SOME WEIGHTED SUMS OF POWERS OF FIBONACCI POLYNOMIALS

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Abstract

We obtain closed formulas for some weighted sums of powers of bivariate Fibonacci and Lucas polynomials of the form $\sum_{n=0}^{q} \lambda^n (x,y) F_{tsn+l}^k (x,y)$ and $\sum_{n=0}^{q} \lambda^n (x,y) L_{tsn+l}^k (x,y)$, in the cases $k = 1, 2, 3, 4$, and some specific values of the parameters t and l. We express these sums as linear combinations of the Fibopolynomials $\binom{q+m}{tk}_{F_s(x,y)}$, $m = 1, 2, \ldots, tk$.

1. Introduction

The sequence of generalized bivariate Fibonacci polynomials $G_n(x,y)$ is defined by the second-order recurrence $G_{n+2}(x,y) = xG_{n+1}(x,y) + yG_n(x,y)$, with arbitrary initial conditions $G_0(x,y)$ and $G_1(x,y)$. When $G_0(x,y) = 0$ and $G_1(x,y) = 1$ we have the bivariate Fibonacci polynomials $F_n(x,y)$, and when $G_0(x,y) = 2$ and $G_1(x,y) = x$ we have the bivariate Lucas polynomials $L_n(x,y)$. (What we will use about these polynomials is contained in reference [2].) The corresponding extensions to negative indices are given by $F_{-n}(x,y) = -(-y)^{-n} F_n(x,y)$ and $L_{-n}(x,y) = (-y)^{-n} L_n(x,y)$, $n \in \mathbb{N}$, respectively. In the case $y = 1$, we have the Fibonacci and Lucas polynomials (in the variable x), $F_n(x,1)$ and $L_n(x,1)$, denoted simply as $F_n(x)$ and $L_n(x)$. By setting $x = 1$ in these polynomials, we obtain the numerical sequences $F_n(1)$ and $L_n(1)$, denoted as F_n and L_n, corresponding to the Fibonacci sequence $F_n = (0, 1, 1, 2, 3, 5, \ldots)$ and the Lucas sequence $L_n = (2, 1, 3, 4, 7, 11, \ldots)$.

In a recent work [7] we showed that the sequence $G_{tsn+\mu}^k(x,y)$, where $t, s, k \in \mathbb{N}$ and $\mu \in \mathbb{Z}$ are given parameters, can be written as a linear combination of the (s-)Fibopolynomials $\binom{n+tk-i}{tk}_{F_s(x,y)}$, $i = 0, 1, \ldots, tk$, according to

$$G_{tsn+\mu}^k(x,y) = (-1)^{s+1} \sum_{i=0}^{tk} \sum_{j=0}^{i} (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x,y)} \times G_{ts(i-j)+\mu}^k(x,y)\, y^{\frac{sj(j-1)}{2}} \binom{n+tk-i}{tk}_{F_s(x,y)}. \quad (1)$$

Recall that for integers $n, p \geq 0$, we have $\binom{n}{0}_{F_s(x,y)} = \binom{n}{n}_{F_s(x,y)} = 1$ and

$$\binom{n}{p}_{F_s(x,y)} = \frac{F_{sn}(x,y) F_{s(n-1)}(x,y) \cdots F_{s(n-p+1)}(x,y)}{F_s(x,y) F_{2s}(x,y) \cdots F_{ps}(x,y)}, \qquad 0 < p < n.$$

If n or p are negative, or $p > n$, we have $\binom{n}{p}_{F_s(x,y)} = 0$. It is known that $\binom{n}{p}_{F_s(x,y)}$ are indeed polynomials in x and y. When $x = y = 1$ we have the s-Fibonomials $\binom{n}{p}_{F_s}$ (see [6]), and when $x = y = s = 1$, we have the (usual) Fibonomials $\binom{n}{p}_F$, introduced and studied by Hoggatt [3] in 1967.

If $\lambda(x,y)$ is a (non-zero) given real function of the real variables x and y, we see from (1) that

$$\begin{aligned}
&\sum_{n=0}^{q} \lambda^n(x,y) G^k_{tsn+\mu}(x,y) \\
&\quad = (-1)^{s+1} \sum_{i=0}^{tk} \sum_{j=0}^{i} (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x,y)} \\
&\qquad \times G^k_{ts(i-j)+\mu}(x,y)\, y^{\frac{sj(j-1)}{2}} \sum_{n=0}^{q} \lambda^n(x,y) \binom{n+tk-i}{tk}_{F_s(x,y)}. \qquad (2)
\end{aligned}$$

Expression (2) can be written as

$$\begin{aligned}
&\sum_{n=0}^{q} \lambda^n(x,y) G^k_{tsn+\mu}(x,y) \\
&\quad = (-1)^{s+1} \sum_{m=1}^{tk} \sum_{i=0}^{tk-m} \sum_{j=0}^{i} (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x,y)} \\
&\qquad \times G^k_{ts(i-j)+\mu}(x,y)\, y^{\frac{sj(j-1)}{2}} \lambda^{i-tk+q+m}(x,y) \binom{q+m}{tk}_{F_s(x,y)} \\
&\qquad + (-1)^{s+1} \lambda^{-tk}(x,y) \sum_{i=0}^{tk} \sum_{j=0}^{i} (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x,y)} \\
&\qquad \times G^k_{ts(i-j)+\mu}(x,y)\, y^{\frac{sj(j-1)}{2}} \lambda^i(x,y) \sum_{n=0}^{q} \lambda^n(x,y) \binom{n}{tk}_{F_s(x,y)}. \qquad (3)
\end{aligned}$$

This simple observation derived from (1) allows us to obtain closed formulas, in terms of Fibopolynomials, for the weighted sums $\sum_{n=0}^{q} \lambda^n(x,y) G^k_{tsn+\mu}(x,y)$, as the following proposition states (with a straightforward proof using (3)).

Proposition 1. *If $z = \lambda(x,y)$ is a root of*

$$\sum_{i=0}^{tk} \sum_{j=0}^{i} (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x,y)} G^k_{ts(i-j)+\mu}(x,y)\, y^{\frac{sj(j-1)}{2}} z^i = 0, \qquad (4)$$

then the weighted sum of k-th powers $\sum_{n=0}^{q} \lambda^{n}(x,y)\, G_{tsn+\mu}^{k}(x,y)$ *can be expressed as a linear combination of the s-Fibopolynomials* $\binom{q+m}{tk}_{F_s(x,y)}$, $m = 1, 2, \ldots, tk$, *according to*

$$\begin{aligned}
&\sum_{n=0}^{q} \lambda^{n-q}(x,y)\, G_{tsn+\mu}^{k}(x,y) \\
&\quad = (-1)^{s+1} \sum_{m=1}^{tk} \sum_{i=0}^{tk-m} \sum_{j=0}^{i} (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x,y)} \\
&\qquad \times G_{ts(i-j)+\mu}^{k}(x,y)\, y^{\frac{sj(j-1)}{2}} \lambda^{i-tk+m}(x,y) \binom{q+m}{tk}_{F_s(x,y)}. \qquad (5)
\end{aligned}$$

Moreover, suppose expression (5) is valid for some weight function $\lambda(x,y)$. *Then* $z = \lambda(x,y)$ *is a root of (4).*

Let us consider the simplest case $t = k = 1$. Equation (4) is in this case

$$\begin{aligned}
0 &= \sum_{i=0}^{1} \sum_{j=0}^{i} (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{2}{j}_{F_s(x,y)} G_{s(i-j)+\mu}(x,y)\, y^{\frac{sj(j-1)}{2}} z^{i} \qquad (6) \\
&= G_{\mu}(x,y) + \left(G_{s+\mu}(x,y) - L_s(x,y)\, G_{\mu}(x,y)\right) z.
\end{aligned}$$

If $G_{s+\mu}(x,y) - L_s(x,y)\, G_{\mu}(x,y) \neq 0$, we have from (6) the weight

$$\lambda(x,y) = \frac{G_{\mu}(x,y)}{L_s(x,y)\, G_{\mu}(x,y) - G_{s+\mu}(x,y)}. \qquad (7)$$

Expression (5) for the corresponding weighted sum is in this case

$$\sum_{n=0}^{q} \lambda^{n}(x,y)\, G_{sn+\mu}(x,y) = G_{\mu}(x,y)\, \lambda^{q}(x,y) \binom{q+1}{1}_{F_s(x,y)}. \qquad (8)$$

In the Fibonacci case we have, from (7) and (8), that if $\mu \neq s$ then

$$\sum_{n=0}^{q} \left(\frac{F_{\mu}(x,y)}{(-y)^{s} F_{\mu-s}(x,y)} \right)^{n-q} F_{sn+\mu}(x,y) = \frac{F_{\mu}(x,y)}{F_s(x,y)} F_{s(q+1)}(x,y). \qquad (9)$$

Similarly, from (7) and (8), we have in the Lucas case that

$$\sum_{n=0}^{q} \left(\frac{L_{\mu}(x,y)}{(-y)^{s} L_{\mu-s}(x,y)} \right)^{n-q} L_{sn+\mu}(x,y) = \frac{L_{\mu}(x,y)}{F_s(x,y)} F_{s(q+1)}(x,y). \qquad (10)$$

Thus, if $\mu \neq 0, s$ we can combine (9) and (10) together as

$$\begin{aligned} & \frac{1}{F_\mu(x,y)} \sum_{n=0}^{q} \left(\frac{F_\mu(x,y)}{(-y)^s F_{\mu-s}(x,y)} \right)^{n-q} F_{sn+\mu}(x,y) \\ = & \frac{1}{L_\mu(x,y)} \sum_{n=0}^{q} \left(\frac{L_\mu(x,y)}{(-y)^s L_{\mu-s}(x,y)} \right)^{n-q} L_{sn+\mu}(x,y) \\ = & \frac{1}{F_s(x,y)} F_{s(q+1)}(x,y) . \qquad (11) \end{aligned}$$

We call attention to the fact that the term $\frac{1}{F_s(x,y)} F_{s(q+1)}(x,y)$ in (11) *does not depend on the parameter* μ. That is, *in (11) we have infinitely many weighted sums of* $q+1$ *bivariate Fibonacci and Lucas polynomials that are equal to* $\frac{1}{F_s(x,y)} F_{s(q+1)}(x,y)$.

We can have a slight generalization of (11) if we substitute x by $L_r(x,y)$ and y by $(-1)^{r+1} y^r$, where $r \in \mathbb{Z}$. By using the equations

$$\begin{aligned} F_r(x,y) F_n\left(L_r(x,y), (-1)^{r+1} y^r\right) &= F_{rn}(x,y), \\ L_n\left(L_r(x,y), (-1)^{r+1} y^r\right) &= L_{rn}(x,y), \end{aligned}$$

we get for $r \neq 0$ (and $\mu \neq 0, s$ in the Fibonacci sums)

$$\begin{aligned} & \frac{1}{F_{r\mu}(x,y)} \sum_{n=0}^{q} \left(\frac{F_{r\mu}(x,y)}{(-y)^{rs} F_{r(\mu-s)}(x,y)} \right)^{n-q} F_{r(sn+\mu)}(x,y) \\ = & \frac{1}{L_{r\mu}(x,y)} \sum_{n=0}^{q} \left(\frac{L_{r\mu}(x,y)}{(-y)^{rs} L_{r(\mu-s)}(x,y)} \right)^{n-q} L_{r(sn+\mu)}(x,y) \\ = & \frac{1}{F_{rs}(x,y)} F_{rs(q+1)}(x,y) . \qquad (12) \end{aligned}$$

Some examples from (12) (corresponding to some specific values of μ) are the following:

- With $s = 1$ we have

$$\begin{aligned} & \frac{1}{F_{-2r}(x,y)} \sum_{n=0}^{q} \left(\frac{F_{2r}(x,y)}{F_{3r}(x,y)} \right)^{n-q} F_{r(n-2)}(x,y) \qquad (13) \\ = & \frac{1}{F_{-r}(x,y)} \sum_{n=0}^{q} \left(\frac{1}{L_r(x,y)} \right)^{n-q} F_{r(n-1)}(x,y) \\ = & \frac{1}{F_{2r}(x,y)} \sum_{n=0}^{q} \left(\frac{L_r(x,y)}{(-y)^r} \right)^{n-q} F_{r(n+2)}(x,y) \\ = & \frac{1}{F_{3r}(x,y)} \sum_{n=0}^{q} \left(\frac{F_{3r}(x,y)}{(-y)^r F_{2r}(x,y)} \right)^{n-q} F_{r(n+3)}(x,y) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{L_{-2r}(x,y)}\sum_{n=0}^{q}\left(\frac{L_{2r}(x,y)}{L_{3r}(x,y)}\right)^{n-q}L_{r(n-2)}(x,y)\\
&= \frac{1}{2}\sum_{n=0}^{q}\left(\frac{2}{L_r(x,y)}\right)^{n-q}L_{rn}(x,y)=\frac{1}{L_r(x,y)}\sum_{n=0}^{q}\left(\frac{L_r(x,y)}{2\,(-y)^r}\right)^{n-q}L_{r(n+1)}(x,y)\\
&= \frac{1}{L_{-r}(x,y)}\sum_{n=0}^{q}\left(\frac{L_r(x,y)}{L_{2r}(x,y)}\right)^{n-q}L_{r(n-1)}(x,y)=\frac{F_{r(q+1)}(x,y)}{F_r(x,y)}.
\end{aligned}$$

- With $s=2$ we have

$$\begin{aligned}
&\frac{1}{F_{-3r}(x,y)}\sum_{n=0}^{q}\left(\frac{F_{3r}(x,y)}{F_{5r}(x,y)}\right)^{n-q}F_{r(2n-3)}(x,y) \qquad (14)\\
&= \frac{1}{F_{-r}(x,y)}\sum_{n=0}^{q}\left(\frac{F_r(x,y)}{F_{3r}(x,y)}\right)^{n-q}F_{r(2n-1)}(x,y)\\
&= \frac{1}{F_{3r}(x,y)}\sum_{n=0}^{q}\left(\frac{F_{3r}(x,y)}{y^{2r}F_r(x,y)}\right)^{n-q}F_{r(2n+3)}(x,y)\\
&= \frac{1}{L_{-3r}(x,y)}\sum_{n=0}^{q}\left(\frac{L_{3r}(x,y)}{L_{5r}(x,y)}\right)^{n-q}L_{r(2n-3)}(x,y)\\
&= \frac{1}{L_{-r}(x,y)}\sum_{n=0}^{q}\left(\frac{L_r(x,y)}{L_{3r}(x,y)}\right)^{n-q}L_{r(2n-1)}(x,y)\\
&= \frac{(-1)^{rq}}{L_r(x,y)}\sum_{n=0}^{q}\frac{(-1)^{rn}L_{r(2n+1)}(x,y)}{y^{r(n-q)}}=\frac{(-1)^{(r+1)q}}{F_r(x,y)}\sum_{n=0}^{q}\frac{(-1)^{(r+1)n}F_{r(2n+1)}(x,y)}{y^{r(n-q)}}\\
&= \frac{1}{L_{3r}(x,y)}\sum_{n=0}^{q}\left(\frac{L_{3r}(x,y)}{y^{2r}L_r(x,y)}\right)^{n-q}L_{r(2n+3)}(x,y)=\frac{F_{2r(q+1)}(x,y)}{F_{2r}(x,y)}.
\end{aligned}$$

- With $s=3$ we have

$$\begin{aligned}
&\frac{1}{F_r(x,y)}\sum_{n=0}^{q}\left(\frac{-1}{(-y)^r L_r(x,y)}\right)^{n-q}F_{r(3n+1)}(x,y) \qquad (15)\\
&= \frac{1}{F_{2r}(x,y)}\sum_{n=0}^{q}\left(-\frac{L_r(x,y)}{y^{2r}}\right)^{n-q}F_{r(3n+2)}(x,y)\\
&= \frac{1}{F_{4r}(x,y)}\sum_{n=0}^{q}\left(\frac{L_r(x,y)\,L_{2r}(x,y)}{(-y)^{3r}}\right)^{n-q}F_{r(3n+4)}(x,y)\\
&= -\frac{(-y)^r}{F_r(x,y)}\sum_{n=0}^{q}\left(\frac{1}{L_r(x,y)\,L_{2r}(x,y)}\right)^{n-q}F_{r(3n-1)}(x,y)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{L_{-r}(x,y)} \sum_{n=0}^{q} \left(\frac{L_r(x,y)}{L_{4r}(x,y)} \right)^{n-q} L_{r(3n-1)}(x,y) \\
&= \frac{1}{L_r(x,y)} \sum_{n=0}^{q} \left(\frac{L_r(x,y)}{(-y)^r L_{2r}(x,y)} \right)^{n-q} L_{r(3n+1)}(x,y) \\
&= \frac{1}{L_{2r}(x,y)} \sum_{n=0}^{q} \left(\frac{L_{2r}(x,y)}{y^{2r} L_r(x,y)} \right)^{n-q} L_{r(3n+2)}(x,y) \\
&= \frac{1}{L_{4r}(x,y)} \sum_{n=0}^{q} \left(\frac{L_{4r}(x,y)}{(-y)^{3r} L_r(x,y)} \right)^{n-q} L_{r(3n+4)}(x,y) = \frac{F_{3r(q+1)}(x,y)}{F_{3r}(x,y)}.
\end{aligned}
$$

In particular we have the following numerical identities (from (13), (14) and (15) with $x = y = r = 1$)

$$
\begin{aligned}
F_{q+1} &= -\sum_{n=0}^{q} 2^{q-n} F_{n-2} = \sum_{n=0}^{q} F_{n-1} = \sum_{n=0}^{q} (-1)^{n+q} F_{n+2} \qquad (16) \\
&= \frac{1}{2} \sum_{n=0}^{q} (-2)^{n-q} F_{n+3} = \frac{1}{3} \sum_{n=0}^{q} \left(\frac{3}{4} \right)^{n-q} L_{n-2} = \sum_{n=0}^{q} \frac{-1}{3^{n-q}} L_{n-1} \\
&= \sum_{n=0}^{q} 2^{n-q-1} L_n = \sum_{n=0}^{q} (-2)^{q-n} L_{n+1}.
\end{aligned}
$$

$$
\begin{aligned}
F_{2(q+1)} &= \frac{1}{2} \sum_{n=0}^{q} \left(\frac{2}{5} \right)^{n-q} F_{2n-3} = \sum_{n=0}^{q} 2^{q-n} F_{2n-1} = \sum_{n=0}^{q} F_{2n+1} \qquad (17) \\
&= \sum_{n=0}^{q} 2^{n-q-1} F_{2n+3} = -\frac{1}{4} \sum_{n=0}^{q} \left(\frac{4}{11} \right)^{n-q} L_{2n-3} = -\sum_{n=0}^{q} 4^{q-n} L_{2n-1} \\
&= \sum_{n=0}^{q} (-1)^{n+q} L_{2n+1} = \sum_{n=0}^{q} 4^{n-q-1} L_{2n+3}.
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2} F_{3(q+1)} &= \sum_{n=0}^{q} F_{3n+1} = \sum_{n=0}^{q} (-1)^{n-q} F_{3n+2} = \frac{1}{3} \sum_{n=0}^{q} (-3)^{n-q} F_{3n+4} \qquad (18) \\
&= \sum_{n=0}^{q} \frac{1}{3^{n-q}} F_{3n-1} = -\sum_{n=0}^{q} \frac{1}{7^{n-q}} L_{3n-1} = \sum_{n=0}^{q} \left(-\frac{1}{3} \right)^{n-q} L_{3n+1} \\
&= \sum_{n=0}^{q} 3^{n-q-1} L_{3n+2} = \frac{1}{7} \sum_{n=0}^{q} (-7)^{n-q} L_{3n+4}.
\end{aligned}
$$

Of course, some of the numerical results shown in (16), (17) and (18) are well-known identities. For example, in [1]: (i) formula $\sum_{n=0}^{q} 2^{n-q-1} L_n = F_{q+1}$ of (16) is identity 236, (ii) formula $\sum_{n=0}^{q} F_{2n+1} = F_{2(q+1)}$ of (17) is essentially identity 2, and (iii) formula $\sum_{n=0}^{q} F_{3n+1} = \frac{1}{2} F_{3(q+1)}$ in (18) is essentially identity 24 (corrected).

In the remaining sections of this work we consider only the case $y = 1$ of Proposition 1. The corresponding weight function $\lambda(x, 1)$ will be denoted as $\lambda(x)$.

2. Weighted Sums of Cubes

In this section we will obtain expressions for some weighted sums of cubes of Fibonacci polynomials. More specifically, we will consider the Fibonacci case of Proposition 1 when $k = 3$, $\mu = 0$ and $t = 1$.

In this case, equation (4) is (after some simplifications)

$$F_s^3(x)\, z \left(z^2 + 2L_s(x)\, z + (-1)^s\right) = 0. \tag{19}$$

Thus, we have two weights, namely

$$\lambda_1(x) = -L_s(x) + \sqrt{L_s^2(x) - (-1)^s} \;\;, \;\; \lambda_2(x) = -L_s(x) - \sqrt{L_s^2(x) - (-1)^s}. \tag{20}$$

The corresponding weighted sum (5) can be written as

$$\sum_{n=0}^{q} \lambda^{n-q}(x)\, F_{sn}^3(x) = F_s^3(x) \left((-1)^{s+1} \lambda(x) \binom{q+1}{3}_{F_s(x)} + \binom{q+2}{3}_{F_s(x)} \right), \tag{21}$$

where $\lambda(x)$ is any of the weights (20).

In the case $x = 1$, we have in particular the following numerical identities (obtained by setting $s = 1$ and $s = 2$ in (21) with the corresponding weights (20))

$$\sum_{n=0}^{q} \left(-1 \pm \sqrt{2}\right)^{n-q} F_n^3 = \frac{F_{q+1}F_q}{2} \left(\left(-1 \pm \sqrt{2}\right) F_{q-1} + F_{q+2} \right). \tag{22}$$

$$\sum_{n=0}^{q} \left(-3 \pm 2\sqrt{2}\right)^{n-q} F_{2n}^3 = \frac{F_{2(q+1)}F_{2q}}{24} \left(-\left(-3 \pm 2\sqrt{2}\right) F_{2(q-1)} + F_{2(q+2)} \right). \tag{23}$$

We will show now some different versions of (21), involving Chebyshev polynomials of the first kind $T_n(x)$, or of the second kind $U_n(x)$. The results are the following.

Proposition 2. *(a) If s is even, we have the following weighted sums of cubes of Fibonacci polynomials, valid for any integer $l \geq 0$*

$$\sum_{n=1}^{q} T_{n+l}\left(-L_s(x)\right) F_{sn}^3(x) \tag{24}$$

$$= F_s^3(x) \left(-T_{q+l+1}\left(-L_s(x)\right) \binom{q+1}{3}_{F_s(x)} + T_{q+l}\left(-L_s(x)\right) \binom{q+2}{3}_{F_s(x)} \right).$$

$$\sum_{n=1}^{q} U_{n+l-1}\left(-L_s(x)\right) F_{sn}^3(x) \tag{25}$$
$$= F_s^3(x)\left(-U_{q+l}\left(-L_s(x)\right)\binom{q+1}{3}_{F_s(x)} + U_{q+l-1}\left(-L_s(x)\right)\binom{q+2}{3}_{F_s(x)}\right).$$

(b) If s is odd, we have the following weighted sums of cubes of Fibonacci polynomials (where $i^2 = -1$), valid for any integer $l \geq 0$

$$\sum_{n=1}^{q} (-i)^n T_{n+l}\left(-iL_s(x)\right) F_{sn}^3(x) \tag{26}$$
$$= F_s^3(x)\left((-i)^{q+1} T_{q+l+1}\left(-iL_s(x)\right)\binom{q+1}{3}_{F_s(x)} + (-i)^q T_{q+l}\left(-iL_s(x)\right)\binom{q+2}{3}_{F_s(x)}\right).$$

$$\sum_{n=1}^{q} (-i)^{n-1} U_{n+l-1}\left(-iL_s(x)\right) F_{sn}^3(x) \tag{27}$$
$$= F_s^3(x)\left((-i)^q U_{q+l}\left(-iL_s(x)\right)\binom{q+1}{3}_{F_s(x)} + (-i)^{q-1} U_{q+l-1}\left(-iL_s(x)\right)\binom{q+2}{3}_{F_s(x)}\right).$$

Proof. Recall that Chebyshev polynomials of the first kind $T_n(x)$ can be calculated as

$$T_n(x) = \frac{1}{2}\left(\left(x+\sqrt{x^2-1}\right)^n + \left(x-\sqrt{x^2-1}\right)^n\right), \tag{28}$$

and that Chebyshev polynomials of the second kind $U_n(x)$ can be calculated as

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}\left(\left(x+\sqrt{x^2-1}\right)^{n+1} - \left(x-\sqrt{x^2-1}\right)^{n+1}\right). \tag{29}$$

We can write (21) as

$$\sum_{n=0}^{q} \lambda^{n+l}(x) F_{sn}^3(x) \tag{30}$$
$$= F_s^3(x)\left((-1)^{s+1}\lambda^{q+l+1}(x)\binom{q+1}{3}_{F_s(x)} + \lambda^{q+l}(x)\binom{q+2}{3}_{F_s(x)}\right),$$

where l is a non-negative integer. If s is even, the weights (20) are

$$\lambda_1(x) = -L_s(x) + \sqrt{L_s^2(x) - 1} \quad , \quad \lambda_2(x) = -L_s(x) - \sqrt{L_s^2(x) - 1}. \tag{31}$$

Thus, (24) follows from (28), (30) and (31). Similarly, (25) follows from (29), (30) and (31).

On the other hand, if s is odd the weights (20) are $\lambda_1(x) = -L_s(x) + \sqrt{L_s^2(x) + 1}$, $\lambda_2(x) = -L_s(x) - \sqrt{L_s^2(x) + 1}$. These weights can be written as

$$\begin{aligned} \lambda_1(x) &= -i\left(-iL_s(x) + \sqrt{(-iL_s(x))^2 - 1}\right), \\ \lambda_2(x) &= -i\left(-iL_s(x) - \sqrt{(-iL_s(x))^2 - 1}\right). \end{aligned} \tag{32}$$

From (28) and (32) we see that

$$(-i)^n T_n(-iL_s(x)) = \frac{1}{2}\left(\lambda_1^n(x) + \lambda_2^n(x)\right), \tag{33}$$

and from (29) and (32) we see that

$$(-i)^{n+1} U_n(-iL_s(x)) = \frac{1}{2i\sqrt{L_s^2(x) + 1}}\left(\lambda_1^{n+1}(x) - \lambda_2^{n+1}(x)\right). \tag{34}$$

Thus, (26) follows from (30), (32) and (33). Similarly, (27) follows from (30), (32) and (34). □

We note that the integer parameter $l \geq 0$ gives us, in (24) and (25), infinitely many weighted sums of cubes of Fibonacci polynomials for each even s, and in (26) and (27) gives us infinitely many weighted sums of cubes of Fibonacci polynomials for each odd s.

We show some numerical examples from formulas (24) to (27). We set $x = 1$ and $q = 5$ in them.

If $s = 2$, the sequence $(T_n(-L_2))_{n=1}^{\infty}$ involved in (24) is

$$(T_n(-L_2))_{n=1}^{\infty} = (-3, 17, -99, 577, -3363, 19601, -114243, 665857, \ldots).$$

(See [5, A001541] for the unsigned version.) Thus, we have the following weighted sums of cubes of Fibonacci numbers (corresponding to $l = 0, 1$)

$$\begin{aligned} -3F_2^3 + 17F_4^3 - 99F_6^3 + 577F_8^3 - 3363F_{10}^3 &= \frac{F_{10}F_{12}}{L_2F_6}\left(-19601F_8 - 3363F_{14}\right), \\ 17F_2^3 - 99F_4^3 + 577F_6^3 - 3363F_8^3 + 19601F_{10}^3 &= \frac{F_{10}F_{12}}{L_2F_6}\left(114243F_8 + 19601F_{14}\right). \end{aligned}$$

Similarly, the sequence $(U_n(-L_2))_{n=1}^{\infty}$ involved in (25) is

$$(U_n(-L_2))_{n=1}^{\infty} = (-6, 35, -204, 1189, -6930, 40391, -235416, 1372105, \ldots).$$

(See [5, A001109] for the unsigned version.) Thus, we have (for $l = 1, 2$)

$$-6F_2^3 + 35F_4^3 - 204F_6^3 + 1189F_8^3 - 6930F_{10}^3 = \frac{F_{10}F_{12}}{L_2F_6}\left(-40391F_8 - 6930F_{14}\right),$$

$$35F_2^3 - 204F_4^3 + 1189F_6^3 - 6930F_8^3 + 40391F_{10}^3 = \frac{F_{10}F_{12}}{L_2F_6}\left(235416F_8 + 40391F_{14}\right).$$

If $s = 1$, the sequence $\left((-i)^n T_n\left(-iL_1\right)\right)_{n=1}^{\infty}$ involved in (26) is

$$\left((-i)^n T_n\left(-iL_1\right)\right)_{n=1}^{\infty} = \left(-1, 3, -7, 17, -41, 99, -239, 577, -1393, 3363, -8119, \ldots\right).$$

(See [5, A001333] for the unsigned version.) Then, we have the weighted sums (for $l = 0, 1$)

$$\begin{aligned} -F_1^3 + 3F_2^3 - 7F_3^3 + 17F_4^3 - 41F_5^3 &= \frac{F_6F_5}{2}\left(99F_4 - 41F_7\right), \\ 3F_1^3 - 7F_2^3 + 17F_3^3 - 41F_4^3 + 99F_5^3 &= \frac{F_6F_5}{2}\left(-239F_4 + 99F_7\right). \end{aligned}$$

Similarly, if $s = 3$, the sequence $\left((-i)^n U_n\left(-iL_3\right)\right)_{n=1}^{\infty}$ involved in (27) is

$$\left((-i)^n U_n\left(-iL_3\right)\right)_{n=1}^{\infty} = \left(-8, 65, -528, 4289, -34840, 283009, -2298912, 18674305, \ldots\right).$$

(See [5, A041025] for the unsigned version.) Then, we have the weighted sums (for $l = 1, 2$)

$$-8F_3^3 + 65F_6^3 - 528F_9^3 + 4289F_{12}^3 - 34840F_{15}^3 = \frac{F_3^2F_{18}F_{15}}{F_6F_9}\left(283009F_{12} - 34840F_{21}\right),$$

$$65F_3^3 - 528F_6^3 + 4289F_9^3 - 34840F_{12}^3 + 283009F_{15}^3 = \frac{F_3^2F_{18}F_{15}}{F_6F_9}\left(-2298912F_{12} + 283009F_{21}\right).$$

3. Other Weighted Sums

In this section we will obtain expressions for some other weighted sums of Fibonacci and Lucas polynomials. More specifically, we will set $\mu = 0$ in Proposition 1 and consider the following cases: (i) $k = t = 2$ and $G = F$ or $G = L$; (ii) $k = 4$, $t = 1$, $G = F$.

Let us begin with the case (i). When $G = F$, equation (4) is

$$\begin{aligned} 0 &= \sum_{i=0}^{4}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{5}{j}_{F_s(x)} F_{2s(i-j)}^2(x)\, z^i \\ &= (-1)^{s+1} F_{2s}^2(x,y)\, z\,(z+1)\left(z^2-(-1)^s L_{2s}(x)\, z+1\right), \end{aligned} \tag{35}$$

and with $G = L$ is

$$\begin{aligned} 0 &= \sum_{i=0}^{4}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{5}{j}_{F_s(x)} L_{2s(i-j)}^2(x)\, z^i \\ &= (-1)^{s+1}\left(L_{2s}^2(x)\, z^2-(3L_{4s}(x)+2)\, z+4\right)\left(z^2-(-1)^s L_{2s}(x)\, z+1\right). \end{aligned} \tag{36}$$

Observe that

$$z^2-(-1)^s L_{2s}(x)\, z+1 = \left(z-(-1)^s\alpha^{2s}(x)\right)\left(z-(-1)^s\beta^{2s}(x)\right),$$

where $\alpha(x) = \frac{1}{2}\left(x+\sqrt{x^2+4}\right)$ and $\beta(x) = \frac{1}{2}\left(x-\sqrt{x^2+4}\right)$.

Since $z^2-(-1)^s L_{2s}(x)\, z+1$ is a factor of the right-hand sides of (35) and (36), we have for both, the Fibonacci and the Lucas cases, the following weights

$$\lambda_1(x) = (-1)^s\alpha^{2s}(x) \quad , \quad \lambda_2(x) = (-1)^s\beta^{2s}(x). \tag{37}$$

In the Fibonacci case we have in addition the weight $\lambda(x) = -1$ (from the factor $z+1$ of the right-hand side of (35)). In the Lucas case we have in addition the following weights (from the factor $L_{2s}^2(x)\, z^2-(3L_{4s}(x)+2)\, z+4$ of the right-hand side of (36))

$$\begin{aligned} \lambda_3(x) &= \frac{2}{L_{2s}(x)}\left(\frac{3L_{4s}(x)+2}{4L_{2s}(x)}+\sqrt{\left(\frac{3L_{4s}(x)+2}{4L_{2s}(x)}\right)^2-1}\right), \\ \lambda_4(x) &= \frac{2}{L_{2s}(x)}\left(\frac{3L_{4s}(x)+2}{4L_{2s}(x)}-\sqrt{\left(\frac{3L_{4s}(x)+2}{4L_{2s}(x)}\right)^2-1}\right). \end{aligned} \tag{38}$$

In the Fibonacci case the corresponding sum (5) is

$$\begin{aligned} \frac{1}{F_{2s}^2(x)}\sum_{n=0}^{q}\lambda^{n-q}(x)\, F_{2sn}^2(x) &= -\lambda(x)\binom{q+1}{4}_{F_s(x)}+\binom{q+3}{4}_{F_s(x)} \\ &\quad +\left(\lambda^{-1}(x)+(-1)^{s+1}\frac{L_{3s}(x)}{L_s(x)}\right)\binom{q+2}{4}_{F_s(x)}, \end{aligned} \tag{39}$$

(where $\lambda(x)$ is any of the weights (37) or $\lambda(x) = -1$). In the Lucas case the sum (5) is

$$\sum_{n=0}^{q} \lambda^{n-q}(x) L_{2sn}^2(x) = -L_{2s}^2(x)\lambda(x)\binom{q+1}{4}_{F_s(x)} + 4\binom{q+4}{4}_{F_s(x)} + \left((-1)^s L_s^3(x) L_{3s}(x)\lambda(x) - L_{2s}^2(x)\lambda^2(x)\right)\binom{q+2}{4}_{F_s(x)} + \left(4\lambda^{-1}(x) - L_s^2(x)\left(3L_{2s}(x) - 2(-1)^s\right)\right)\binom{q+3}{4}_{F_s(x)}, \tag{40}$$

(where $\lambda(x)$ is any of the weights (37) or (38)).

For the weight $\lambda(x) = -1$ of the Fibonacci case, we have from (39) the following alternating sum of squares of Fibonacci polynomials

$$\sum_{n=0}^{q} (-1)^{n+q} F_{2sn}^2(x) = F_{2s}^2(x)\left(\binom{q+1}{4}_{F_s(x)} + (-1)^{s+1} L_{2s}(x)\binom{q+2}{4}_{F_s(x)} + \binom{q+3}{4}_{F_s(x)}\right). \tag{41}$$

When $x = s = 1$, the weights (37) are $-\frac{3\pm\sqrt{5}}{2}$. In this case we have the following numerical formulas for weighted sums of squares of Fibonacci and Lucas numbers in terms of Fibonomials:

$$\sum_{n=0}^{q}\left(-\tfrac{3\pm\sqrt{5}}{2}\right)^{n-q} F_{2n}^2 = \tfrac{3\pm\sqrt{5}}{2}\binom{q+1}{4}_F + \tfrac{5\pm\sqrt{5}}{2}\binom{q+2}{4}_F + \binom{q+3}{4}_F,$$

$$\sum_{n=0}^{q}\left(-\tfrac{3\pm\sqrt{5}}{2}\right)^{n-q} L_{2n}^2 = \tfrac{9\left(3\pm\sqrt{5}\right)}{2}\binom{q+1}{4}_F - \tfrac{51\pm23\sqrt{5}}{2}\binom{q+2}{4}_F + \left(-17\pm2\sqrt{5}\right)\binom{q+3}{4}_F + 4\binom{q+4}{4}_F.$$

By using the weights (37) in (39) and (40), we can obtain expressions for weighted sums of squares of Fibonacci and Lucas polynomials, in which the weight functions are in turn certain Fibonacci or Lucas polynomials. This is the content of the following proposition.

Proposition 3. *For $l \in \mathbb{Z}$ we have the following weighted sums of squares of Fibonacci and Lucas polynomials*

(a)

$$\frac{1}{F_{2s}^2(x)}\sum_{n=0}^{q}(-1)^{s(n+q+1)+1}F_{2s(n+l)}(x)F_{2sn}^2(x) \tag{42}$$
$$= F_{2s(q+l+1)}(x)\binom{q+1}{4}_{F_s(x)} + F_s(x)L_{s(2q+2l+1)}(x)\binom{q+2}{4}_{F_s(x)}$$
$$+ (-1)^{s+1}F_{2s(q+l)}(x)\binom{q+3}{4}_{F_s(x)}.$$

(b)

$$\frac{1}{F_{2s}^2(x)}\sum_{n=0}^{q}(-1)^{s(n+q+1)+1}L_{2s(n+l)}(x)F_{2sn}^2(x) \tag{43}$$
$$= L_{2s(q+l+1)}(x)\binom{q+1}{4}_{F_s(x)} + \left(x^2+4\right)F_s(x)F_{s(2q+2l+1)}(x)\binom{q+2}{4}_{F_s(x)}$$
$$+ (-1)^{s+1}L_{2s(q+l)}(x)\binom{q+3}{4}_{F_s(x)}.$$

(c)

$$\sum_{n=0}^{q}(-1)^{s(n+q)}F_{2s(n+l)}(x)L_{2sn}^2(x) \tag{44}$$
$$= L_{2s}^2(x)(-1)^{s+1}F_{2s(q+l+1)}(x)\binom{q+1}{4}_{F_s(x)} + 4F_{2s(q+l)}(x)\binom{q+4}{4}_{F_s(x)}$$
$$+ \left(L_s^3(x)L_{3s}(x)F_{2s(q+l+1)}(x) - L_{2s}^2(x)F_{2s(q+l+2)}(x)\right)\binom{q+2}{4}_{F_s(x)}$$
$$+ \left(4(-1)^s F_{2s(q+l-1)}(x) - L_s^2(x)\left(3L_{2s}(x) - 2(-1)^s\right)F_{2s(q+l)}(x)\right)\binom{q+3}{4}_{F_s(x)}.$$

(d)

$$\sum_{n=0}^{q}(-1)^{s(n+q)}L_{2s(n+l)}(x)L_{2sn}^2(x) \tag{45}$$
$$= (-1)^{s+1}L_{2s}^2(x)L_{2s(q+l+1)}(x)\binom{q+1}{4}_{F_s(x)} + 4L_{2s(q+l)}(x)\binom{q+4}{4}_{F_s(x)}$$
$$+ \left(L_s^3(x)L_{3s}(x)L_{2s(q+l+1)}(x) - L_{2s}^2(x)L_{2s(q+l+2)}(x)\right)\binom{q+2}{4}_{F_s(x)}$$
$$+ \left(4(-1)^s L_{2s(q+l-1)}(x) - L_s^2(x)\left(3L_{2s}(x) - 2(-1)^s\right)L_{2s(q+l)}(x)\right)\binom{q+3}{4}_{F_s(x)}.$$

Proof. First write the sum (39) as

$$\begin{aligned}\frac{1}{F_{2s}^{2}(x)}\sum_{n=0}^{q}\lambda^{n+l}(x)F_{2sn}^{2}(x) \qquad (46)\\ = -\lambda^{q+l+1}(x)\binom{q+1}{4}_{F_s(x)}+\lambda^{q+l}(x)\binom{q+3}{4}_{F_s(x)}\\ +\left(\lambda^{q+l-1}(x)+(-1)^{s+1}\frac{L_{3s}(x)}{L_s(x)}\lambda^{q+l}(x)\right)\binom{q+2}{4}_{F_s(x)},\end{aligned}$$

where $l\in\mathbb{Z}$. Substitute the weights (37) in (46), take the difference of the resulting expressions, multiply both sides of this difference by $\left(x^2+4\right)^{-\frac{1}{2}}$, and use the Binet's formula $F_r(x)=\frac{1}{\sqrt{x^2+4}}\left(\alpha^r(x)-\beta^r(x)\right)$, to obtain that

$$\begin{aligned}\frac{1}{F_{2s}^{2}(x)}\sum_{n=0}^{q}(-1)^{s(n+q+1)+1}F_{2s(n+l)}(x)\,F_{2sn}^{2}(x)\\ = F_{2s(q+l+1)}(x)\binom{q+1}{4}_{F_s(x)}+(-1)^{s+1}F_{2s(q+l)}(x)\binom{q+3}{4}_{F_s(x)}\\ +\left(\frac{L_{3s}(x)}{L_s(x)}F_{2s(q+l)}(x)-F_{2s(q+l+1)}(x)\right)\binom{q+2}{4}_{F_s(x)}.\end{aligned}$$

Finally, use the identity

$$\frac{L_{3s}(x)}{L_s(x)}F_{2s(q+l)}(x)-F_{2s(q+l-1)}(x)=F_s(x)\,L_{s(2q+2l+1)}(x),$$

to obtain (42). Similarly, if we substitute the weights (37) in (46), then take the sum of the resulting expressions, then use the Binet's formula $L_r(x)=\alpha^r(x)+\beta^r(x)$, and then use the identity

$$\frac{L_{3s}(x)}{L_s(x)}L_{2s(q+l)}(x)-L_{2s(q+l-1)}(x)=\left(x^2+4\right)F_s(x)\,F_{s(2q+2l+1)}(x),$$

we obtain (43).

In a similar fashion, if we begin now with (40), written as

$$\begin{aligned}\sum_{n=0}^{q}\lambda^{n+l}(x)L_{2sn}^{2}(x) \qquad (47)\\ = -L_{2s}^{2}(x)\,\lambda^{q+l+1}(x)\binom{q+1}{4}_{F_s(x)}+4\lambda^{q+l}(x)\binom{q+4}{4}_{F_s(x)}\\ +\left((-1)^s L_s^3(x)\,L_{3s}(x)\,\lambda^{q+l+1}(x)-L_{2s}^{2}(x)\,\lambda^{q+l+2}(x)\right)\binom{q+2}{4}_{F_s(x)}\\ +\left(4\lambda^{q+l-1}(x)-L_s^2(x)\left(3L_{2s}(x)-2(-1)^s\right)\lambda^{q+l}(x)\right)\binom{q+3}{4}_{F_s(x)},\end{aligned}$$

we see that (44) and (45) are obtained by using the weights (37) in (47), together with Binet's formulas. □

Observe that the case $l = 0$ of (42) and (45) gives us sums of cubes of Fibonacci and Lucas polynomials. More precisely, if s is even we have the following (unweighted) sums of cubes

$$\frac{1}{F_{2s}^{2}(x)}\sum_{n=0}^{q}F_{2sn}^{3}(x) \tag{48}$$
$$= -F_{2s(q+1)}(x)\binom{q+1}{4}_{F_s(x)} - F_s(x)\,L_{s(2q+1)}(x)\binom{q+2}{4}_{F_s(x)} + F_{2sq}(x)\binom{q+3}{4}_{F_s(x)}.$$

$$\begin{aligned}\sum_{n=0}^{q}L_{2sn}^{3}(x) &= -L_{2s}^{2}(x)\,L_{2s(q+1)}(x)\binom{q+1}{4}_{F_s(x)} + 4L_{2sq}(x)\binom{q+4}{4}_{F_s(x)} \\ &\quad + \left(L_s^3(x)\,L_{3s}(x)\,L_{2s(q+1)}(x) - L_{2s}^{2}(x)\,L_{2s(q+2)}(x)\right)\binom{q+2}{4}_{F_s(x)} \\ &\quad + \left(4L_{2s(q-1)}(x) - L_s(x)\left(3L_{3s}(x)+L_s(x)\right)L_{2sq}(x)\right)\binom{q+3}{4}_{F_s(x)}.\end{aligned} \tag{49}$$

If s is odd, we have the following alternating sums of cubes

$$\frac{1}{F_{2s}^{2}(x)}\sum_{n=0}^{q}(-1)^{n+q}F_{2sn}^{3}(x) \tag{50}$$
$$= F_{2s(q+1)}(x)\binom{q+1}{4}_{F_s(x)} + F_s(x)\,L_{s(2q+1)}(x)\binom{q+2}{4}_{F_s(x)} + F_{2sq}(x)\binom{q+3}{4}_{F_s(x)}.$$

$$\sum_{n=0}^{q}(-1)^{n+q}L_{2sn}^{3}(x) \tag{51}$$
$$\begin{aligned}&= L_{2s}^{2}(x)\,L_{2s(q+1)}(x)\binom{q+1}{4}_{F_s(x)} + 4L_{2sq}(x)\binom{q+4}{4}_{F_s(x)} \\ &\quad + \left(L_s^3(x)\,L_{3s}(x)\,L_{2s(q+1)}(x) - L_{2s}^{2}(x)\,L_{2s(q+2)}(x)\right)\binom{q+2}{4}_{F_s(x)} \\ &\quad - \left(4L_{2s(q-1)}(x) + L_s(x)\left(3L_{3s}(x)-L_s(x)\right)L_{2sq}(x)\right)\binom{q+3}{4}_{F_s(x)}.\end{aligned}$$

For the remaining weights (38) of the Lucas case, we have in particular the following numerical identity (case $s = x = 1$)

$$\sum_{n=0}^{q}\left(\tfrac{23\pm\sqrt{385}}{18}\right)^{n-q}L_{2n}^{2} = -\tfrac{23\pm\sqrt{385}}{2}\tbinom{q+1}{4}_F - \tfrac{61\pm3\sqrt{385}}{2}\tbinom{q+2}{4}_F - \tfrac{-1\pm\sqrt{385}}{2}\tbinom{q+3}{4}_F + 4\tbinom{q+4}{4}_F.$$

In fact, we can proceed in a similar fashion of the proof of Proposition 2 (using now (38) and (40)) to obtain that, if

$$\Delta_s(x) = \frac{3L_{4s}(x)+2}{4L_{2s}(x)},$$

we have the following weighted sum of squares of Lucas polynomials, valid for integers $l \geq 0$,

$$\begin{aligned}
&\sum_{n=0}^{q}\left(\frac{2}{L_{2s}(x)}\right)^{n-q}T_{n+l}(\Delta_s(x))\,L_{2sn}^{2}(x) \qquad (52)\\
&= -2L_{2s}(x)\,T_{q+l+1}(\Delta_s(x))\binom{q+1}{4}_{F_s(x)}\\
&+\left((-1)^s\frac{2L_s^3(x)\,L_{3s}(x)}{L_{2s}(x)}T_{q+l+1}(\Delta_s(x)) - 4T_{q+l+2}(\Delta_s(x))\right)\binom{q+2}{4}_{F_s(x)}\\
&+\left(2L_{2s}(x)\,T_{q+l-1}(\Delta_s(x)) - L_s^2(x)\left(3L_{2s}(x)-2(-1)^s\right)T_{q+l}(\Delta_s(x))\right)\binom{q+3}{4}_{F_s(x)}\\
&+4T_{q+l}(\Delta_s(x))\binom{q+4}{4}_{F_s(x)}.
\end{aligned}$$

For example, if $x = 1$ and $s = 2$, the sequence $\left(T_n\left(\frac{143}{28}\right)\right)_{n=0}^{\infty}$ involved in (52) is

$$\left(T_n\left(\tfrac{143}{28}\right)\right)_{n=0}^{\infty} = \left(1, \tfrac{143}{28}, \tfrac{20057}{392}, \tfrac{2840123}{5488}, \tfrac{402206417}{76832}, \tfrac{56958853523}{1075648}, \tfrac{8066283596057}{15059072}, \tfrac{1142314618945643}{210827008}, \ldots\right).$$

For $q = 4$ and $l = 0, 1$, we have the weighted sums

$$\begin{aligned}
&\left(\tfrac{2}{7}\right)^{-4}L_0^2 + \left(\tfrac{2}{7}\right)^{-3}\tfrac{143}{28}L_4^2 + \left(\tfrac{2}{7}\right)^{-2}\tfrac{20057}{392}L_8^2 + \left(\tfrac{2}{7}\right)^{-1}\tfrac{2840123}{5488}L_{12}^2 + \tfrac{402206417}{76832}L_{16}^2\\
&= -\tfrac{56958853523}{76832}\tbinom{5}{1}_{F_2} + \tfrac{400320800329}{76832}\tbinom{6}{2}_{F_2} - \tfrac{68220633199}{76832}\tbinom{7}{3}_{F_2} + \tfrac{402206417}{19208}\tbinom{8}{4}_{F_2},
\end{aligned}$$

and

$$\begin{aligned}
&\left(\tfrac{2}{7}\right)^{-4}\tfrac{143}{28}L_0^2 + \left(\tfrac{2}{7}\right)^{-3}\tfrac{20057}{392}L_4^2\\
&\quad + \left(\tfrac{2}{7}\right)^{-2}\tfrac{2840123}{5488}L_8^2 + \left(\tfrac{2}{7}\right)^{-1}\tfrac{402206417}{76832}L_{12}^2 + \tfrac{56958853523}{1075648}L_{16}^2\\
&= -\tfrac{8066283596057}{1075648}\tbinom{5}{1}_{F_2} + \tfrac{56691820586491}{1075648}\tbinom{6}{2}_{F_2} - \tfrac{9661131494701}{1075648}\tbinom{7}{3}_{F_2}\\
&\quad + \tfrac{56958853523}{268912}\tbinom{8}{4}_{F_2}.
\end{aligned}$$

Finally, let us consider the case (ii) mentioned at the beginning of this section (Fibonacci case of Proposition 1, with $\mu = 0$, $k = 4$ and $t = 1$). In this case, equation (4) is

$$\begin{aligned} 0 &= \sum_{i=0}^{4}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{5}{j}_{F_s(x)} F_{s(i-j)}^4(x)\, z^i \\ &= (-1)^{s+1} F_s^4(x)\, z(z+1)\left(z^2+(-1)^s\left(3L_{2s}(x)+4(-1)^s\right)z+1\right). \end{aligned} \tag{53}$$

Thus we have the weights $\lambda_1(x) = -1$ and

$$\begin{aligned} \lambda_2(x) &= -\frac{3L_{2s}(x)+4(-1)^s}{2(-1)^s} + \sqrt{\left(\frac{3L_{2s}(x)+4(-1)^s}{2}\right)^2 - 1}\,, \\ \lambda_3(x) &= -\frac{3L_{2s}(x)+4(-1)^s}{2(-1)^s} - \sqrt{\left(\frac{3L_{2s}(x)+4(-1)^s}{2}\right)^2 - 1}\,. \end{aligned} \tag{54}$$

The corresponding weighted sum (5) (of fourth powers of Fibonacci polynomials) is in this case

$$\begin{aligned} &\frac{1}{F_s^4(x)}\sum_{n=0}^{q}\lambda^{n-q}(x)F_{sn}^4(x) \qquad (55)\\ &\qquad = -\lambda(x)\binom{q+1}{4}_{F_s(x)} + \binom{q+3}{4}_{F_s(x)} \\ &\qquad\quad - \left(\left(3(-1)^s L_{2s}(x)+5\right)\lambda(x)+\lambda^2(x)\right)\binom{q+2}{4}_{F_s(x)}. \end{aligned}$$

With the weight $\lambda_1(x) = -1$ we obtain from (55) the following alternating sum of fourth powers of Fibonacci polynomials

$$\begin{aligned} &\sum_{n=0}^{q}(-1)^n F_{sn}^4(x) \qquad (56)\\ &= (-1)^q F_s^4(x)\left(\binom{q+1}{4}_{F_s(x)} + \left(3(-1)^s L_{2s}(x)+4\right)\binom{q+2}{4}_{F_s(x)} + \binom{q+3}{4}_{F_s(x)}\right). \end{aligned}$$

(The case $x = 1$ of (56), after some transformations, is contained in [4].) By setting $s = 1$ in the weights (54), we have in particular the numerical identity from (55)

$$\sum_{n=0}^{q}\left(\tfrac{5\pm\sqrt{21}}{2}\right)^{n-q}F_n^4 = -\tfrac{5\pm\sqrt{21}}{2}\tbinom{q+1}{4}_F - \tfrac{3\pm\sqrt{21}}{2}\tbinom{q+2}{4}_F + \tbinom{q+3}{4}_F. \tag{57}$$

However, following the ideas of the proof of Proposition 2, we can see from (54) and (55) that for any integer $l \geq 0$ we have

$$\begin{aligned}\frac{1}{F_s^4(x)}\sum_{n=1}^{q} T_{n+l}(\Theta_s(x)) F_{sn}^4(x) \qquad (58)\\ = -T_{q+l+1}(\Theta_s(x))\binom{q+1}{4}_{F_s(x)} + T_{q+l}(\Theta_s(x))\binom{q+3}{4}_{F_s(x)}\\ -((3(-1)^s L_{2s}(x)+5)T_{q+l+1}(\Theta_s(x)) + T_{q+l+2}(\Theta_s(x)))\binom{q+2}{4}_{F_s(x)},\end{aligned}$$

where

$$\Theta_s(x) = -\frac{3L_{2s}(x)+4(-1)^s}{2(-1)^s}.$$

For example, if $x = 1$ and $s = 2$, we have $\Theta_2(1) = -\frac{25}{2}$, and the sequence $\left(T_n\left(-\frac{25}{2}\right)\right)_{n=1}^{\infty}$ involved in (58) is

$$\begin{aligned}&\left(T_n\left(-\tfrac{25}{2}\right)\right)_{n=1}^{\infty}\\ &= \left(-\tfrac{25}{2}, \tfrac{623}{2}, -7775, \tfrac{388127}{2}, -\tfrac{9687625}{2}, 120901249, -\tfrac{6035374825}{2}, \tfrac{150642568127}{2}, \ldots\right).\end{aligned}$$

(See [5, A090733] for $2T_n\left(\frac{25}{2}\right)$.) With $q = 5$ and $l = 0, 1$ we have the weighted sums

$$\begin{aligned}&-\tfrac{23}{2}F_2^4 + \tfrac{623}{2}F_4^4 - 7775F_6^4 + \tfrac{388127}{2}F_8^4 - \tfrac{9687625}{2}F_{10}^4\\ &\qquad = -120901249\tbinom{6}{2}_{F_2} - \tfrac{251490123}{2}\tbinom{7}{3}_{F_2} - \tfrac{9687625}{2}\tbinom{8}{4}_{F_2},\end{aligned}$$

$$\begin{aligned}&\tfrac{623}{2}F_2^4 - 7775F_4^4 + \tfrac{388127}{2}F_6^4 - \tfrac{9687625}{2}F_8^4 + 120901249F_{10}^4\\ &\qquad = \tfrac{6035374825}{2}\tbinom{6}{2}_{F_2} + \tfrac{6277177323}{2}\tbinom{7}{3}_{F_2} + 120901249\tbinom{8}{4}_{F_2}.\end{aligned}$$

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