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FINE-WILF GRAPHS AND THE GENERALIZED FINE-WILF THEOREM

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Abstract

The Fine-Wilf theorem was generalized to finite sequences with three periods by M. G. Castelli, F. Mignosi, and A. Restivo. They introduced a function f from the set of all ordered triples of nonnegative integers to the set of positive integers which was critical to their analysis, and they introduced the graphs that we shall refer to as Fine-Wilf graphs. The work of Castelli et al. was generalized by R. Tijdeman and L. Zamboni, who introduced a function *fw* from the set of all sequences of nonnegative integers to the set of positive integers that was essential to their analysis. In this paper, we obtain an alternative formulation of f and *fw*, and we use this formulation to establish important properties of f and *fw*, obtaining in particular new upper and lower bounds for each. We also carry out an investigation of Fine-Wilf graphs for arbitrary finite sequences, showing how they are related to f and *fw*.

1. Introduction

A finite sequence $w = (a_1, a_2, \ldots, a_n)$ is said to have period $r \geq 1$, or to be r-periodic, if for every positive integer i for which $i + r \leq n$, $a_i = a_{i+r}$. In 1962, R. C. Lyndon and M. P. Schützenberger [4] established that for any integers $r, s \geq 1$, if w is both r-periodic and s-periodic and $|w| \geq r + s$, then w is $\gcd(r, s)$-periodic. Shortly thereafter (1965), N. J. Fine and H. S. Wilf [2] proved that for any integers $r, s \geq 1$, if $\{a_i\}$ is an infinite sequence of period r and $\{b_i\}$ is an infinite sequence of period s such that $a_i = b_i$ for all i with $1 \leq i \leq r + s - \gcd(r, s)$, then $a_i = b_i$ for all i. This is equivalent to the following result, which is commonly referred to as the Fine-Wilf theorem: for any integers $r, s \geq 1$, if w is a finite sequence that is both r-periodic and s-periodic, and $|w| \geq r + s - \gcd(r, s)$, then w is $\gcd(r, s)$-periodic. It was also shown in [2] that this bound is best possible, in the sense that for any positive integers r and s (with neither a divisor of the other), there exists a sequence w of length $r + s - \gcd(r, s) - 1$ that is both r-periodic and s-periodic, but not $\gcd(r, s)$-periodic. It is known that for such r and s, there is a unique (up to relabelling) two-symbol sequence of length $r+s-\gcd(r,s)-1$ that is both r-periodic and s-periodic, but not $\gcd(r,s)$-periodic. For example, for $r=2$ and $s=3$, the sequence is $(0,1,0)$.

Nearly thirty-five years later (1999), the Fine-Wilf theorem was generalized to finite sequences with three periods by M. G. Castelli, F. Mignosi, and A. Restivo [1]. They introduced a function f from the set of all ordered triples of positive integers to the set $\mathbb{Z}^+$ of positive integers with the property that if w is a finite sequence with periods p_1, p_2, and p_3, and $|w| \geq f(p)$, where $p=(p_1,p_2,p_3)$, then w is $\gcd(p)$-periodic as well. They further established a condition on p under which the bound $f(p)$ is best possible. The sequences p that met this condition were precisely those for which the unique (up to relabelling) finite sequence of greatest length and with the greatest possible number of distinct entries that had periodicity p_1, p_2, and p_3, but not $\gcd(p_1,p_2,p_3)$ had exactly three distinct entries. In support of their work, they introduced the difference graphs that we shall refer to as Fine-Wilf graphs $G_p(n)$, where $p=(p_1,p_2,p_3)$ and $p_1<p_2<p_3$ and n are positive integers and $G_p(n)$ denotes the graph with vertex set $\{1,2,\ldots,n\}$ and edge set

$$\{\{i,j\} \mid |i-j| \in \{p_1,p_2,p_3\}\}.$$

The work of Castelli et al. was followed immediately (2000) by work of J. Justin [3], who extended the definition of the function f to all finite sequences of positive integers, with analogous results.

A broader generalization of the work of Castelli et al. was then given by R. Tijdeman and L. Zamboni [5] (2003). They introduced a function, which we shall denote as fw, from the set of all finite sequences of positive integers to $\mathbb{Z}^+$, and they proved that for a sequence $p=(p_1,p_2,\ldots,p_n)$, a finite sequence w with periods p_i, $i=1,2,\ldots,n$ and length at least $fw(p)$ must be $\gcd(p)$-periodic as well, and that there exists a sequence w of length $fw(p)-1$ that is p_i-periodic for all i, but not $\gcd(p)$-periodic.

In this paper, we establish new properties of the functions f and fw. In particular, we introduce new upper and lower bounds for f. We also begin an investigation of Fine-Wilf graphs for arbitrary $p_1,p_2,\ldots,p_n$, with a view to understanding how the graph depends on the values $p_1,p_2,\ldots,p_n$, and how the properties of fw are reflected in the graphs.

2. Generalization of the Fine-Wilf Theorem

Let $\mathscr{F}$ denote the set of all strictly increasing finite sequences with entries from $\mathbb{Z}^+$. For $p \in \mathscr{F}$, let $\gcd(p)$ denote the greatest common divisor of the entries in p, let $|p|$ denote the length of p, and for i such that $1 \leq i \leq |p|$, let p_i denote the i^{th} entry of p and let $p|_i$ denote the truncated sequence $(p_1,p_2,\ldots,p_i)$. In particular, for $p \in \mathscr{F}$

with $|p| > 1$, we shall let $p^- = p|_{|p|-1}$. Finally, let $\max(p) = p_{|p|}$ and $\min(p) = p_1$.

Definition 2.1. *For $p \in \mathscr{F}$, let $p' = p$ if $|p| = 1$, otherwise let p' denote the element of $\mathscr{F}$ whose entries form the set $\{\min(p), p_2-\min(p), \ldots, \max(p)-\min(p)\}$. Moreover, define $p^{(i)} \in \mathscr{F}$ for $i \geq 0$ as follows: $p^{(0)} = p$, and for $k \geq 0$, $p^{(k+1)} = (p^{(k)})'$. Finally, let $ht(p)$, the height of p, be the least nonnegative integer m for which $|p^{(m)}| = 1$, and $wt(p)$, the weight of p, be given by $wt(p) = \sum_{i=1}^{|p|} p_i$.*

Note that $|p'| = |p| - 1$ if $p_i = 2\min(p)$ for some i; otherwise, $|p'| = |p|$. As well, for any $p \in \mathscr{F}$, $\gcd(p) = \gcd(p')$, and this is the single entry in $p^{(ht(p))}$.

Definition 2.2. *Define $f\colon \mathscr{F} \to \mathbb{Z}^+$ by $f(p) = \sum_{i=0}^{ht(p)} \min(p^{(i)})$ for $p \in \mathscr{F}$. Moreover, the column of sequences whose i^{th} row is $p^{(i)}$, $0 \leq i \leq ht(p)$, shall be called the* tableau *for the calculation of $f(p)$.*

For example, the tableaux for the calculation of $f(p)$ for $p = (4, 7)$ and $p = (4, 7, 9)$ are shown below. In each case, $ht(p) = 4$, and $f(p) = 10$.

$p^{(0)}$	4,7	4,7,9
$p^{(1)}$	3,4	3,4,5
$p^{(2)}$	1,3	1,2,3
$p^{(3)}$	1,2	1,2
$p^{(4)}$	1	1

Note that for any $p \in \mathscr{F}$ with $|p| > 1$, $f(p) = \min(p) + f(p')$.

Lemma 2.3. *For any $p \in \mathscr{F}$, $f(p) \geq \max(p)$; and if $|p| > 1$, then $f(p) \geq 2\min(p)$.*

Proof. We use induction on $ht(p)$. If $p \in \mathscr{F}$ has $ht(p) = 0$, then $|p| = 1$ and $\max(p) = \min(p) = f(p)$. Suppose now that $p \in \mathscr{F}$ has $ht(p) > 0$, and the result holds for lower sequences. Note that $|p| > 1$ since $ht(p) > 0$. Thus $\min(p) < \max(p)$ and $f(p) = \min(p) + f(p')$. Since $ht(p') = ht(p) - 1$, the inductive hypothesis applies to p' and we obtain $f(p') \geq \max(p')$. If $\max(p) - \min(p) > \min(p)$, then $\max(p') = \max(p) - \min(p)$ and then $f(p) = \min(p) + f(p')) \geq \min(p) + \max(p) - \min(p) = \max(p) > 2\min(p)$. Otherwise, $\max(p) - \min(p) \leq \min(p)$, so $\max(p') = \min(p)$ and thus $f(p) = \min(p) + f(p') \geq \min(p) + \max(p') = 2\min(p) \geq \max(p)$. The result follows now by induction. □

Definition 2.4. *Define $fw\colon \mathscr{F} \to \mathbb{Z}^+$ as follows. For $p \in \mathscr{F}$,*

$$fw(p) = \begin{cases} fw(p^-) & \text{if } |p| > 1,\ \gcd(p^-) = \gcd(p), \text{ and } \max(p) \geq f(p^-) \\ f(p) & \text{otherwise.} \end{cases}$$

We shall show later (see Proposition 3.12) that if $p \in \mathscr{F}$ satisfies $|p| > 1$ and $\gcd(p^-) = \gcd(p)$, then $fw(p) \leq fw(p^-)$.

The following lemma can be easily proven by induction on $\max(p)$.

Lemma 2.5. *Let $c \in \mathbb{Z}^+$, and $p \in \mathscr{F}$, say $p = (p_1, p_2, \ldots, p_n)$. Let $cp = (cp_1, cp_2, \ldots,$
$cp_n)$. Then $cp \in \mathscr{F}$ and $f(cp) = c\,f(p)$ and $fw(cp) = c\,fw(p)$.*

Proposition 2.6. *Let $p \in \mathscr{F}$. If $min(p) = \gcd(p)$, then $fw(p) = \min(p)$ and $f(p) = \max(p)$, while if $\min(p) \neq \gcd(p)$, then $fw(p) \geq 2\min(p)$.*

Proof. First, we prove by induction on $ht(p)$ that if $\min(p) = \gcd(p)$, then $f(p) = \max(p)$. The base case of $ht(p) = 0$ is immediate, so suppose that $ht(p) > 1$ with $\min(p) = \gcd(p)$, and the assertion holds for all elements of $\mathscr{F}$ of lower height. Then $f(p) = \min(p) + f(p')$ and $ht(p') < ht(p)$. Since every element of p is a multiple of $\gcd(p) = \min(p)$, it follows that $\min(p') = \min(p) = \gcd(p) = \gcd(p')$, and so we may apply the induction hypothesis to p' to obtain that $f(p') = \max(p') = \max(p) - \min(p)$. Thus $f(p) = \max(p)$, as required.

Next, we prove by induction on $|p|$ that if $\min(p) = \gcd(p)$, then $fw(p) = \gcd(p)$, while if $\min(p) > \gcd(p)$, then $fw(p) \geq 2\min(p)$. The base case $|p| = 1$ is immediate, so suppose that $p \in \mathscr{F}$ has $|p| > 1$, and the assertion holds for all shorter sequences. Consider first the case when $\min(p) = \gcd(p)$. Then $\gcd(p) = \gcd(p^-)$ and so $\min(p^-) = \min(p) = \gcd(p^-)$. Since $|p^-| < |p|$, we may apply the induction hypothesis to p^- to obtain that $fw(p^-) = \gcd(p^-)$. Since $\gcd(p) = \gcd(p^-)$, and by the preceding part, $f(p^-) = \max(p^-) < \max(p)$, the definition of fw yields $fw(p) = fw(p^-) = \gcd(p)$. Now suppose that $\min(p) > \gcd(p)$. If $fw(p) = f(p)$, the result follows from Lemma 2.3, so we may suppose that $fw(p) \neq f(p)$. Then by definition of fw, $|p| > 1$ and $\gcd(p^-) = \gcd(p)$, and $fw(p) = fw(p^-)$. We thereby obtain that $\min(p^-) = \min(p) > \gcd(p) = \gcd(p^-)$, and so upon application of the induction hypothesis to p^-, we obtain $fw(p) = fw(p^-) \geq 2\min(p^-) = 2\min(p)$. The result follows now by induction. □

The next result gives an important lower bound for $f(p)$, and this result can be viewed as one generalization of the Fine-Wilf theorem. We will later obtain an upper bound (see Proposition 2.14, also Proposition 4.11) for $f(p)$ and the combination of these upper and lower bounds, when applied in the case of $|p| = 2$, yields the classical Fine-Wilf theorem.

Theorem 2.7. *Let $p \in \mathscr{F}$. If $|p| > 1$, then $f(p) \geq \dfrac{-\gcd(p) + wt(p)}{|p| - 1}$, and if equality holds and i is such that $p_i = 2\min(p)$, then $i = |p|$ and $f(p) = 2\min(p) = \max(p)$.*

Proof. The proof is by induction on $ht(p)$. If $p \in \mathscr{F}$ has $ht(p) = 0$, then $|p| = 1$, and the assertion is vacuously true. Suppose now that $p \in \mathscr{F}$ has $ht(p) > 0$, so $|p| > 1$, and that the assertion holds for all elements of $\mathscr{F}$ of lower height. In particular, the assertion holds for p', since $ht(p') = ht(p) - 1$.

Case 1: $|p'| = |p| > 1$. Then

$$wt(p') = \min(p) + \sum_{i=2}^{|p|}(p_i - \min(p)) = wt(p) - (|p|-1)\min(p),$$

and by the induction hypothesis and the facts $|p'| = |p|$ and $\gcd(p') = \gcd(p)$, we have $f(p) = \min(p) + f(p') \geq \min(p) + \frac{-\gcd(p')+wt(p')}{|p'|-1} = \frac{-\gcd(p)+wt(p)}{|p|-1}$. In this case, there is no value of i such that $p_i = 2\min(p)$.

Case 2: $|p'| = |p| - 1$. Then there exists i with $1 \leq i \leq |p|$ such that $p_i = 2\min(p)$, and $wt(p') = \sum_{j=2}^{|p|}(p_j - \min(p)) = wt(p) - |p|\min(p)$.

If $|p'| = 1$, then $p = (\min(p), 2\min(p))$ and the result holds. Suppose that $|p'| \geq 2$. Then $f(p') \geq \frac{-\gcd(p')+wt(p')}{|p'|-1} = \frac{-\gcd(p)+wt(p)-|p|\min(p)}{|p|-2}$ by the induction hypothesis applied to p', and so $f(p) \geq \min(p) + \frac{-\gcd(p)+wt(p)-|p|\min(p)}{|p|-2} = \frac{-2\min(p)-\gcd(p)+wt(p)}{|p|-2}$.

Consider first the possibility that $\frac{-2\min(p)-\gcd(p)+wt(p)}{|p|-2} \geq \frac{-\gcd(p)+wt(p)}{|p|-1}$, in which case $f(p) \geq \frac{-\gcd(p)+wt(p)}{|p|-1}$, as required. If as well, $f(p) = \frac{-\gcd(p)+wt(p)}{|p|-1}$, then $\frac{-2\min(p)-\gcd(p)+wt(p)}{|p|-2} = \frac{-\gcd(p)+wt(p)}{|p|-1}$, and thus $-\gcd(p)+wt(p) = (|p|-1)2\min(p)$, yielding $f(p) = 2\min(p)$. Then by Lemma 2.3, $2\min(p) = f(p) \geq \max(p) \geq p_i = 2\min(p)$ and thus $p_i = \max(p)$.

Now suppose that $\frac{-2\min(p)-\gcd(p)+wt(p)}{|p|-2} < \frac{-\gcd(p)+wt(p)}{|p|-1}$. Then $-\gcd(p) + wt(p)$ is less than $(|p|-1)2\min(p)$ and so $\frac{-\gcd(p)+wt(p)}{|p|-1} < 2\min(p) = p_i \leq \max(p) \leq f(p)$. This completes the proof of the inductive step, and so the result follows. $\square$

Proposition 2.8. *For any $p \in \mathscr{F}$, the following hold.*

1. *If $|p| > 1$, $\gcd(p) = \gcd(p^-)$, and $\max(p) \geq f(p^-)$, then $f(p) = \max(p)$.*
2. *$f(p) \geq fw(p)$.*

Proof. We prove (1) by induction on $ht(p)$, with trivial base case of $ht(p) = 0$. Suppose now that $p \in \mathscr{F}$ has $ht(p) > 0$ with $\gcd(p) = \gcd(p^-)$ and $\max(p) \geq f(p^-)$, and that the result holds for all sequences of lower height. If $|p| = 2$, then $\min(p) = \gcd(p)$, and so by Proposition 2.6, $f(p) = \max(p)$. Suppose that $|p| > 2$. Then by Lemma 2.3, we have $f(p^-) \geq 2\min(p)$, so $\max(p) \geq 2\min(p)$ and thus $\max(p') = \max(p) - \min(p)$. Suppose first that $\max(p) > 2\min(p)$. Then we have $(p')^- = (p^-)'$, so $\gcd((p')^-) = \gcd((p^-)') = \gcd(p^-) = \gcd(p) = \gcd(p')$ and $\max(p') = \max(p) - \min(p) \geq f(p^-) - \min(p) = f((p^-)') = f((p')^-)$. Since $ht(p') < ht(p)$, we may apply the induction hypothesis to p' to obtain that $f(p') = \max(p') = \max(p) - \min(p)$, and thus $f(p) = \max(p)$ in this case. Otherwise, $\max(p) = 2\min(p)$ and then $p' = (p^-)'$ and from $\max(p) \geq f(p^-) \geq 2\min(p)$ it follows that $f(p^-) = 2\min(p)$. We have $f(p') = f((p^-)') = f(p^-) - \min(p^-) =$

$2\min(p) - \min(p) = \min(p)$ and so $f(p) = \min(p) + f(p') = 2\min(p) = \max(p)$, as required. Statement (1) of the theorem follows now by induction.

Next, we prove (2) by induction on $|p|$, with obvious base case $|p| = 1$. Suppose now that $p \in \mathscr{F}$ has $|p| > 1$, and that the result holds for all shorter sequences. Suppose that $f(p) \neq fw(p)$, so $\gcd(p) = \gcd(p^-)$ and $\max(p) \geq f(p^-)$. Then by (1) and the induction hypothesis, we have $f(p) = \max(p) \geq f(p^-) \geq fw(p^-) = fw(p)$. The result follows now by induction. $\square$

Definition 2.9. *$p \in \mathscr{F}$ is said to be* trim *if either $|p| = 1$ or else $|p| > 1$ and either $\gcd(p) \neq \gcd(p^-)$ or $\gcd(p) = \gcd(p^-)$ but $\max(p) < f(p^-)$. For any $p \in \mathscr{F}$, there exists i with $1 \leq i \leq |p|$ such that $p|_i$ is trim. The trimmed form of p, denoted by p^t, is $p|_i$, where i is maximal with respect to the property $p|_i$ is trim.*

Note that p is not trim if and only if $|p| > 1$, $\gcd(p) = \gcd(p^-)$, and $\max(p) \geq f(p^-)$. We remark that even if p is trim, there may exist i with $1 < i < |p|$ such that $p|_i$ is not trim. We shall examine such situations in the final section.

By Definition 2.4, we have

$$fw(p) = \begin{cases} fw(p^-) & \text{if } p \text{ is not trim} \\ f(p) & \text{if } p \text{ is trim.} \end{cases}$$

Corollary 2.10. *Let $p \in \mathscr{F}$. If p is not trim, then $f(p^{(i)}) = \max(p^{(i)})$ for every i, $0 \leq i \leq ht(p)$.*

Proof. Since p is not trim, we have $|p| > 1$, $\gcd(p) = \gcd(p^-)$, and $\max(p) \geq f(p^-)$, so we may apply Proposition 2.8 to obtain $f(p) = \max(p)$. We now prove by induction on $ht(p)$ that if $f(p) = \max(p)$, then $f(p^{(i)}) = \max(p^{(i)})$ for every i, $0 \leq i \leq ht(p)$. The base case of $ht(p) = 0$ is true by definition, so suppose that $p \in \mathscr{F}$ has $ht(p) > 0$ and $f(p) = \max(p)$, and that the result holds for all sequences of lower height. By Lemma 2.3, $f(p) \geq 2\min(p)$ and so we have $\max(p) = f(p) \geq 2\min(p)$. Hence $\max(p) - \min(p) \geq \min(p)$ and so $\max(p') = \max(p) - \min(p)$. Now, $f(p') = f(p) - \min(p) = \max(p) - \min(p) = \max(p')$, and since $ht(p') < ht(p)$, we may apply the inductive hypothesis to p' to obtain that $f((p')^{(i)}) = \max((p')^{(i)})$ for every i with $0 \leq i \leq ht(p')$, and so $f(p^{(i)}) = \max(p^{(i)})$ for every i with $0 \leq i \leq ht(p)$. The result follows now by induction. $\square$

If $p \in \mathscr{F}$ is not trim, then $|p| > 1$, $p^t = (p^-)^t$, and $fw(p) = fw(p^-)$. We shall say that p^- is obtained by trimming p. Evidently, for any $p \in \mathscr{F}$, we may iteratively apply the trimming operation to obtain p^t, and it follows that $fw(p) = fw(p^t)$.

Proposition 2.11. *Let $p \in \mathscr{F}$. Then $fw(p) = fw(p^t) = f(p^t)$, and $\gcd(p) = \gcd(p^t)$. Furthermore, if $|p^t| > 1$, then $\min(p) > \gcd(p^t)$.*

Proof. If p is trim, then by definition of fw, $fw(p) = f(p)$. Otherwise, $fw(p) = fw(p^-)$, and so the first assertion follows by induction on $|p|$. Next, note that $\min(p) = \min(p^t) \geq \gcd(p^-)$. Suppose that $|p^t| > 1$. If $\min(p^t) = \gcd(p^t)$, then by Proposition 2.6, $fw(p^t) = \min(p^t) < \max(p^t) = f(p^t)$, which is not the case, and so $\min(p) > \gcd(p^t)$. That $\gcd(p) = \gcd(p^t)$ is immediate from the definition of p^t. ◻

Corollary 2.12. *Let $p \in \mathscr{F}$. If p is not trim, then $p_{|p^t|+1} \geq fw(p)$.*

Proof. Suppose that p is not trim. Then by Definition 2.9, $p_{|p^t|+1} \geq fw(p^t)$, and by Proposition 2.11, $fw(p^t) = fw(p)$. ◻

Proposition 2.13. *Let $p \in \mathscr{F}$. If p is trim and $|p| > 1$, then $f(p) > \max(p)$. In addition, if p' is not trim, then $f(p) = 2\min(p)$.*

Proof. The proof is by induction on $ht(p)$, with vacuous base case. Suppose that $p \in \mathscr{F}$ is trim with $ht(p) > 1$, and the implication holds for every sequence of lower height. Since $ht(p) > 1$, we have $|p| > 1$, and so $f(p) = \min(p) + f(p')$. Consider first the situation when p' is trim. If $|p'| = 1$, then $\max(p) - \min(p) = \min(p)$ and $p = (\min(p), 2\min(p))$, which is not possible since p is trim. Thus $|p'| > 1$, and then by the inductive hypothesis, we have $f(p') > \max(p') \geq \max(p) - \min(p)$, and so $f(p) > \max(p)$.

Now suppose that p' is not trim, in which case $|p'| > 1$, $\gcd(p') = \gcd((p')^-)$, and $\max(p') \geq f((p')^-)$, and by Corollary 2.10, $f(p') = \max(p')$. Thus $f(p) = \min(p) + \max(p')$. If $\max(p) < 2\min(p)$, then $\max(p') = \min(p)$, and so $f(p) = 2\min(p) > \max(p)$, as required. The proof of the inductive step will be complete once we prove that $\max(p) \geq 2\min(p)$ is not possible. Indeed, suppose that $\max(p) \geq 2\min(p)$. Then $\max(p') = \max(p) - \min(p)$. If $\max(p) - \min(p) = \min(p)$, then $p' = (p^-)'$, while if $\max(p) - \min(p) > \min(p)$, then $(p')^- = (p^-)'$. In the first case, we have $\gcd(p^-) = \gcd((p^-)') = \gcd(p') = \gcd(p)$, while in the second case, we have $\gcd(p^-) = \gcd((p^-)') = \gcd((p')^-) = \gcd(p') = \gcd(p)$. In either case therefore, we have $\gcd(p^-) = \gcd(p)$. Since p is trim, this implies that $|p| > 2$ and $\max(p) < f(p^-) = \min(p^-) + f((p^-)') = \min(p) + f((p^-)')$. Now, $(p^-)' = p'$ or $(p')^-$, so either $f((p^-)') = f(p') = \max(p')$, or else $f((p^-)') = f((p')^-) \leq \max(p')$. Thus $\max(p) < \min(p) + \max(p') = \max(p)$, which is impossible. ◻

The following result gives an upper bound for f that is reminiscent of the Fine-Wilf theorem. Later (see Proposition 4.11), we shall establish a generalization of this which for p trim with $|p| \geq 3$ offers a slightly improved upper bound for fw.

Proposition 2.14. *For $p \in \mathscr{F}$, $f(p) \leq \min(p) + \max(p) - \gcd(p)$.*

Proof. The proof is by induction on $ht(p)$, with immediate base case. Suppose now that $p \in \mathscr{F}$ has $ht(p) > 0$, and the result holds for all lower elements of $\mathscr{F}$.

Since $ht(p) > 0$, the induction hypothesis applies to p', and since $ht(p) > 0$ implies that $|p| > 1$, we have $f(p) = \min(p) + f(p') \leq \min(p) + \min(p') + \max(p') - \gcd(p') = \min(p) + \min(p') + \max(p') - \gcd(p)$. It therefore suffices to prove that $\min(p') + \max(p') \leq \max(p)$. We consider three cases. The first occurs when $\min(p) \leq p_2 - \min(p)$, in which case $\min(p') = \min(p)$ and $\max(p') = \max(p) - \min(p)$, so $\min(p') + \max(p') = \max(p)$. Next, suppose that $p_2 - \min(p) < \min(p) \leq \max(p) - \min(p)$. Then $\min(p') = p_2 - \min(p)$ and $\max(p') = \max(p) - \min(p)$, so $\min(p') + \max(p') = \max(p) + p_2 - 2\min(p) < \max(p)$. Finally, suppose that $\max(p) - \min(p) < \min(p)$, so that $\min(p') = p_2 - \min(p)$ and $\max(p') = \min(p)$ and thus $\min(p') + \max(p') = p_2 \leq \max(p)$. This completes the proof of the inductive step. $\square$

Theorem 2.15 (Fine-Wilf)**.** *Let $p \in \mathscr{F}$ with $|p| = 2$. Then $f(p) = \min(p) + \max(p) - \gcd(p)$. Furthermore, $fw(p) = \min(p) + \max(p) - \gcd(p)$ if p is trim, otherwise $fw(p) = \min(p) < f(p)$.*

Proof. By Theorem 2.7, $f(p) \geq \min(p) + \max(p) - \gcd(p)$, while by Proposition 2.14, $f(p) \leq \min(p) + \max(p) - \gcd(p)$. If p is trim, then $fw(p) = f(p)$, while if p is not trim, then $fw(p) = fw(p^-) = \min(p)$, while $f(p) = \max(p)$. $\square$

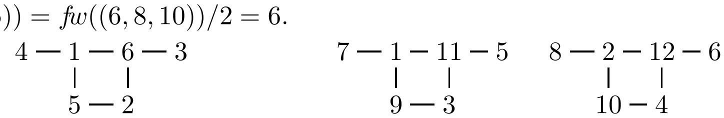
3. The Fine-Wilf Graphs $G_p(k)$

Definition 3.1. *Let $p \in \mathscr{F}$. For any $k \in \mathbb{Z}^+$, $G_p(k)$ denotes the simple graph with vertex set $\{1, \ldots, k\}$ and edge set $\{\{i,j\} \mid |i-j| = p_t$ for some $t = 1, 2, \ldots, |p|\}$. The values $k = fw(p)$ and $k = fw(p) - 1$ feature prominently in the development of the theory, and we shall let $G_p = G_p(fw(p))$ and $G_p^- = G_p(fw(p) - 1)$.*

Proposition 3.2. *Let $p \in \mathscr{F}$. Then for any $k \leq fw(p)$, $G_p(k) = G_{p^t}(k)$.*

Proof. We need only consider p not trim. In such a case, by Corollary 2.12, $fw(p^t) \leq p_{|p^t|+1}$. Thus for any i, j, k such that $1 \leq i < j \leq k \leq fw(p) = fw(p^t)$, $j - i \leq fw(p^t) - 1 < p_{|p^t|+1}$. It follows that if $\{i,j\}$ is an edge of $G_p(k)$, then it is an edge $G_{p^t}(k)$ (the converse is obvious) and so $G_p(k) = G_{p^t}(k)$. $\square$

Our first goal in this section is to establish that for $p \in \mathscr{F}$, the graph G_p has exactly $d = \gcd(p)$ connected components, each isomorphic to $G_{p/d}$. For example, for $p = (6, 8, 10)$, $fw(p) = 12$, $\gcd(p) = 2$, and G_p has two connected components, each isomorphic to the connected graph $G_{(3,4,5)}$, where by Lemma 2.5, $fw((3,4,5)) = fw((6,8,10))/2 = 6$.



$G_{(3,4,5)}$ $G_{(6,8,10)}$

The second major objective of the section is to establish that $\kappa(G_p^-) > d$.

Proposition 3.3. *Let $p \in \mathscr{F}$ with $|p| > 1$, and let $k \geq 1$. Then for any i and j with $1 \leq i < j \leq k$, i and j belong to the same connected component of $G_{p'}(k)$ if and only if $i + \min(p)$ and $j + \min(p)$ belong to the same connected component of $G_p(k + \min(p))$.*

Proof. Let i and j be such that $1 \leq i < j \leq k$. For the first implication, it suffices to show that if $\{i, j\}$ is an edge in $G_{p'}(k)$, then $i+\min(p)$ and $j+\min(p)$ are connected in $G_p(k + \min(p))$. Suppose that $\{i, j\}$ is an edge in $G_{p'}(k)$. If $j - i = \min(p)$, then $\{j, i\}$, $\{i, i + \min(p)\}$ and $\{j, j + \min(p)\}$ are edges in $G_p(k + \min(p))$, and so $i + \min(p)$ and $j + \min(p)$ are connected in $G_p(k + \min(p))$. Otherwise, $j - i = p_r - \min(p)$ for some r, and then $\{j + \min(p), i\}$ and $\{i, i + \min(p)\}$ are edges in $G_p(k + \min(p))$, so $i + \min(p)$ and $j + \min(p)$ are connected in $G_p(\min(p) + k)$.

We prove the converse by induction on path length. Our hypothesis is that if $i + \min(p)$ and $j + \min(p)$ are connected by a path of length n in $G_p(k + \min(p))$, then i and j are connected in $G_{p'}(k)$. We consider first the case $n = 1$; that is, $\{i+\min(p), j+\min(p)\}$ is an edge in $G_p(k+\min(p))$. Then $(j+\min(p))-(i+\min(p)) = p_r$ for some r, and so $j-i = p_r \geq \min(p)$. But then $(j-\min(p))-i = p_r-\min(p) \geq 0$. Thus either $j - \min(p) = i$ or $\{j - \min(p), i\}$ is an edge in $G_{p'}(k)$. In either case, since $\{j, j-\min(p)\}$ is an edge in $G_{p'}(k)$, i and j are connected in $G_{p'}(k)$. Suppose now that $n \geq 1$ is an integer and the hypothesis holds for all smaller integers. We consider i and j with $1 \leq i < j \leq k$ such that $i + \min(p)$ and $j + \min(p)$ are joined by a path of length $n+1$ in $G_p(k+\min(p))$, say $i+\min(p), i_1, \ldots, i_{n+1} = j+\min(p)$.
Case 1: $i_1 > \min(p)$. Since $i_1 \leq k+\min(p)$, we have $i' = i_1 - \min(p) \leq k$. But then $1 \leq i', j \leq k$ and $i' + \min(p) = i_1$ and $j + \min(p)$ are connected in $G_p(k + \min(p))$ by a path of length n, so by hypothesis, i' and j are connected in $G_{p'}(k)$. We prove now that i and i' are connected in $G_{p'}(k)$. We have $|i + \min(p) - i_1| = p_r$ for some r. If $i \geq i_1 - \min(p)$, then $i = i_1 + p_r - \min(p) \geq i_1 > \min(p)$, which means that $(i - \min(p)) - (i_1 - \min(p)) = p_r - \min(p)$, and $i_1 - \min(p) \leq k$, so $\{i_1 - \min(p), i - \min(p)\}$ and $\{i - \min(p), i\}$ are edges in $G_{p'}(k)$, whence i and $i' = i_1 - \min(p)$ are connected in $G_{p'}(k)$. Otherwise, $i < i_1 - \min(p)$, and so $i_1 - \min(p) - i = p_r \geq \min(p)$. Then $i_1 - 2\min(p) - i = p_r - \min(p)$, and so $\{i_1 - \min(p), i_1 - 2\min(p)\}$ is an edge in $G_{p'}(k)$ and either $i_1 - 2\min(p) = i$ or else $\{i_1 - 2\min(p), i\}$ is an edge in $G_{p'}(k)$. Thus i and i' are connected in $G_{p'}(k)$, and so i and j are connected in $G_{p'}(k)$.
Case 2: $i_1 \leq \min(p)$. Then $i + \min(p) - i_1 = p_r$ for some r, and so $i - i_1 = p_r - \min(p) \geq 0$. Thus $k \geq i \geq i_1$. As well, we have $|i_2 - i_1| = p_t \geq \min(p)$ for some t, which means that $i_2 - i_1 = p_t$ since $i_1 - i_2 < \min(p)$. Thus $(i_2 - \min(p)) - i_1 = p_t - \min(p)$, which means that $\{i_2 - \min(p), i_1\}$ is an edge in $G_{p'}(k)$. By hypothesis, since $(i_2 - \min(p)) + \min(p) = i_2, \ldots, i_{n+1} = j + \min(p)$ is a path of length $n-1$ in $G_p(k + \min(p))$, $i_2 - \min(p)$ and j are connected in $G_{p'}(k)$. Since $\{i_2 - \min(p), i_1\}$

and $\{i, i_1\}$ are edges in $G_{p'}(k)$ (possibly $i = i_1$ or $i_2 - \min(p) = i_1$), i and j are connected in $G_{p'}(k)$.

This completes the proof of the inductive step, and so the result follows. □

Definition 3.4. *Let $p \in \mathscr{F}$ with $|p| > 1$. For any $k \geq 1$, define the function $\alpha_p : G_{p'}(k) \to G_p(\min(p) + k)$ by $\alpha_p(i) = \min(p) + i$.*

We remark that in general, α_p is not a graph homomorphism. However, the preceding proposition establishes that if C is a connected component of $G_{p'}(k)$, then $\alpha_p(C)$ is contained in a component of $G_p(\min(p) + k)$, and more generally, α_p induces an injective map from the set of components of $G_{p'}(k)$ into the set of components of $G_p(k + \min(p))$.

Our next objective is to show that for any $p \in \mathscr{F}$ with $\gcd(p) = 1$, G_p is connected. For this, the following lemma will be useful.

Lemma 3.5. *Let $p \in \mathscr{F}$ and $k \in \mathbb{Z}^+$ be such that $\min(p) \leq k$. If the interval $\{1, 2, \ldots, \min(p)\}$ is contained within a component of $G_p(k)$, then $G_p(k)$ is connected.*

Proof. Every $i > \min(p)$ is connected to $j \leq \min(p)$ by a path of length q, where $i = q \min(p) + j$ and $0 < j \leq \min(p)$. □

Proposition 3.6. *Let $p \in \mathscr{F}$ and $k \geq fw(p)$. If $\gcd(p) = 1$, then $G_p(k)$ is connected.*

Proof. The proof is by induction on $\max(p)$. The base case occurs when $\max(p) = 1$, in which case $p = (1)$ and $G_p(k)$ is the chain graph on k vertices. Suppose now that $\max(p) > 1$ and $\gcd(p) = 1$, and the result holds for all sequences with smaller maximum entry. Note that $\gcd(p) = 1$ and $\max(p) > 1$ imply that $|p| > 1$.

We consider first the case when p is not trim. Then $1 = \gcd(p) = \gcd(p^-)$ and $fw(p^-) = fw(p) \leq f(p) = \max(p)$, so $G_p = G_p(fw(p^-)) = G_{p^-}$. As well, $\max(p^-) < \max(p)$, so by hypothesis, $G_p = G_{p^-}$ is connected. Suppose now that $k \geq fw(p)$. Then G_p is a connected subgraph of $G_p(k)$, and since $fw(p) \geq \min(p)$, Lemma 3.5 implies that $G_p(k)$ is connected.

Now suppose that p is trim. Since $|p| > 1$, we have $fw(p) = f(p) = \min(p) + f(p')$, so $G_p = G_p(\min(p) + f(p'))$. Now, since $1 = \gcd(p) = \gcd(p')$ and $\max(p') < \max(p)$, we may apply the induction hypothesis to p' to obtain that $G_{p'}(f(p'))$ is connected since $f(p') \geq fw(p')$. Thus $\{i + \min(p) \mid 1 \leq i \leq f(p')\}$, the image of $G_{p'}(f(p'))$ under α_p, is contained within a connected component of $G_p(f(p))$. Since $f(p') \geq \min(p)$, it follows that $\{1, 2, \ldots, \min(p)\}$ is contained within a connected component of $G_p(f(p))$, and thus by Lemma 3.5, G_p is connected. But then for any $k \geq fw(p)$, G_p is a connected subgraph of $G_p(k)$ and thus by Lemma 3.5, $G_p(k)$ is connected. □

Lemma 3.7. *Let $p \in \mathscr{F}$, and $i, j, k \in \mathbb{Z}^+$. If i and j belong to the same connected component of $G_p(k)$, then $\gcd(p)$ divides $i - j$. Moreover, if $k \geq \gcd(p)$, then $\kappa(G_p(k)) \geq \gcd(p)$.*

Proof. It suffices to observe that if i and j are adjacent in $G_p(k)$, then $|i - j| = p_r$ for some r with $1 \leq r \leq |p|$. Thus if $1 \leq i < j \leq \gcd(p)$, it follows that i and j cannot be in the same connected component of $G_p(k)$. □

Proposition 3.8. *Let $p \in \mathscr{F}$, and let $d = \gcd(p)$. Then for each $i = 1, 2, \ldots, d$, the map $\gamma_i : G_{p/d} \to G_p$ defined by $\gamma_i(j) = i + (j-1)d$ for $1 \leq j \leq fw(p/d)$ is a graph isomorphism from $G_{p/d}$ onto the subgraph $\gamma_i(G_{p/d})$ of G_p. Moreover, G_p has exactly d components, whose vertex sets are the images of γ_i, $i = 1, 2, \ldots, d$; that is, the congruence classes of the interval $\{1, 2, \ldots, fw(p)\}$ modulo d.*

Proof. It is immediate from Lemma 3.7 that each component of G_p is contained in the image of γ_i for some i with $1 \leq i \leq d$, and by Proposition 3.6, $G_{p/d}$ is connected. Let j, k be vertices of $G_{p/d}$. Since $|\gamma_i(j) - \gamma_i(k)| = |(j-1)d - (k-1)d| = |j - k|d$, it follows that $|j - k| = p_t/d$ if and only if $|\gamma_i(j) - \gamma_i(k)| = p_t$. Thus γ_i is a graph isomorphism from $G_{p/d}$ onto the subgraph $\gamma_i(G_{p/d})$ of G_p. □

Corollary 3.9. *Let $p \in \mathscr{F}$ and let $k \geq fw(p)$. Then $G_p(k)$ has exactly $\gcd(p)$ components, the congruence classes of the interval $\{1, 2, \ldots, k\}$ modulo $\gcd(p)$.*

Proof. G_p is a subgraph of $G_p(k)$, and by Proposition 3.8, $\kappa(G_p) = \gcd(p)$. Moreover, by Lemma 3.7, $\kappa(G_p(k)) \geq \gcd(p)$, and since each vertex i of $G_p(k)$ for which $fw(p) < i \leq k$ is connected to a vertex in the subgraph G_p, the result follows. □

Proposition 3.10. *Let $p \in \mathscr{F}$. If $\gcd(p) < \min(p) \leq k < fw(p)$, then $\kappa(G_p(k)) > \gcd(p)$.*

Proof. Suppose that $\gcd(p) < \min(p)$. Then by Proposition 2.6, $\min(p) < fw(p)$. Since for any k with $\min(p) \leq k < fw(p)$, $\kappa(G_p(k)) \geq \kappa(G_p(k+1))$ (since $k \geq \min(p)$, in $G_p(k)$, any $i > \min(p)$ is connected to either $\min(p)$ if i is a multiple of $\min(p)$, or some $r < \min(p)$ if i is not a multiple of p), it suffices to prove that $\kappa(G_p^-) > \gcd(p)$. The proof will be by induction on $\max(p)$, and the result is vacuously true for $\max(p) = 1$. Suppose now that $p \in \mathscr{F}$ has $\max(p) > 1$ and the result holds for all elements of $\mathscr{F}$ with smaller maximum entry. Suppose further that $\gcd(p) < \min(p)$. We first consider the case when p is not trim. Since $\max(p) > \max(p^-)$, we may apply the induction hypothesis to p^-. Since p is not trim, we have $\gcd(p) = \gcd(p^-)$, and $fw(p) = fw(p^-)$, so $\min(p^-) = \min(p) > \gcd(p) = \gcd(p^-)$. Thus $\kappa(G_{p^-}^-) > \gcd(p^-) = \gcd(p)$. By Corollary 2.10, $\max(p) = f(p) > f(p) - 1 \geq fw(p) - 1 = fw(p^-) - 1$, and so $G_{p^-}^- = G_p^-$. Thus $\kappa(G_p^-) > \gcd(p)$.

Suppose now that p is trim. There are two possibilities, p' trim or not. Suppose first that p' is trim. By Proposition 2.11, either $\min(p') > \gcd(p')$ or $|p'| = 1$. If $|p'| = 1$, then $\gcd(p') = \min(p') = \min(p) > \gcd(p) = \gcd(p')$, which is not possible. Thus $\min(p') > \gcd(p')$. Now since $\max(p') < \max(p)$, we may apply the induction hypothesis to p' to obtain that $G^-_{p'} = G_{p'}(\mathit{fw}(p')-1)$ has more than $\gcd(p') = \gcd(p)$ components. By Proposition 3.3, this implies that $G_p(\min(p) + \mathit{fw}(p') - 1)) = G_p(\mathit{fw}(p) - 1) = G^-_p$ has more than $\gcd(p)$ components, as required.

Now consider the case when p is trim but p' is not trim. By Proposition 2.13, $f(p) = 2\min(p)$. Since p is trim, we have $\mathit{fw}(p) = f(p) = 2\min(p)$, and so it follows that $G^-_p = G_p(\mathit{fw}(p) - 1) = G_p(2\min(p) - 1)$ has $\{\min(p)\}$ as a component. By Proposition 3.3, the map $\alpha_p : G_{p'}(\min(p) - 1) = G_{p'}(\mathit{fw}(p) - \min(p) - 1) \to G^-_p$ induces an injective map on components. Since $\min(p)$ is not in the image of α_p, $\kappa(G^-_p) \geq 1 + \kappa(G_{p'}(\min(p) - 1))$. We have $\min(p) = f(p') \geq \mathit{fw}(p')$. By Corollary 3.9, if $\mathit{fw}(p') < \min(p)$, then $\kappa(G_{p'}(\min(p) - 1)) = \gcd(p') = \gcd(p)$, which then implies that $\kappa(G^-_p) > \gcd(p)$. Suppose now that $\mathit{fw}(p') = \min(p)$. Then $G_{p'}(\min(p) - 1) = G^-_{p'}$. If $\min(p') = \gcd(p')$, then by Proposition 2.6, $\mathit{fw}(p') = \min(p')$ and so $\min(p) = \mathit{fw}(p') = \min(p') = \gcd(p') = \gcd(p)$, which is not the case. Thus $\min(p') > \gcd(p')$. Again by Proposition 2.6, $\mathit{fw}(p') \geq 2\min(p') > \min(p')$, and so we may apply the induction hypothesis to p' to obtain that $G^-_{p'} = G_{p'}(\mathit{fw}(p') - 1)$ has more than $\gcd(p') = \gcd(p)$ components. By Proposition 3.3, this implies that $\kappa(G_p(\min(p) + \mathit{fw}(p') - 1)) > \gcd(p)$. Since $\mathit{fw}(p') = \min(p)$, $\min(p) + \mathit{fw}(p') - 1 = 2\min(p) - 1 = \mathit{fw}(p) - 1$, and so $G_p(\min(p) + \mathit{fw}(p') - 1) = G^-_p$. Thus $\kappa(G^-_p) > \gcd(p)$ in this case was well, and this completes the proof of the inductive step. □

Corollary 3.11. *For $p \in \mathscr{F}$, $\mathit{fw}(p) = \min\{k \mid k \geq \min(p),\ \kappa(G_p(k)) = \gcd(p)\}$.*

Proof. If $\min(p) > \gcd(p)$, the result follows from Proposition 3.10 and Corollary 3.9. Suppose that $\min(p) = \gcd(p)$. Then $\mathit{fw}(p) = \gcd(p)$, and for $k < \gcd(p) = \min(p)$, $G_p(k)$ is a null graph, so has k components, while for $k \geq \gcd(p) = \mathit{fw}(p)$, Corollary 3.9 asserts that $\kappa(G_p(k)) = \gcd(p)$. □

Proposition 3.12. *If $p \in \mathscr{F}$ has $|p| > 1$ and $\gcd(p^-) = \gcd(p)$, then $\mathit{fw}(p^-) \geq \mathit{fw}(p)$.*

Proof. By Lemma 2.5, we need only consider $p \in \mathscr{F}$ for which $\gcd(p) = 1$. Note that by Proposition 2.6, the result holds if $\min(p) = 1$. Suppose that $p \in \mathscr{F}$ is such that $|p| > 1$, $\gcd(p^-) = \gcd(p) = 1 < \min(p)$, and, contrary to our claim, $\mathit{fw}(p^-) < \mathit{fw}(p)$. Then G_{p^-} is a subgraph of G^-_p, and by Corollary 3.9, G_{p^-} is connected. Thus $\{1, 2, \ldots, \min(p^-)\}$ is contained within a component of G^-_p, and since $\min(p^-) = \min(p)$, G^-_p is connected by Lemma 3.5. However, by Corollary 3.11, G^-_p is not connected, and so we have obtained a contradiction. □

We remark that if $\gcd(p^-) = \gcd(p)$ and $\max(p) \geq fw(p^-)$, then p is not trim and so $fw(p) = fw(p^-)$. However, when $\gcd(p^-) = \gcd(p)$ and $\max(p) < fw(p^-)$, it is possible that we may actually have $fw(p^-) > fw(p)$. The lexically first such example is $p = (5, 7, 8)$, where $fw(p) = 10$, while $fw((5, 7)) = 11$.

It might be tempting to believe that fw grows monotonically with respect to the product order on sequences of a given length and greatest common divisor 1, and, as the Fine-Wilf theorem tells us, this is indeed the case for sequences of length 2. However, this observation does not hold even for sequences of length 3. For example, $fw((7, 9, 11)) = 15$, while $fw((7, 9, 13)) = 14$.

Our next observation relates $\kappa(G_p(f(p)-1))$ to the tableau for the computation of $f(p)$. Let $p \in \mathscr{F}$ with $\gcd(p) = 1$ and $|p| > 1$, and consider the tableau for the computation of $f(p)$. Let $m = ht(p)$. Then $p^{(m)} = (1)$, and $p^{(m-1)} = (1, 2)$. For each i with $0 \leq i \leq m$, we shall call $p^{(i)}$ a *jump* if $f(p^{(i)}) = 2\min(p^{(i)})$, and in the tableau for the computation of $f(p)$, we shall prefix each jump with a plus sign (+). Furthermore, let $J(p)$ denote the number of jumps in the tableau for the calculation of $f(p)$. For example, $p = (6, 10, 13)$ has tableau

	6,10,13
+	4,6,7
+	2,3,4
+	1,2
	1

and so $J(p) = 3$. We observe that $p^{(m)}$ is never a jump, while $p^{(m-1)}$ is always a jump. For each $i = 0, \ldots, m$, let $G^i = G_{p^{(m-i)}}(f(p^{(m-i)})-1)$, so that G^0 is the null graph on a single vertex, and $G^m = G_p(f(p) - 1)$. Now for each $i = 1, \ldots, m$, let $\alpha_i : G^{i-1} \to G^i$ denote $\alpha_{p^{(m-(i-1))}}$, so that for a vertex j, $\alpha_i(j) = \min(p^{(m-i)})+j$. By Proposition 3.3, for each i, α_i induces an injective map from the set of components of G^{i-1} into the set of components of G^i, and G^0 has a single component. Moreover, the image of α_i is the set $\{\min(p^{(m-i)}) + 1, \ldots, \min(p^{(m-i)}) + f(p^{(m-i+1)}) - 1\}$, which is equal to $\{\min(p^{(m-i)})+1, \ldots, f(p^{(m-i)})-1\}$. If $f(p^{(m-i)}) > 2\min(p^{(m-i)})$, then for each $k \in \{1, 2, \ldots, \min(p^{(m-i)})\}$, $k + \min(p^{(m-i)}) \leq 2\min(p^{(m-i)}) \leq f(p^{(m-i)}) - 1$, and thus $\{k, k + \min(p^{(m-i)})\}$ is an edge in G^i joining k to a vertex in the image of α_i. Consequently, $\kappa(G^i) = \kappa(G^{i-1})$. On the other hand, if $f(p^{(m-i)}) = 2\min(p^{(m-i)})$, then $\min(p^{(m-i)})$ has degree 0 in G^i, so $\{\min(p^{(m-i)})\}$ is a component of G^i that is not contained in the image of α_i. For any k with $1 \leq k < \min(p^{(m-i)})$, $\{k, k+\min(p^{(m-i)})\}$ is an edge in G^i joining k to a vertex in the image of α_i, and so $\kappa(G^i) = 1 + \kappa(G^{i-1})$. This proves the following result.

Proposition 3.13. *If* $p \in \mathscr{F}$ *has* $|p| > 1$ *and* $\gcd(p) = 1$, *then* $\kappa(G_p(f(p) - 1)) = J(p)$*, the number of jumps in the tableau for the computation of* $f(p)$*.*

Note that for p trim, $fw(p) = f(p)$ and so $G_p^- = G_p(f(p) - 1)$, and the finite sequence a of length $fw(p) - 1$ formed by labelling the components of G_p^-, then setting a_i equal to the label of the component containing vertex i of G_p^-, is the unique sequence (up to labelling) of length $fw(p) - 1$ with the greatest number of distinct entries that has periods the entries of p, but not $\gcd(p)$. By Proposition 3.13, the number of distinct entries in a is equal to $J(p)$. We observe as well that a can be calculated from the tableau for the calculation of $f(p)$. Let $m = ht(p)$. We begin at row $p^{(m-1)}$ with sequence 0. Then at stage $p^{(i)}$, shift the preceding word $\min(p^{(i)})$ spaces to the right. If $p^{(i)}$ is not a jump, then the preceding sequence has length at least $\min(p^{(i)})$ and we fill in the first $\min(p^{(i)})$ locations of the new sequence with the first $\min(p^{(i)})$ entries in the preceding sequence, while if $p^{(i)}$ is a jump, then the preceding sequence has length $\min(p^{(i)}) - 1$, and we fill in the first $\min(p^{(i)}) - 1$ spaces with the entries of the preceding sequence and then introduce a new symbol for the vertex at position $\min(p(i))$.

4. Reduction

In this final section, we show that the reduction concept first introduced with the trimming operation can be developed further.

Definition 4.1. *For $p \in \mathscr{F}$ and j such that $2 \le j \le |p|$ and $\gcd(p|_j) = \gcd(p|_{j-1})$, we shall say that p_j is* type I redundant *in p if p_j is a multiple of p_i for some i with $1 \le i < j$, or* type II redundant *in p if $p_j \ge f(p|_{j-1})$. If p_j is either type I or type II redundant in p, we shall say that p_j is* redundant *in p.*

Definition 4.2. *For $p \in \mathscr{F}$ and j such that $1 \le j \le |p|$, let $p - p_j$ denote the element of $\mathscr{F}$ that is formed by deleting p_j from p.*

Proposition 4.3. *Let $p \in \mathscr{F}$ and p_j be type I redundant in p. Then*

1. *For every $k \in \mathbb{Z}^+$, the vertex sets of the components of $G_p(k)$ are identical to those of $G_{p-p_j}(k)$.*
2. *$fw(p) = fw(p - p_j)$.*

Proof. Let i be such that $1 \le i < j$ and $p_j = tp_i$ for some integer t. Then every edge of $G_p(k)$ that is determined by p_j has its endpoints joined by a path of length t with edges determined by p_i, so the edges of $G_p(k)$ that are determined by p_j can be deleted (thereby forming $G_{p-p_j}(k)$) with no change in component vertex sets. Thus the vertex sets of the components of $G_p(k)$ are identical to those of $G_{p-p_j}(k)$. By Corollary 3.11, since $\gcd(p) = \gcd(p - p_j)$, $fw(p) = \min\{k \mid \kappa(G_p(k)) = \gcd(p)\} = \min\{k \mid \kappa(G_{p-p_j}(k)) = \gcd(p - p_j)\} = fw(p - p_j)$. □

One might suspect that if p_j is type I redundant in p, then p_j is type II redundant as well, but this is not necessarily the case. For example, if $p = (5, 13, 15)$, then

$f(p) = 17 = f(p^-) > max(p)$, while $\gcd(p) = 1 = \gcd(p^-)$, so p is trim; that is, $\max(p)$ is type I redundant but not type II redundant.

Lemma 4.4. *If $p \in \mathscr{F}$ and j is such that $1 < j < |p|$ and p_j is type II redundant in p, then $f(p) = f(p - p_j)$.*

Proof. Let $q = p|_j$. Since p_j is type II redundant in p, q is not trim. By Lemma 2.10 and Lemma 2.3, $\max(q^{(i)}) = f(q^{(i)}) \geq 2\min(q^{(i)})$ for every i, $0 \leq i < ht(q)$, and so for each $i = 0, 1, \ldots, ht(q) - 1$, $\max(q^{(i)}) - \min(q^{(i)}) \geq \min(q^{(i)})$. Thus in the formation of the tableau for the calculation of $f(q)$, if $\min(q^{(i)})$ needed to be inserted in the formation of $q^{(i+1)}$, it would be inserted prior to the last entry of $q^{(i)}$. Let $n = ht(q)$, and let $d = \gcd(q)$. Then $q^{(n)} = (d)$, and $q^{(n-1)} = (d, 2d)$. Since $p_j < p_{j+1}$, there exists $e \in \mathbb{Z}^+$ such that the tableau for the calculation of $f(p)$ looks like

$$\begin{aligned} &p \\ &\vdots \\ &p^{(n-1)} = d, 2d, e, \ldots \\ &p^{(n)} = d, e - d, \ldots \\ &\vdots \end{aligned}$$

where the entry $2d$ in $p^{(n-1)}$ is derived from p_j in p, while the entry e is derived from p_{j+1}. Thus for each $i \geq n$, $p^{(i)} = (p - p_j)^{(i)}$, while for each i with $1 \leq i \leq n$, $\min(p^{(i)}) = \min((p - p_j)^{(i)})$. It follows that $ht(p) = ht(p - p_j)$ and $f(p) = \sum_{i=1}^{ht(p)} \min(p^{(i)}) = \sum_{i=1}^{ht(p-p_j)} \min((p - p_j)^{(i)}) = f(p - p_j)$. □

We note that the requirement $j < |p|$ in the preceding lemma is essential, as the example discussed prior to Lemma 3.5 illustrates.

Proposition 4.5. *If $p \in \mathscr{F}$ and p_j is redundant in p, then $fw(p) = fw(p - p_j)$.*

Proof. If p_j is type I redundant, then the result follows from Proposition 4.3, so we may assume that p_j is type II redundant and thus $\gcd(p|_j) = \gcd(p|_{j-1})$ and $p_j \geq f(p|_{j-1})$. If $j \geq |p^t|$, then by Lemma 4.4, $f(p|_i) = f(p|_i - p_j)$ for all $i > j$, and thus $p^t = (p - p_j)^t$, in which case by Proposition 2.11, we have $fw(p) = fw(p^t) = fw((p - p_j)^t) = fw(p - p_j)$. Thus we may further assume that p is trim (and thus $j < |p|$).

Suppose that $fw(p - p_j) > fw(p)$. Then by Proposition 3.10, $G_{p-p_j}(fw(p))$ is not connected, and by Proposition 2.13, $p_j < \max(p) < f(p) = fw(p)$. Thus $fw(p|_{j-1}) \leq f(p|_{j-1}) \leq p_j < fw(p)$, and so $G_{p|_{j-1}}(fw(p|_{j-1}))$ is a connected subgraph of $G_{p-p_j}(fw(p))$. Since $fw(p|_{j-1}) \geq \min(p)$, the interval $\{1, 2, \ldots, \min(p)\}$ is contained within a component of $G_{p-p_j}(fw(p))$, and so by Lemma 3.5, $G_{p-p_j}(fw(p))$ is connected. This contradiction implies that $fw(p - p_j) \leq fw(p)$. Suppose now that $fw(p - p_j) < fw(p)$. Then $G_{p-p_j}(fw(p - p_j))$ is a connected subgraph of G_p^-. By

Proposition 2.6 applied to $p - p_j$, $fw(p - p_j) \geq 2\min(p) > \min(p)$ and so the interval $\{1, 2, \ldots, \min(p)\}$ is contained within a component of G_p^-. But then by Lemma 3.5, G_p^- is connected, which is not the case. This last contradiction implies that $fw(p) = fw(p - p_j)$, as required. □

Definition 4.6. *For $p \in \mathscr{F}$, let p^r denote the sequence obtained from p by deleting all type I redundant entries of p. Then p^r shall be called the* reduced form *of p, and is said to be obtained by* reducing *p. If $p = p^r$, we say that p is* reduced.

The next result is an immediate consequence of Proposition 4.5, since it is clear that we may form p^r by removing type I redundant entries one after the other in any order (that is, no new type I redundant entry can be formed nor can an existing type I redundant entry be made non-redundant by the deletion of an existing type I redundant entry).

Corollary 4.7. *For $p \in \mathscr{F}$, $fw(p) = fw(p^r)$.*

Proposition 4.8. *Let $p \in \mathscr{F}$ and p_j be type II redundant in p, where $1 < j < |p|$. Then for any $i \neq j$ with $1 \leq i \leq |p|$, p_i is type II redundant in p if and only if p_i is type II redundant in $p - p_j$.*

Proof. We have $\gcd(p|_j) = \gcd(p|_{j-1})$ and $p_j \geq f(p|_{j-1})$. The result is obvious for $i < j$, so we consider the case when $i > j$. Suppose first of all that p_i is type II redundant in p. Then $\gcd(p|_i) = \gcd(p|_{i-1})$, and $p_i \geq f(p|_{i-1})$. Note that the index of p_i in $p - p_j$ is $i - 1$. Since $i > j$, we have $\gcd((p - p_j)|_{i-1}) = \gcd(p|_i - p_j) = \gcd(p|_i) = \gcd(p|_{i-1}) = \gcd(p|_{i-1} - p_j) = \gcd((p - p_j)|_{i-2})$. If $i = j+1$, then $(p-p_j)_{i-1} = p_i > p_j \geq f(p|_{j-1}) = f((p-p_j)|_{j-1}) = f((p-p_j)|_{i-2})$, while if $i > j + 1$, then we have $(p - p_j)_{i-1} = p_i \geq f(p|_{i-1})$ and by Lemma 4.4 (since $j < i - 1 = |p|_{i-1}|$), $f(p|_{i-1}) = f(p|_{i-1} - p_j) = f((p - p_j)|_{i-2})$, so $(p - p_j)_{i-1} \geq f((p - p_j)|_{i-2})$. Thus in either case, p_i is a type II redundant entry of $p - p_j$.

Conversely, suppose that p_i is type II redundant in $p - p_j$. Then $p_i = (p-p_j)_{i-1} \geq f((p - p_j)|_{i-2})$ and $\gcd((p - p_j)|_{i-1}) = \gcd((p - p_j)|_{i-2})$. We have $\gcd(p|_i) = \gcd(p|_i - p_j) = \gcd((p-p_j)|_{i-1}) = \gcd((p-p_j)|_{i-2}) = \gcd(p|_{i-1} - p_j) = \gcd(p|_{i-1})$. As well, if $i > j + 1$, then by Proposition 4.4, we have $f(p|_{i-1}) = f(p|_{i-1} - p_j)$, and so $p_i \geq f((p - p_j)|_{i-2}) = f(p|_{i-1} - p_j) = f(p|_{i-1})$, while if $i = j + 1$, then by Proposition 2.8 applied to $p|_j$, $p_i > p_j = f(p|_j) = f(p|_{i-1})$. In either case, p_i is a type II redundant entry of p. □

Definition 4.9. *For $p \in \mathscr{F}$, let $\hat{p}$ denote the totally reduced form of p; namely the sequence obtained from p^r by deleting all type II redundant entries of p^r, and let $r(p) = |p| - |\hat{p}|$. If $p = \hat{p}$, we say that p is totally reduced.*

Note that $\hat{p}$ contains no redundant entries and is thus totally reduced. As well, note that as a result of Proposition 4.8, we may form $\hat{p}$ by deleting the type II redundant elements of p^r one by one, in any order whatsoever, and so induction on $r(p)$ proves the next result.

Proposition 4.10. *Let $p \in \mathscr{F}$. Then $fw(p) = fw(\hat{p})$.*

We are now in a position to give an upper bound for $fw(p)$ that is an improvement over that given in Proposition 2.14 (provided that $r(p) < |p| - 1$, its maximum possible value).

Theorem 4.11. *For $p \in \mathscr{F}$, $fw(p) \le \min(p) + \max(p) - \gcd(p)(|p| - 1 - r(p))$.*

Proof. Let $d = \gcd(p)$. By Lemma 2.5, $r(p) = r(p/d)$, $d\,fw(p/d) = fw(p)$, while by definition of $p/d, \min(p) = d \min(p/d)$, $\max(p) = d \max(p/d)$ and $|p| = |p/d|$. It suffices therefore to prove the result for $p \in \mathscr{F}$ with $\gcd(p) = 1$, and this we shall do by induction on $\max(p)$. If $\gcd(p) = 1$ and $\max(p) = 1$, then $p = (1)$ and the result holds.

Suppose now that $\gcd(p) = 1$, $\max(p) > 1$, and the result holds for all elements of $\mathscr{F}$ with greatest common divisor 1 and smaller maximum entry. Since $|p| = 1$ would imply that $1 = \gcd(p) = \max(p) > 1$, it follows that $|p| > 1$. Consider $\hat{p}$, the totally reduced form of p. Since $\min(\hat{p}) = \min(p)$ and $\max(\hat{p}) \le \max(p)$, it follows that if we are able to prove that $fw(\hat{p}) \le \min(\hat{p}) + \max(\hat{p}) - (|\hat{p}| - 1)$, then by Proposition 4.10, $fw(p) = fw(\hat{p}) \le \min(\hat{p}) + \max(\hat{p}) - (|\hat{p}| - 1) \le \min(p) + \max(p) - (|p| - r(p) - 1)$, as required. Thus we may assume that p is totally reduced, and we are to prove that $fw(p) \le \min(p) + \max(p) - (|p| - 1)$. Since p is totally reduced, it is in particular trim, and so $fw(p) = f(p) = \min(p) + f(p')$. If p' is not trim, then by Proposition 2.13, $f(p) = 2\min(p)$, and since $\max(p) - \min(p) \ge |p| - 1$, $fw(p) = f(p) = 2\min(p) \le \min(p) + \max(p) - (|p| - 1)$, as required. Thus we may assume that p' is trim, so $fw(p') = f(p')$. Furthermore, by Proposition 4.3, the fact that p is totally reduced means in particular that no entry of p is a multiple of $\min(p)$. Thus $\min(p) \ne p_j - \min(p)$ for every j with $2 \le j \le |p|$, and so $|p'| = |p|$. Apply the induction hypothesis to p' to obtain $fw(p) = \min(p) + f(p') = \min(p) + fw(p') \le \min(p) + \min(p') + \max(p') - (|p'| - 1 - r(p')) = \min(p) + \min(p') + \max(p') - (|p| - 1 - r(p'))$. It will suffice to prove that $\min(p') + \max(p') + r(p') \le \max(p)$. Let us first treat the case when p' is totally reduced; that is, $r(p') = 0$. There are three subcases to consider. If $\min(p) \le p_2 - \min(p)$, then $\min(p') = \min(p)$ and $\max(p') = \max(p) - \min(p)$, so $\min(p') + \max(p') + r(p') = \max(p)$. Suppose now that $p_2 - \min(p) < \min(p) \le \max(p) - \min(p)$, so that $\min(p') = p_2 - \min(p)$ and $\max(p') = \max(p) - \min(p)$. Then $\min(p') + \max(p') + r(p') = p_2 - \min(p) + \max(p) - \min(p) = \max(p) + p_2 - 2\min(p)$. But from $p_2 - \min(p) < \min(p)$, we have $p_2 - 2\min(p) < 0$ and so $\max(p) + p_2 - 2\min(p) < \max(p)$. Finally, suppose that $\max(p) - \min(p) < \min(p)$, so that $\min(p') = p_2 - \min(p)$ and $\max(p') = \min(p)$, which implies that $\min(p')+\max(p')+r(p') = p_2-\min(p)+\min(p) = p_2 \leq \max(p)$, as required.

We now treat the case when p' is not totally reduced, so that $r(p') > 0$. Let j be such that $p_j - \min(p)$ is redundant in p'. Since p is totally reduced and therefore reduced, $\min(p) \neq p_i - \min(p)$ for every i, and so in particular, $\min(p) \neq p_j - \min(p)$. Consider first the possibility that $\min(p) < p_j - \min(p)$. Then $(p|_j)' = p'|_j$ has maximum entry $p_j - \min(p)$ and is not trim, so by Proposition 2.13, $f(p|_j) = 2\min(p)$. As p is totally reduced, $p|_j$ is trim, and since $j > 1$, Proposition 2.13 implies that $f(p|_j) > \max(p|_j) = p_j$. Thus $2\min(p) > p_j$, contradicting our assumption that $p_j - \min(p) > \min(p)$. Hence $p_j - \min(p) < \min(p)$ (which implies that $j > 2$ since $p_2 - \min(p) = \min(p')$), and so we have established that if j is any index such that $p_j - \min(p)$ is redundant in p', then $p_2 < p_j < 2\min(p)$. Thus $r(p') \leq |\{j \mid p_2 < p_j < 2\min(p)\}| + 1$, where we have added 1 to acknowledge that $\min(p)$ might be redundant in p'. Thus $0 < r(p') \leq (2\min(p) - p_2 - 1) + 1 = 2\min(p) - p_2$, and so $p_2 < 2\min(p)$, which means that $p_2 - \min(p) = \min(p')$. There are two subcases to consider.

Subcase 1: $\min(p) < \max(p) - \min(p)$. Then $\max(p') = \max(p) - \min(p)$, and so $\min(p')+\max(p')+r(p') \leq p_2-\min(p)+\max(p)-\min(p)+2\min(p)-p_2 = \max(p)$.

Subcase 2: $\min(p) > \max(p) - \min(p)$. Then $\max(p') = \min(p)$, and $\max(p) < 2\min(p)$. Since p' is trim, $\max(p')$ is not type II redundant in p' and thus in this case, we have $r(p') \leq |\{j \mid p_2 < p_j < 2\min(p)\}| = |\{j \mid p_2 < p_j \leq \max(p)\}| = |p| - 2$ and so $\min(p')+\max(p')+r(p') \leq p_2-\min(p)+\min(p)+|p|-2 = p_2+|p|-2 \leq \max(p)$. This completes the proof of the inductive step. □

Corollary 4.12. *Let $p \in \mathscr{F}$ be totally reduced. Then $fw(p) \leq \min(p) + \max(p) - \gcd(p)(|p| - 1)$.*

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