DISJUNCTIVE RADO NUMBERS FOR $ax_1 + x_2 = x_3$ AND $bx_1 + x_2 = x_3$

Liz Lane-Harvard
Department of Mathematics, Colorado State University, Fort Collins, Colorado
lane@math.colostate.edu

Daniel Schaal
Dept. of Mathematics, South Dakota State University, Brookings, South Dakota
daniel.schaal@sdstate.edu

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Abstract

If S is a set of linear equations, the 2-color *disjunctive Rado number* for the set S is the least integer n, provided it exists, such that every coloring of the set $\{1, 2, ..., n\}$ with two colors admits a monochromatic solution to at least one equation in the set S. If no such integer n exists, then the 2-color disjunctive Rado number for the set S is infinite. In this paper, the disjunctive Rado number for the set consisting of the two equations $ax_1 + x_2 = x_3$ and $bx_1 + x_2 = x_3$ is determined for all integers a and b, where $1 \leq a < b$.

1. Introduction

Let $\mathbb{N}$ represent the set of natural numbers and let $[a, b]$ denote the set $\{n \in \mathbb{N} : a \leq n \leq b\}$. For every $t \in \mathbb{N}$, a function $\Delta : [1, n] \to [0, t-1]$ is referred to as a t-coloring of the set $[1, n]$, and the set $[0, t-1]$ is referred to as the set of colors. Given a t-coloring Δ and a system L of linear equations or inequalities in m variables, a solution $(x_1, x_2, ..., x_m)$ to the system L is *monochromatic* if and only if

$$\Delta(x_1) = \Delta(x_2) = ... = \Delta(x_m).$$

In 1916, I. Schur [31] proved that for every $t \geq 2$, there exists a least integer $n = S(t)$ such that for every t-coloring of the set $[1, n]$, there exists a monochromatic solution to $x_1 + x_2 = x_3$.

The integers $S(t)$ are called *Schur numbers*. It is known that $S(2) = 5$, $S(3) = 14$, and $S(4) = 45$, but no other Schur numbers are known [33]. In 1933, R. Rado generalized the concept of Schur numbers to arbitrary systems of linear equations. Rado found necessary and sufficient conditions to determine if an arbitrary system of linear equations admits a monochromatic solution under every t-coloring of the natural numbers [16, 17, 18]. For a given system L of linear equations, the least integer n, provided that it exists, such that for every t-coloring of the set $[1, n]$ there exists a monochromatic solution to L is called the t-color *Rado number* (or t-color *generalized Schur number*) for the system L. If such an integer n does not exist, then the t-color Rado number for the system L is infinite. In recent years, the exact Rado numbers for several families of equations and inequalities have been found [4, 6, 7, 8, 11, 12, 15, 23, 24, 25, 27, 28, 32].

We will need the following notation to proceed. For all integers $m \geq 3$, let $L(m)$ represent the equation

$$L(m)\colon\ x_1 + x_2 + ... + x_{m-1} = x_m,$$

and let $r(L(m))$ represent the 2-color Rado number for this equation. In [1], it was determined that for every $m \geq 3$, the 2-color Rado number for the equation $L(m)$ is

$$r(L(m)) = m^2 - m - 1.$$

It is routine to verify that for every $m \geq 3$, it is impossible to 2-color the values in the set

$$\{1, m-1, m, m^2 - 2m + 1, m^2 - m - 1\}$$

and avoid a monochromatic solution to $L(m)$, showing that $r(L(m)) \leq m^2 - m - 1$.

Several variations of $L(m)$ have recently been considered. One such variation will require the following notation. For all integers $a \geq 1$, let $S(a)$ represent the equation

$$S(a)\colon\ ax_1 + x_2 = x_3,$$

and let $r(S(a))$ represent the 2-color Rado number for this equation. For every integer $a \geq 1$, if (α, β, γ) is a solution to $S(a)$, then the $(a+2)$-tuple $(\alpha, \alpha, ..., \alpha, \beta, \gamma)$ is a solution to $L(a+2)$. So, every 2-coloring that admits a monochromatic solution to $S(a)$ also admits a monochromatic solution to $L(a+2)$. It follows that for every integer $a \geq 1$,

$$r(S(a)) \geq r(L(a + 2)).$$

In [10], it was shown that for every integer $a \geq 1$,

$$r(S(a)) = a^2 + 3a + 1 = (a + 2)^2 - (a + 2) - 1 = r(L(a + 2)).$$

It is also routine to verify that for every $a \geq 1$, it is impossible to 2-color the values in the set

$$\{1, a + 1, a + 2, a^2 + 2a + 1, a^2 + 3a + 1\}$$

and avoid a monochromatic solution to $S(a)$, showing that $r(S(a)) \leq a^2+3a+1$. It should be noted that this result determines the 2-color Rado numbers for a more general family of equations. Let positive integers a and l be given and let $T(a)$ represent any equation of the form

$$T(a)\colon\ a_1x_1 + a_2x_2 + \ldots + a_lx_l + x_{l+1} = x_{l+2}$$

where $a_i \in \mathbb{N}$ for $i \in [1,l]$ and $\sum_{i=1}^{l} a_i = a$. Let $r(T(a))$ represent the 2-color Rado number for this equation. For reasons similar to the argument above, every 2-coloring that admits a monochromatic solution to $S(a)$ also admits a monochromatic solution to $T(a)$, and every 2-coloring that admits a monochromatic solution to $T(a)$ also admits a monochromatic solution to $L(a+2)$. Hence,

$$r(S(a)) \geq r(T(a)) \geq r(L(a+2)) \quad \text{and} \quad r(T(a)) = a^2+3a+1.$$

Recently, several variations of the classical Rado numbers have been developed [2, 3, 5, 14, 19, 20, 21, 26, 29, 30]. Specifically, the concept of t-color disjunctive Rado numbers has been introduced [9, 13, 22]. Given a set S of linear equations, the least integer n, provided that it exists, such that for every t-coloring of the set $[1,n]$ there exists a monochromatic solution to at least one equation in S is called the t-color *disjunctive Rado number* for the set S. If such an integer n does not exist, then the t-color disjunctive Rado number for the set S is infinite. Given a set of linear equations, it is clear that the t-color disjunctive Rado number for this set is less than or equal to the t-color Rado number for each equation in the set.

Specifically, the 2-color disjunctive Rado numbers for the set $S = \{L(m), L(n)\}$ were recently considered. Let $r_d(L(m), L(n))$ represent the 2-color disjunctive Rado number for this set. In [13], it was determined that

$$r_d(L(m),L(n)) = \begin{cases} m^2-m-1 & \text{if } n = m+1 \\ m^2-2m+1 & \text{if } m+2 \leq n \leq m^2-2m+2 \\ n-1 & \text{if } m^2-2m+3 \leq n \leq m^2-m-1 \\ m^2-m-1 & \text{if } n \geq m^2-m \end{cases}$$

for all integers m and n such that $3 \leq m < n$. In this paper we consider the 2-color disjunctive Rado numbers for the set $S = \{S(a), S(b)\}$. Let $r_d(S(a), S(b))$ represent the 2-color disjunctive Rado number for this set. Just as it was easy to see that $r(S(a)) \geq r(L(a+2))$, it is also easy to see that for all integers a and b where $1 \leq a < b$,

$$r_d(S(a), S(b)) \geq r_d(L(a+2), L(b+2)).$$

One may wonder if equality also holds in this situation. The answer is sometimes, but not always. For ease of comparison, we will express the formula for $r_d(L(m), L(n))$ as $r_d(L(a+2), L(b+2))$. For all integers a and b where $1 \leq a < b$,

$$r_d(L(a+2),L(a+2)) = \begin{cases} a^2+3a+1 & \text{if } b=a+1 \\ a^2+2a+1 & \text{if } a+2 \le b \le a^2+2a \\ b+1 & \text{if } a^2+2a+1 \le b \le a^2+3a-1 \\ a^2+3a+1 & \text{if } b \ge a^2+3a \end{cases}$$

The main result of this paper is the following theorem. We will need to introduce the value r, which is the remainder when a^2+2a+1 is divided by b.

Theorem 1. *For all integers a and b where $1 \le a < b$,*
$r_d(S(a),S(b)) = r_d(L(a+2),L(b+2))+$

$$\begin{cases} 1 & \textit{if } a+3 \le b \le \frac{a^2+3a-4}{2} \textit{ and } r=0 \\ a & \textit{if } a+3 \le b \le \frac{a^2+3a-4}{2} \textit{ and } r \ge a+1 \textit{ and } r \le b-a+1 \\ b-r+1 & \textit{if } a+3 \le b \le \frac{a^2+3a-4}{2} \textit{ and } r \ge a+1 \textit{ and } r \ge b-a+2 \\ a-2 & \textit{if } b = \frac{a^2+3a-2}{2} \\ a & \textit{if } \frac{a^2+3a}{2} \le b \le a^2+a \\ 0 & \textit{otherwise} \end{cases}$$

where $a^2+2a+1 = qb+r$ and $0 \le r \le b-1$.

It should be noted that this theorem also gives upper bounds for the 2-color disjunctive Rado numbers for a more general family of sets of equations. Let positive integers a and b be given where $a < b$, and let equations $T(a)$ and $T(b)$ of the form described above be given. Let $r_d(T(a),T(b))$ represent the 2-color disjunctive Rado number for the set $\{T(a),T(b)\}$. If (α,β,γ) is a solution to $\{S(a),S(b)\}$, then (α,β,γ) is a solution to $S(a)$ or $S(b)$. For reasons similar to the argument above, we have that

$$r_d(S(a),S(b)) \ge r_d(T(a),T(b)) \ge r_d(L(a+2),L(b+2)).$$

This gives upper and lower bounds for $r_d(T(a),T(b))$ for all values of a and b, with the difference between the bounds the difference between $r_d(S(a),S(b))$ and $r_d(L(a+2),L(b+2))$ given in Theorem 1. For values of a and b where the difference is zero, the inequality gives the exact values of $r_d(T(a),T(b))$.

II. Main Result

Before proving Theorem 1, we will first state and prove two lemmas. For the remainder of this paper we will assume, without loss of generality, that every 2-coloring Δ colors the first integer 0. We will need the following definition.

Definition 1. For all positive integers a and n, a 2-coloring $\Delta : [1, n] \to [0, 1]$ is said to be *a-good* if Δ avoids a monochromatic solution to the equation $S(a)$.

Lemma 2. *For every positive integer a, every a-good coloring $\Delta : [1, a^2+2a+1] \to [0, 1]$ satisfies the property*

$$\Delta(x) = \begin{cases} 0 & \text{if } 1 \leq x \leq a \\ 1 & \text{if } a+1 \leq x \leq a^2+a+1 \\ 0 & \text{if } x = a^2+2a+1. \end{cases} \tag{1}$$

Proof. Let a positive integer a be given and let an a-good coloring $\Delta : [1, a^2 + 2a + 1] \to [0, 1]$ be given. If $\Delta(a + 1) = 0$, then $(1, 1, a + 1)$ is a solution to $S(a)$ monochromatic in color 0. So, we are able to conclude that $\Delta(a + 1) = 1$.

If $\Delta(a^2 + 2a + 1) = 1$, then $(a + 1, a + 1, a^2 + 2a + 1)$ is a solution to $S(a)$ monochromatic in color 1, so we have $\Delta(a^2 + 2a + 1) = 0$.

If $\Delta(a^2 + a + 1) = 0$, then $(1, a^2 + a + 1, a^2 + 2a + 1)$ is a solution to $S(a)$ monochromatic in color 0, so we have

$\Delta(a^2 + 2a + 1) = 1$.

We will now prove the following claim.

Claim. *For every $k \in [1, a-1]$, we have $\Delta(a + 1 - k) = 0$.*

We will prove this via induction on k. First assume that $k = 1$. If $\Delta(a+1-1) = \Delta(a) = 1$, then $(a, a+1, a^2+a+1)$ would be a solution to $S(a)$ monochromatic is color 1, so we have $\Delta(a) = 0$. Now, let $k \in [1, a-2]$ and assume that $\Delta(a+1-k) = 0$. If $\Delta(a^2 + a(1 - k) + 1) = 0$, then $(a + 1 - k, 1, a^2 + a(1 - k) + 1)$ is a solution to $S(a)$ monochromatic in color 0, so we have

$$\Delta(a^2 + a(1 - k) + 1) = 1.$$

Now, if $\Delta(a - k) = 1$, then $(a - k, a + 1, a^2 + a(1 - k) + 1)$ is a solution to $S(a)$ monochromatic is color 1, so we have $\Delta(a - k) = 0$ and the proof of the claim is complete.

To complete the proof of Lemma 2 it remains to be shown that $\Delta(x) = 1$ for $x \in [a + 2, a^2 + a]$. Let $x \in [a + 2, a^2 + a]$ be given, and note that there exists unique integers $\alpha, \beta \in [1, a]$ such that $x = a\alpha + \beta$. If $\Delta(x) = 0$, then (α, β, x) is a solution to $S(a)$ monochromatic in color 0, so $\Delta(x) = 1$ and the proof of Lemma 2 is complete. □

It can be verified that property (1) characterizes all a-good 2-colorings on the set $[1, a^2 + 2a + 1]$. In fact, we can arbitrarily color all the numbers in the set $[a^2 + a + 2, a^2 + 2a]$ and the colorings will be a-good, as long as property (1) is satisfied. As we will see in Lemma 3, the number of a-good colorings decreases as the length of the interval being colored increases until there is a unique a-good 2-coloring of the set $[1, a^2 + 3a]$.

Lemma 3. *For every positive integer a and every integer $p \in [1, a-1]$, every a-good coloring $\Delta : [1, a^2+2a+1+p] \to [0,1]$ satisfies the property*

$$\Delta(x) = \begin{cases} 0 & \text{if } 1 \le x \le a \\ 1 & \text{if } a+1 \le x \le a^2+a+1+p \\ 0 & \text{if } a^2+2a+1 \le x \le a^2+2a+1+p. \end{cases} \tag{2}$$

Proof. Let $a \in \mathbb{N}$ and $p \in [1, a-1]$ be given and let an a-good coloring $\Delta : [1, a^2+2a+1+p] \to [0,1]$ be given. Now, the coloring Δ restricted to the domain $[1, a^2+2a+1+p]$ will also be a-good, and hence, satisfies property (1). So, we will only need to show that

$$\Delta(a^2+a+1+q) = 1 \text{ and } \Delta(a^2+2a+1+q) = 0 \text{ for } q \in [1,p].$$

Let $q \in [1,p]$ be given. If $\Delta(a^2+2a+1+q) = 1$, then

$$(a+1, a+1+q, a^2+2a+1+q)$$

is a solution to $S(a)$ monochromatic is color 1, so we have

$$\Delta(a^2+2a+1+q) = 0.$$

If $\Delta(a^2+a+1+q) = 0$, then

$$(1, a^2+a+1+q, a^2+2a+1+q)$$

is a solution to $S(a)$ monochromatic is color 0, so we have

$$\Delta(a^2+a+1+q) = 1.$$

The proof of Lemma 3 is complete. □

Before proving Theorem 1, we first restate the theorem in a more expanded form.

Theorem 4. *For all integers a and b where $1 \le a < b$, $r_d(S(a), S(b)) =$*

$$\begin{cases} a^2+3a+1 & \text{if } b = a+1 \\ a^2+2a+1 & \text{if } b = a+2 \\ a^2+2a+2 & \text{if } a+3 \le b \le \frac{a^2+3a-4}{2} \text{ and } r = 0 \\ a^2+2a+1 & \text{if } a+3 \le b \le \frac{a^2+3a-4}{2} \text{ and } 1 \le r \le a \\ a^2+3a+1 & \text{if } a+3 \le b \le \frac{a^2+3a-4}{2} \text{ and } r \ge a+1 \text{ and } r \le b-a+1 \\ a^2+2a+b-r+2 & \text{if } a+3 \le b \le \frac{a^2+3a-4}{2} \text{ and } r \ge a+1 \text{ and } r \ge b-a+2 \\ a^2+3a-1 & \text{if } b = \frac{a^2+3a-2}{2} \\ a^2+3a+1 & \text{if } \frac{a^2+3a}{2} \le b \le a^2+a \\ a^2+2a+1 & \text{if } a^2+a+1 \le b \le a^2+2a \\ b+1 & \text{if } a^2+2a+1 \le b \le a^2+3a-1 \\ a^2+3a+1 & \text{if } a^2+3a \le b \end{cases}$$

where $a^2+2a+1 = qb+r$ *and* $0 \le r \le b-1$.

Proof. Let integers a and b be given where $1 \le a < b$. Let r and q be the unique integers such that $a^2+2a+1 = qb+r$ and $0 \le r \le b-1$. Note that when $a+3 \le b \le \frac{a^2+3a-4}{2}$, it follows that $q \in [2, a-1]$. Recall from the introduction that

$$r_d(S(a), S(b)) \le r(S(a)) = a^2+3a+1 \tag{3}$$

and $r_d(S(a), S(b)) \ge r_d(L(a+2), L(b+2)) =$

$$\begin{cases} a^2+3a+1 & \text{if } b = a+1 \\ a^2+2a+1 & \text{if } a+2 \le b \le a^2+2a \\ b+1 & \text{if } a^2+2a+1 \le b \le a^2+3a-1 \\ a^2+3a+1 & \text{if } b \ge a^2+3a. \end{cases} \tag{4}$$

We will use eleven cases based on the values of b and r. Note that when $a \in [1,2]$, some cases have no values of b that satisfy the assumption. Also note that $r_d(S(1), S(2))$, $r_d(S(1), S(3))$, and $r_d(S(2), S(4))$ are each in two cases, but the values given in the two cases agree. For all the cases where the lower bound is not given by (4), the lower bound will be established by the coloring $\widehat{\Delta} : [1, a^2+3a] \to [0,1]$ defined by

$$\widehat{\Delta}(x) = \begin{cases} 0 & \text{if } 1 \le x \le a \\ 1 & \text{if } a+1 \le x \le a^2+2a \\ 0 & \text{if } a^2+2a+1 \le x \le a^2+3a \end{cases}$$

or a truncation of this coloring. It is easy to check that the coloring $\widehat{\Delta}$ is a-good, so every truncation of $\widehat{\Delta}$ is also a-good. In fact, from Lemma 2, we see that $\widehat{\Delta}$ is the unique a-good 2-coloring of the set $[1, a^2+3a]$, and if we are considering only colorings where 1 is colored 0 then $\widehat{\Delta}$ is the unique coloring. For every $\theta \in [a^2+2a+1, a^2+3a]$, let $\widehat{\Delta}\restriction_{[1,\theta]} : [1,\theta] \to [0,1]$ represent the coloring $\widehat{\Delta}$ restricted to the set $[1,\theta]$. To establish the lower bounds we will need to show that the coloring $\widehat{\Delta}\restriction_{[1,\theta]}$ is b-good for some values of b and θ. That is, we will need to show that if the triple (x_1, x_2, x_3) is monochromatic, then (x_1, x_2, x_3) is not a solution to $S(b)$. Let $b \ge a+1$ and $\theta \in [a^2+2a, a^2+3a]$ be given. If the triple (x_1, x_2, x_3) is monochromatic in color 1, then

$$bx_1 + x_2 \ge b(a+1) + (a+1) = ba + b + a + 1 > a^2+2a \ge x_3,$$

so (x_1, x_2, x_3) is not a solution to $S(b)$. If the triple (x_1, x_2, x_3) is monochromatic in color 0 and $x_i \in [1,a]$ for $i \in [1,3]$, then

$$bx_1 + x_2 \ge b(1) + 1 > a \ge x_3,$$

so (x_1, x_2, x_3) is not a solution to $S(b)$. If the triple (x_1, x_2, x_3) is monochromatic in color 0 and either $x_1 \in [a^2 + 2a + 1, \theta]$ or $x_2 \in [a^2 + 2a + 1, \theta]$, then

$$bx_1 + x_2 \geq b(1) + (a^2 + 2a + 1) > a^2 + 3a + 1 > x_3,$$

so (x_1, x_2, x_3) is not a solution to $S(b)$. Hence, to show the coloring $\widehat{\Delta}\restriction[1, \theta]$ is both a-good and b-good, it only remains to be demonstrated that if the triple (x_1, x_2, x_3) is monochromatic in color 0 and $x_i \in [1, a]$ for $i \in [1, 2]$ and $x_3 \in [a^2 + 2a + 1, \theta]$, then (x_1, x_2, x_3) is not a solution to $S(b)$.

`Case 1:` Assume that $b = a + 1$. The upper bound follows from (3) and the lower bound follows from (4).

`Case 2:` Assume that $b = a + 2$. The lower bound follows from (4). For the upper bound, let a coloring $\Delta : [1, a^2 + 2a + 1] \to [0, 1]$ be given. We must show that Δ is either not a-good or not b-good. Since $b = a + 2$, the triple $(a, 1, a^2 + 2a + 1)$ is a solution to $S(b)$. If Δ is a-good, then from Lemma 2, this triple is monochromatic in color 0, so Δ is not b-good.

`Case 3:` Assume that $a+3 \leq b \leq \frac{a^2+3a-4}{2}$ and $r = 0$. It follows that $a^2+2a+1 = qb$. For the lower bound, we will show that the coloring $\widehat{\Delta}\restriction[1, a^2 + 2a + 1]$ is both a-good and b-good. If $x_1 \in [q, q - 1]$, $x_2 \in [1, a]$, and $x_3 = a^2 + 2a + 1$, then

$$bx_1 + x_2 \leq b(q - 1) + a = a^2 + 2a + 1 - b + a < a^2 + 2a + 1 = x_3.$$

If $x_1 \in [q, a]$, $x_2 \in [1, a]$, and $x_3 = a^2 + 2a + 1$, then

$$bx_1 + x_2 \geq bq + 1 = a^2 + 2a + 2 > x_3.$$

From this and the argument above, the lower bound is complete. For the upper bound, let a coloring $\Delta : [1, a^2 + 2a + 2] \to [0, 1]$ be given. We must show that Δ is either not a-good or not b-good. Since $a^2+2a+1 = qb$, the triple $(a, 1, a^2+2a+2)$ is a solution to $S(b)$. If Δ is a-good, then from Lemma 2, this triple is monochromatic in color 0, so Δ is not b-good.

`Case 4:` Assume that $a+3 \leq b \leq \frac{a^2+3a-4}{2}$ and $1 \leq r \leq a$. The lower bound follows from (4). For the upper bound, let a coloring $\Delta : [1, a^2 + 2a + 1] \to [0, 1]$ be given. We must show that Δ is either not a-good or not b-good. Since $a^2 + 2a + 1 = qb + r$, the triple $(q, r, a^2 + 2a + 1)$ is a solution to $S(b)$. If Δ is a-good, then from Lemma 2 and the fact that $r \leq a$, this triple is monochromatic in color 0. Thus, Δ is not b-good.

`Case 5:` Assume that $a + 3 \leq b \leq \frac{a^2+3a-4}{2}$ and $r \geq a + 1$ and $r \leq b - a + 1$. The upper bound follows from (3). For the lower bound, we will show that the coloring $\widehat{\Delta}$ is both a-good and b-good. If $x_1 \in [1, q]$, $x_2 \in [1, a]$, and $x_3 \in [a^2+2a+1, a^2+3a]$, then, since $r \geq a + 1$,

$$bx_1 + x_2 \leq bq + a = a^2 + 2a + 1 - r + a < a^2 + 2a + 1 \leq x_3.$$

If $x_1 \in [q+1, a]$, $x_2 \in [1, a]$, and $x_3 \in [a^2+2a+1, a^2+3a]$, then since $r \le b-a+1$,

$$bx_1 + x_2 \ge b(q+1) + 1 \ge bq + r + a = a^2 + 3a + 1 \ge x_3.$$

From this and the argument above, the lower bound is complete.

`Case 6:` Assume that $a+3 \le b \le \frac{a^2+3a-4}{2}$ and $r \ge a+1$ and $r \ge b-a+2$. For the lower bound we will show that the coloring $\widehat{\Delta}\restriction[1, a^2+2a+b-r+1]$ is both a-good and b-good. Note that since $r \ge b-a+2$, it follows that

$$a^2 + 2a + b - r + 1 \in [a^2+2a+1, a^2+3a+1].$$

If $x_1 \in [1, q]$, $x_2 \in [1, a]$, and $x_3 \in [a^2+2a+1, a^2+2a+b-r+1]$, then since $r \ge a+1$,

$$bx_1 + x_2 \le bq + a = a^2 + 2a + 1 - r + a < a^2 + 2a + 1 \le x_3.$$

If $x_1 \in [q+1, a]$, $x_2 \in [1, a]$, and $x_3 \in [a^2+2a+1, a^2+2a+b-r+1]$, then

$$bx_1 + x_2 \ge b(q+1) + 1 = bq + r + b - r + 1 = a^2 + 2a + b - r + 2 > x_3.$$

From this and the argument above, the lower bound is complete. For the upper bound let a coloring $\Delta : [1, a^2+2a+b-r+2] \to [0,1]$ be given. We must show that Δ is either not a-good or not b-good. Since $qb + r = a^2 + 2a + 1$, the triple $(q+1, 1, a^2+2a+b-r+2)$ is a solution to $S(b)$. If Δ is a-good, then from Lemma 2, this triple is monochromatic in color 0, so Δ is not b-good.

`Case 7:` Assume $b = \frac{a^2+3a-2}{2}$. For the lower bound we will show that the coloring $\widehat{\Delta}\restriction[1, a^2+3a-2]$ is both a-good and b-good. If $x_1 = 1$, $x_2 \in [1, a]$, and $x_3 \in [a^2+2a+1, a^2+3a-2]$, then since $b = \frac{a^2+3a-2}{2}$,

$$bx_1 + x_2 \le b + a = \tfrac{a^2+5a-2}{2} < a^2 + 2a + 1 \le x_3.$$

If $x_1 \in [2, a]$, $x_2 \in [1, a]$, and $x_3 \in [a^2+2a+1, a^2+3a-2]$, then since $b = \frac{a^2+3a-2}{2}$,

$$bx_1 + x_2 \ge 2b + 1 = a^2 + 3a - 1 > a^2 + 3a - 2 \ge x_3.$$

From this and the argument above, the lower bound is complete. For the upper bound, let a coloring $\Delta : [1, a^2+3a-1] \to [0,1]$ be given. We must show that Δ is either not a-good or not b-good. Since $b = \frac{a^2+3a-2}{2}$, the triple $(2, 1, a^2+3a-1)$ is a solution to $S(b)$. If Δ is a-good, then from Lemma 2 this triple is monochromatic in color 0, so Δ is not b-good.

`Case 8:` Assume that $\frac{a^2+3a}{2} \le b \le a^2 + a$. The upper bound follows from (3). For the lower bound we will show that the coloring $\widehat{\Delta}$ is both a-good and b-good. If $x_1 = 1$, $x_2 \in [1, a]$, and $x_3 \in [a^2+2a+1, a^2+3a]$, then since $b \le a^2 + a$,

$$bx_1 + x_2 \le b + a \le a^2 + 2a < a^2 + 2a + 1 \le x_3.$$

If $x_1 \in [2, a]$, $x_2 \in [1, a]$, and $x_3 \in [a^2 + 2a + 1, a^2 + 3a]$, then since $b \geq \frac{a^2+3a}{2}$,

$$bx_1 + x_2 \geq 2b + 1 \geq a^2 + 3a + 1 > a^2 + 3a \geq x_3.$$

From this and the argument above, the lower bound is complete.

`Case 9:` Assume that $a^2 + a + 1 \leq b \leq a^2 + 2a$. The lower bound follows from (4). For the upper bound, let a coloring $\Delta : [1, a^2 + 2a + 1] \to [0, 1]$ be given. We must show that Δ is either not a-good or not b-good. Let $b = a^2 + a + k$ for $k \in [1, a]$, so the triple $(1, a + 1 - k, a^2 + 2a + 1)$ is a solution to $S(b)$. If Δ is a-good, then from Lemma 2, this triple is monochromatic in color 0, so Δ is not b-good.

`Case 10:` Assume that $a^2+2a+1 \leq b \leq a^2+3a-1$. The lower bound follows from (4). For the upper bound let a coloring $\Delta : [1, b + 1] \to [0, 1]$ be given. We must show that Δ is either not a-good or not b-good. The triple $(1, 1, b+1)$ is a solution to $S(b)$. Note that $b + 1 = a^2 + 2a + 1 + p$ where $p \in [1, a - 1]$, so if Δ is a-good, then from Lemma 3, this triple is monochromatic in color 0, so Δ is not b-good.

`Case 11:` Assume that $a^2 + 3a \leq b$. The upper bound follows from (3) and the lower bound follows from (4).

Since we have shown upper and lower bounds for all cases, the proof is complete. $\square$

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