{36};q^{36})_\infty(q^6;q^6)_\infty}{(q^3;q^3)_\infty^3}.
\end{aligned} \tag{62}$$

Now, multiplying both sides of (59) by $\psi^2(q^6)/\psi^2(q^2)$, replacing q by q^3, and then employing (38), we deduce that

$$1 + q^3\,\frac{\psi^2(q^{18})}{\psi^2(q^6)} = \frac{(q^6;q^6)_\infty^2(q^9;q^9)_\infty^3(q^{36};q^{36})_\infty}{(q^3;q^3)_\infty(q^{18};q^{18})_\infty^2(q^{12};q^{12})_\infty^3}. \tag{63}$$

Employing (63) in (62), we arrive at (62) to finish the proof. □

3. Proofs of Theorems 1.1, 1.3, and 1.5

Proofs of (2) *and* (3). Using (44) in (1), we have

$$\sum_{n=0}^{\infty} PDO(n)q^n = \frac{(q^8;q^8)_\infty^2(q^{12};q^{12})_\infty^4}{(q^2;q^2)_\infty^2(q^4;q^4)_\infty(q^6;q^6)_\infty^2(q^{24};q^{24})_\infty^2} + q\,\frac{(q^4;q^4)_\infty^6(q^{24};q^{24})_\infty^2}{(q^2;q^2)_\infty^4(q^8;q^8)_\infty^2(q^{12};q^{12})_\infty^2}.$$

Extracting the terms involving q^{2n} and q^{2n+1} in the above, we easily arrive at (2) and (3), respectively. □

Proofs of (4)–(6). Using (57) in (1), we find that

$$\sum_{n=0}^{\infty} PDO(n)q^n = \frac{(q^6;q^6)_\infty^2(q^{18};q^{18})_\infty^4}{(q^3;q^3)_\infty^4(q^{36};q^{36})_\infty^2} + q\,\frac{(q^6;q^6)_\infty^4(q^9;q^9)_\infty^3(q^{36};q^{36})_\infty}{(q^3;q^3)_\infty^5(q^{12};q^{12})_\infty(q^{18};q^{18})_\infty^2} + 2q^2\,\frac{(q^6;q^6)_\infty^3(q^{18};q^{18})_\infty(q^{36};q^{36})_\infty}{(q^3;q^3)_\infty^4(q^{12};q^{12})_\infty}. \tag{64}$$

Extracting from both sides of (64), the terms involving q^{3n}, q^{3n+1}, and q^{3n+2}, respectively, we arrive at (4)–(6), respectively. □

Proofs of (10) *and* (12). Employing (45) in (2), we have

$$\sum_{n=0}^{\infty} PDO(2n)q^n = \frac{(q^4;q^4)_\infty^2(q^8;q^8)^5(q^{24};q^{24})_\infty^5}{(q^2;q^2)_\infty^5(q^6;q^6)_\infty(q^{12};q^{12})_\infty^2(q^{16};q^{16})_\infty^2(q^{48};q^{48})_\infty^2} + 2q\,\frac{(q^4;q^4)_\infty^6(q^{12};q^{12})_\infty^2}{(q^2;q^2)_\infty^6(q^6;q^6)_\infty^2} + 4q^4\,\frac{(q^4;q^4)_\infty^4(q^{16};q^{16})_\infty^4(q^{48};q^{48})_\infty^2}{(q^2;q^2)_\infty^5(q^6;q^6)_\infty(q^8;q^8)_\infty(q^{12};q^{12})_\infty^2(q^{24};q^{24})_\infty}.$$

Extracting the terms involving q^{2n} and q^{2n+1} from both sides of the above we obtain (10) and (12), respectively. □

Proofs of (11) *and* (13). Employing (48) in (3), we have

$$\sum_{n=0}^{\infty} PDO(2n+1)q^n = \frac{(q^2;q^2)_\infty^6(q^{12};q^{12})_\infty^2}{(q^4;q^4)_\infty^2(q^6;q^6)_\infty^2}\Big\{\frac{(q^4;q^4)_\infty^{14}}{(q^2;q^2)_\infty^{14}(q^8;q^8)_\infty^4} + 4q\,\frac{(q^4;q^4)_\infty^2(q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^{10}}\Big\}.$$

Extracting the terms involving q^{2n} and q^{2n+1} from both sides of the above, we easily deduce (11) and (13). □

Proofs of (14) *and* (16). Using (48) in (4), we arrive at

$$\sum_{n=0}^{\infty} PDO(3n)q^n = \frac{(q^2;q^2)_\infty^2 (q^6;q^6)_\infty^4}{(q^{12};q^{12})_\infty^2} \left\{ \frac{(q^4;q^4)_\infty^{14}}{(q^2;q^2)_\infty^{14}(q^8;q^8)_\infty^4} + 4q \frac{(q^4;q^4)_\infty^2 (q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^{10}} \right\}. \tag{65}$$

Now (14) and (16) can be deduced by extracting the terms involving the even and odd powers of q, respectively, of the above. □

Proofs of (15) *and* (17). Employing (48) in (6), we arrive at

$$\begin{aligned} \sum_{n=0}^{\infty} PDO(3n+2)q^n &= 2\ \frac{(q^2;q^2)_\infty^3 (q^6;q^6)_\infty (q^{12};q^{12})_\infty}{(q^4;q^4)_\infty} \\ &\times \left\{ \frac{(q^4;q^4)_\infty^{14}}{(q^2;q^2)_\infty^{14}(q^8;q^8)_\infty^4} + 4q\ \frac{(q^4;q^4)_\infty^2 (q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^{10}} \right\}. \end{aligned} \tag{66}$$

Extracting the even and odd powers of q from both sides of (66), we readily deduce (15) and (17), respectively. □

Proofs of (18) *and* (19). Squaring both sides of (51) and then employing the resultant identity in (4), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} PDO(3n)q^n = \frac{\varphi^2(-q^6)\varphi^6(-q^9)}{\varphi^8(-q^3)} \{ &1 + 4q\ w(q^3) + 12q^2\ w^2(q^3) + 16q^3 w^3(q^3) \\ &+ 16q^4\ w^4(q^3) \}. \end{aligned} \tag{67}$$

Extracting from both sides of (67), those terms involving only q^{3n+1} and q^{3n+2}, respectively, we find that

$$\sum_{n=0}^{\infty} PDO(9n+3)q^n = 4\ \frac{w(q)\varphi^2(-q^2)\varphi^6(-q^3)}{\varphi^8(-q)} \{1 + 4q\ w^3(q)\}$$

and

$$\sum_{n=0}^{\infty} PDO(9n+6)q^n = 12\ \frac{w^2(q)\varphi^2(-q^2)\varphi^6(-q^3)}{\varphi^8(-q)},$$

which by (52) and (56) reduce to (18) and (19), respectively. □

Proofs of (20), (21), (23) *and* (25). Employing (53) in (12), we find that

$$\sum_{n=0}^{\infty} PDO(4n+2)q^n$$
$$= 2\left\{\frac{(q^6;q^6)_\infty^{12}(q^9;q^9)_\infty^{16}}{(q^3;q^3)_\infty^{20}(q^{18};q^{18})_\infty^{8}} + 6q\,\frac{(q^6;q^6)_\infty^{11}(q^9;q^9)_\infty^{13}}{(q^3;q^3)_\infty^{19}(q^{18};q^{18})_\infty^{5}} + 21q^2\,\frac{(q^6;q^6)_\infty^{10}(q^9;q^9)_\infty^{10}}{(q^3;q^3)_\infty^{18}(q^{18};q^{18})_\infty^{2}}\right.$$
$$+ 44q^3\,\frac{(q^6;q^6)_\infty^{9}(q^9;q^9)_\infty^{7}(q^{18};q^{18})_\infty}{(q^3;q^3)_\infty^{17}} + 60q^4\,\frac{(q^6;q^6)_\infty^{8}(q^9;q^9)_\infty^{4}(q^{18};q^{18})_\infty^{4}}{(q^3;q^3)_\infty^{16}}$$
$$\left. + 48q^5\,\frac{(q^6;q^6)_\infty^{7}(q^9;q^9)_\infty(q^{18};q^{18})_\infty^{7}}{(q^3;q^3)_\infty^{15}}\right\}. \tag{68}$$

Extracting from both sides of (68), those terms involving only q^{3n}, q^{3n+1} and q^{3n+2}, respectively, we deduce (21), (23) and (25).

Now we prove (20) and present a second proof of (23).

From (49), we have

$$\frac{(q^3;q^3)_\infty^4}{(q;q)_\infty^{12}} = \frac{(q^4;q^4)_\infty^{24}(q^6;q^6)_\infty^{12}}{(q^2;q^2)_\infty^{36}(q^{12};q^{12})_\infty^{8}} + 12q\,\frac{(q^4;q^4)_\infty^{20}(q^6;q^6)_\infty^{10}}{(q^2;q^2)_\infty^{34}(q^{12};q^{12})_\infty^{4}}$$
$$+ 54q^2\,\frac{(q^4;q^4)_\infty^{16}(q^6;q^6)_\infty^{8}}{(q^2;q^2)_\infty^{32}} + 108q^3\,\frac{(q^4;q^4)_\infty^{12}(q^6;q^6)_\infty^{6}(q^{12};q^{12})_\infty^{4}}{(q^2;q^2)_\infty^{30}}$$
$$+ 81q^4\,\frac{(q^4;q^4)_\infty^{8}(q^6;q^6)_\infty^{4}(q^{12};q^{12})_\infty^{8}}{(q^2;q^2)_\infty^{28}}. \tag{69}$$

Employing (69) in (14), we find that

$$\sum_{n=0}^{\infty} PDO(6n)q^n = \frac{(q^2;q^2)_\infty^{14}}{(q^4;q^4)_\infty^{4}(q^6;q^6)_\infty^{2}}\left\{\frac{(q^4;q^4)_\infty^{24}(q^6;q^6)_\infty^{12}}{(q^2;q^2)_\infty^{36}(q^{12};q^{12})_\infty^{8}}\right.$$
$$+ 12q\,\frac{(q^4;q^4)_\infty^{20}(q^6;q^6)_\infty^{10}}{(q^2;q^2)_\infty^{34}(q^{12};q^{12})_\infty^{4}} + 54q^2\,\frac{(q^4;q^4)_\infty^{16}(q^6;q^6)_\infty^{8}}{(q^2;q^2)_\infty^{32}}$$
$$+ 108q^3\,\frac{(q^4;q^4)_\infty^{12}(q^6;q^6)_\infty^{6}(q^{12};q^{12})_\infty^{4}}{(q^2;q^2)_\infty^{30}}$$
$$\left. + 81q^4\,\frac{(q^4;q^4)_\infty^{8}(q^6;q^6)_\infty^{4}(q^{12};q^{12})_\infty^{8}}{(q^2;q^2)_\infty^{28}}\right\}. \tag{70}$$

Extracting the terms involving q^{2n} and q^{2n+1} from both sides of (70), we readily arrive at (20) and (23) to finish the proof. □

Proofs of (22) *and* (24). Squaring both sides of (50), we have

$$\begin{aligned}\frac{(q^3;q^3)_\infty^4}{(q;q)_\infty^8} &= \frac{(q^4;q^4)_\infty^8(q^6;q^6)_\infty^2(q^8;q^8)_\infty^8(q^{12};q^{12})_\infty^4}{(q^2;q^2)_\infty^{20}(q^{16};q^{16})_\infty^4(q^{24};q^{24})_\infty^2}\\
&+4q\,\frac{(q^4;q^4)_\infty^{10}(q^6;q^6)_\infty^2(q^8;q^8)_\infty^2(q^{12};q^{12})_\infty^4}{(q^2;q^2)_\infty^{20}(q^{24};q^{24})_\infty^2}\\
&+4q\,\frac{(q^4;q^4)_\infty^5(q^6;q^6)_\infty^3(q^8;q^8)_\infty^{10}(q^{12};q^{12})_\infty}{(q^2;q^2)_\infty^{19}(q^{16};q^{16})_\infty^4}\\
&+16q^2\,\frac{(q^4;q^4)_\infty^7(q^6;q^6)_\infty^3(q^8;q^8)_\infty^4(q^{12};q^{12})_\infty}{(q^2;q^2)_\infty^{19}}\\
&+4q^2\,\frac{(q^4;q^4)_\infty^{12}(q^6;q^6)_\infty^2(q^{12};q^{12})_\infty^4(q^{16};q^{16})_\infty^4}{(q^2;q^2)_\infty^{20}(q^8;q^8)_\infty^4(q^{24};q^{24})_\infty^2}\\
&+4q^2\,\frac{(q^4;q^4)_\infty^2(q^6;q^6)_\infty^4(q^8;q^8)_\infty^{12}(q^{24};q^{24})_\infty^2}{(q^2;q^2)_\infty^{18}(q^{12};q^{12})_\infty^2(q^{16};q^{16})_\infty^4}\\
&+16q^3\,\frac{(q^4;q^4)_\infty^9(q^6;q^6)_\infty^3(q^{12};q^{12})_\infty(q^{16};q^{16})_\infty^4}{(q^2;q^2)_\infty^{19}(q^8;q^8)_\infty^2}\\
&+16q^3\,\frac{(q^4;q^4)_\infty^4(q^8;q^8)_\infty^6(q^6;q^6)_\infty^4(q^{24};q^{24})_\infty^2}{(q^2;q^2)_\infty^{18}(q^{12};q^{12})_\infty^2}\\
&+16q^4\,\frac{(q^4;q^4)_\infty^6(q^6;q^6)_\infty^4(q^{16};q^{16})_\infty^4(q^{24};q^{24})_\infty^2}{(q^2;q^2)_\infty^{18}(q^{12};q^{12})_\infty^2}. \qquad (71)\end{aligned}$$

Now, using (71) in (16), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} PDO(6n+3)q^n = 4\,\Bigg\{&\frac{(q^4;q^4)_\infty^{12}(q^8;q^8)_\infty^8(q^{12};q^{12})_\infty^4}{(q^2;q^2)_\infty^{18}(q^{16};q^{16})_\infty^4(q^{24};q^{24})_\infty^2}\\
&+4q\,\frac{(q^4;q^4)_\infty^{14}(q^8;q^8)_\infty^2(q^{12};q^{12})_\infty^4}{(q^2;q^2)_\infty^{18}(q^{24};q^{24})_\infty^2}\\
&+4q\,\frac{(q^4;q^4)_\infty^9(q^6;q^6)_\infty(q^8;q^8)_\infty^{10}(q^{12};q^{12})_\infty}{(q^2;q^2)_\infty^{17}(q^{16};q^{16})_\infty^4}\\
&+16q^2\,\frac{(q^4;q^4)_\infty^{11}(q^6;q^6)_\infty(q^8;q^8)_\infty^4(q^{12};q^{12})_\infty}{(q^2;q^2)_\infty^{17}}\\
&+4q^2\,\frac{(q^4;q^4)_\infty^{16}(q^{12};q^{12})_\infty^4(q^{16};q^{16})_\infty^4}{(q^2;q^2)_\infty^{18}(q^8;q^8)_\infty^4(q^{24};q^{24})_\infty^2}\\
&+4q^2\,\frac{(q^4;q^4)_\infty^6(q^6;q^6)_\infty^2(q^8;q^8)_\infty^{12}(q^{24};q^{24})_\infty^2}{(q^2;q^2)_\infty^{16}(q^{12};q^{12})_\infty^2(q^{16};q^{16})_\infty^4}\end{aligned}$$

$$+ 16q^3 \frac{(q^4;q^4)_\infty^{13}(q^6;q^6)_\infty(q^{12};q^{12})_\infty(q^{16};q^{16})_\infty^4}{(q^2;q^2)_\infty^{17}(q^8;q^8)_\infty^2}$$
$$+ 16q^3 \frac{(q^4;q^4)_\infty^{8}(q^8;q^8)_\infty^6(q^{6};q^{6})_\infty^2(q^{24};q^{24})_\infty^2}{(q^2;q^2)_\infty^{16}(q^{12};q^{12})_\infty^2}$$
$$+ 16q^4 \frac{(q^4;q^4)_\infty^{10}(q^6;q^6)_\infty^2(q^{16};q^{16})_\infty^4(q^{24};q^{24})_\infty^2}{(q^2;q^2)_\infty^{16}(q^{12};q^{12})_\infty^2}\Bigg\}. \quad (72)$$

Extracting the terms involving q^{2n} and q^{2n+1} from both sides of the above, we easily deduce (22) and (24), respectively. □

Proofs of (29)–(32). By the binomial theorem, it is easy to deduce that

$$(q;q)_\infty^2 \equiv (q^2;q^2)_\infty \pmod 2 \quad (73)$$

and

$$(q;q)_\infty^4 \equiv (q^2;q^2)_\infty^2 \pmod 4. \quad (74)$$

Employing (73) in (12), (13), and (19), we find that

$$\sum_{n=0}^\infty \frac{PDO(4n+2)}{2}q^n \equiv (q^2;q^2)_\infty^3(q^6;q^6)_\infty \pmod 2,$$
$$\sum_{n=0}^\infty \frac{PDO(4n+3)}{4}q^n \equiv (q^2;q^2)_\infty^6(q^6;q^6)_\infty \pmod 2,$$

and

$$\sum_{n=0}^\infty \frac{PDO(9n+6)}{12}q^n \equiv \frac{(q^6;q^6)_\infty^3}{(q^2;q^2)} \pmod 2,$$

respectively. Now (29)–(31) are apparent from the above.

Now, from (67), we find that

$$\sum_{n=0}^\infty PDO(3n)q^n \equiv \frac{\varphi^2(-q^6)\varphi^6(-q^9)}{\varphi^8(-q^3)} \pmod 4.$$

Thus,

$$\sum_{n=0}^\infty PDO(9n)q^n \equiv \frac{\varphi^2(-q^2)\varphi^6(-q^3)}{\varphi^8(-q)} \pmod 4,$$

which can be rewritten with the help of (56) as

$$\sum_{n=0}^{\infty} PDO(9n)q^n \equiv \frac{(q^2;q^2)^{12}(q^3;q^3)^{12}}{(q;q)^{16}(q^4;q^4)^2(q^6;q^6)^6} \pmod 4.$$

Employing (74) in the above, we obtain

$$\sum_{n=0}^{\infty} PDO(9n)q^n \equiv 1 \pmod 4,$$

from which (32) follows readily. □

Proofs of (33)–(35). From (72), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} PDO(6n+3)q^n &\equiv 4\,\frac{(q^4;q^4)_\infty^{12}(q^8;q^8)_\infty^{8}(q^{12};q^{12})_\infty^{4}}{(q^2;q^2)_\infty^{18}(q^{16};q^{16})_\infty^{4}(q^{24};q^{24})_\infty^{2}} \pmod{16}. \\ &\equiv 4\,(q^4;q^4)_\infty^{3} \pmod{16}, \qquad (75)\end{aligned}$$

which immediately yields (33)–(35). □

4. Concluding Remarks

1. It is not known whether the congruences in Corollary 1.2, Corollary 1.4, or Theorem 1.5 are part of a family of congruences modulo prime powers or not. It would also be interesting to find proofs of the congruences in this paper by appealing to a rank/crank-type statistic.

2. Identities (1) and (2) imply that

$$\sum_{n=0}^{\infty} PDO(2n)q^n = \left(\sum_{n=0}^{\infty} PDO(n)q^n\right)^2.$$

It would be interesting to find a combinatorial proof of this identity.

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