GAP DISTRIBUTION OF FAREY FRACTIONS UNDER SOME DIVISIBILITY CONSTRAINTS

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Abstract

For a given positive integer ℓ, we show the existence of the limiting gap distribution measure for the sets of Farey fractions $\frac{a}{q}$ of order Q with $\ell \nmid a$, and respectively with $(q,\ell)=1$, as $Q \to \infty$.

1. Introduction

The set $\mathcal{F}_Q$ of Farey fractions of order Q consists of those rational numbers $\frac{a}{q} \in (0,1]$ with $(a,q)=1$ and $q \leqslant Q$. The spacing statistics of the increasing sequence $(\mathcal{F}_Q)$ of finite subsets of $(0,1]$ have been investigated by several authors [9, 1, 7]. Recently Badziahin and Haynes considered a problem related to the distribution of gaps in the subset $\mathcal{F}_{Q,d}$ of $\mathcal{F}_Q$ of those fractions $\frac{a}{q}$ with $(q,d)=1$, where d is a fixed positive integer and $Q \to \infty$. They proved [2] that, for each $k \in \mathbb{N}$, the number $N_{Q,d}(k)$ of pairs $\left(\frac{a}{q}, \frac{a'}{q'}\right)$ of consecutive elements in $\mathcal{F}_{Q,d}$ with $a'q - aq' = k$ satisfies the asymptotic formula

$$N_{Q,d}(k) = c(d,k)Q^2 + O_{d,k}(Q \log Q) \qquad (Q \to \infty), \tag{1.1}$$

for some positive constant $c(d,k)$ that can be expressed using the measure of certain cylinders associated with the area-preserving transformation introduced by Cobeli,

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Zaharescu, and the first author in [4]. The pair correlation function of $(\mathcal{F}_{Q,d})$ was studied and shown to exist by Xiong and Zaharescu [11], even in the more general situation where $d = d_Q$ is no longer constant but increases according to the rules $d_{Q_1} \mid d_{Q_2}$ as $Q_1 < Q_2$ and $d_Q \ll Q^{\log\log Q/4}$.

This paper is concerned with the gap distribution of the sequence of sets $(\mathcal{F}_{Q,d})$, and respectively of $(\widetilde{\mathcal{F}}_{Q,\ell})$, the sequence of sets $\widetilde{\mathcal{F}}_{Q,\ell}$ of Farey fractions $\gamma = \frac{a}{q} \in \mathcal{F}_Q$ with $\ell \nmid a$. Our peculiar interest in $\widetilde{\mathcal{F}}_{Q,\ell}$ arises from the problem studied in [5], concerning the distribution of the free path associated to the linear flow through $(0,0)$ in $\mathbb{R}^2$ in the small scatterer limit, in the case of circular scatterers of radius $\varepsilon > 0$ placed at the points $(m,n) \in \mathbb{Z}^2$ with $\ell \nmid (m-n)$. When $\ell = 3$ this corresponds, after suitable normalization, to the situation of scatterers distributed at the vertices of a honeycomb tessellation, and the linear flow passing through the center of one of the hexagons. When $\ell = 2$ the scatterers are placed at the vertices of a square lattice and the linear flow passes through the center of one the squares. Arithmetic properties of the number ℓ are shown to be explicitly reflected by the gap distribution of the elements of $(\widetilde{\mathcal{F}}_{Q,\ell})$. The symmetry $x \mapsto 1-x$ shows that for the purpose of studying the gap distribution of these fractions on $[0,1]$ one can replace the condition $\ell \nmid (m-n)$ by the more esthetic one $\ell \nmid n$.

The *gap distribution* (or *nearest neighbor distribution*) of a numerical sequence, or more generally of a sequence of finite subsets of $[0,1)$, measures the distribution of lengths of gaps between the elements of the sequence. Let $A = \{x_0 \leqslant x_1 \leqslant \ldots \leqslant x_N\}$ be a finite list of numbers in $[0,1)$, not all equal, scaled to $\tilde{x}_j = \frac{Nx_j}{x_N - x_0}$ with mean spacing $\frac{\tilde{x}_N - \tilde{x}_0}{N} = 1$. The *gap distribution measure* of A is the finitely supported probability measure on $[0,\infty)$ defined by

$$\nu_A(-\infty,\xi] = \nu_A[0,\xi] := \frac{1}{N}\#\big\{j \in [1,N] : \tilde{x}_j - \tilde{x}_{j-1} \leqslant \xi\big\}, \qquad \xi \geqslant 0.$$

If it exists, the weak limit $\nu = \nu_{\mathcal{A}}$ of the sequence (ν_{A_n}) of probability measures associated with an increasing sequence $\mathcal{A} = (A_n)$ of finite lists of numbers in $[0,1)$, is called the *limiting gap measure* of $\mathcal{A}$.

It is elementary (see, e.g., Lemma 1 below) that

$$\#\widetilde{\mathcal{F}}_{Q,\ell} = \widetilde{K}_\ell Q^2 + O_\ell(Q\log Q), \qquad \#\mathcal{F}_{Q,d} = K_d Q^2 + O_d(Q\log Q), \tag{1.2}$$

where

$$\widetilde{K}_\ell = \frac{1}{2\zeta(2)} - \frac{C(\ell)}{2\ell}, \quad K_d = \frac{C(d)}{2}, \quad \text{with} \quad C(\ell) = \frac{1}{\zeta(2)} \prod_{\substack{p\in\mathcal{P} \\ p\mid\ell}} \left(1 + \frac{1}{p}\right)^{-1}.$$

We prove the following result:

Theorem 1. *Given positive integers ℓ and d, the limiting gap measures $\widetilde{\nu}_\ell$ of $(\widetilde{\mathcal{F}}_{Q,\ell})$, and respectively ν_d of $(\mathcal{F}_{Q,d})$, exist. Their densities are continuous on $[0,\infty)$ and real analytic on each component of $(0,\infty) \setminus \mathbb{N}\widetilde{K}_\ell$, and respectively of $(0,\infty) \setminus \mathbb{N}K_d$.*

The existence of $\widetilde{\nu}_\ell$ is proved in Section 2 and the limiting gap distribution is explicitly computed in (2.9) using tools from [4], [8] and [5]. The result on ν_d is proved in Section 4. When d is a prime power, an explicit computation can be done as for $\tilde{\nu}_\ell$. In general the repartition function of ν_d depends on the measure of some cylinders associated with the transformation T from (2.7), and on the length of strings of consecutive elements in $\mathcal{F}_Q$ with at least one denominator relatively prime with d.

The upper bound $4d^3$ for $L(d) = \min\{L : \forall i, \forall Q, \exists j \in [0, L], (q_{i+j}, d) = 1\}$ was found in [2], where $q_i, \ldots, q_{i+L}$ denote the denominators of a string $\gamma_i < \cdots < \gamma_{i+L}$ of consecutive elements in $\mathcal{F}_Q$. Although we expect this bound to be considerably smaller, we could only improve it in a limited number of situations. In Section 3 we lower it to $4\omega(d)^3$ for integers d with the property that the smallest prime divisor of d is $\geqslant \omega(d)$, where $\omega(d)$ denotes as usual the number of distinct prime factors of d. The bound $L(d) = 1$ is trivial when d is a prime power. Employing properties of the transformation T^2 we show that $L(d) \leqslant 5$ when d is the product of two prime powers, which is sharp. Finding better bounds on $L(d)$ when $\omega(d) \geqslant 3$ appears to be an interesting problem in combinatorial number theory.

2. The Gap Distribution of $\widetilde{\mathcal{F}}_{Q,\ell}$

Let $\mathcal{F}_Q^{(\ell)} = \mathcal{F}_Q \setminus \widetilde{\mathcal{F}}_{Q,\ell}$ denote the set of Farey fractions $\gamma = \frac{a}{q} \in \mathcal{F}_Q$ with $\ell \mid a$, and let $N_Q^{(\ell)}$ denote the cardinality of $\mathcal{F}_Q^{(\ell)}$. Consider also:

$$\begin{aligned}
\mathcal{G}_Q(\xi) &:= \left\{(\gamma, \gamma') : \gamma,\, \gamma' \text{ consecutive in } \mathcal{F}_Q, 0 < \gamma' - \gamma \leqslant \frac{\xi}{Q^2}\right\},\\
\mathcal{G}_Q^{(\ell)}(\xi) &:= \left\{(\gamma, \gamma') : \gamma,\, \gamma' \text{ consecutive in } \widetilde{\mathcal{F}}_{Q,\ell}, 0 < \gamma' - \gamma \leqslant \frac{\xi}{Q^2}\right\},\\
N_Q(\xi) &: = \#\mathcal{G}_Q(\xi), \quad N_Q^{(\ell)}(\xi) := \#\mathcal{G}_Q^{(\ell)}(\xi).
\end{aligned}$$

Lemma 1. $N_Q^{(\ell)} = \dfrac{C(\ell)}{2\ell} Q^2 + O_\ell(Q \log Q)$ *as* $Q \to \infty$.

Proof. It is clear that

$$N_Q^{(\ell)} = \#\mathcal{F}_Q^{(\ell)} = \sum_{\substack{q=1\\(\ell,q)=1}}^{Q} \sum_{\substack{a=1\\(a,q)=1\\ \ell \mid a}}^{q} 1.$$

Letting $k = \frac{a}{\ell}$ and noting that, whenever $(\ell, q) = 1$, we have $(k\ell, q) = 1$ if and only

$(k,q)=1$, the sum above becomes

$$\sum_{\substack{q=1\\(\ell,q)=1}}^{Q}\ \sum_{\substack{k=1\\(k,q)=1}}^{[q/\ell]} 1.$$

Standard Möbius summation, cf. (A.1) and (A.2), and $\sum_{q=1}^{Q}\sigma_0(q)=O(Q\log Q)$, where $\sigma_0(q)=\sum_{d|q}1$, yield

$$\sum_{\substack{q=1\\(\ell,q)=1}}^{Q}\ \sum_{\substack{k=1\\(k,q)=1}}^{[q/\ell]} 1=\sum_{\substack{q=1\\(\ell,q)=1}}^{Q}\left(\frac{\varphi(q)}{q}\cdot\frac{q}{\ell}+O(\sigma_0(q))\right)=\frac{C(\ell)}{2\ell}Q^2+O_\ell(Q\log Q),$$

concluding the proof. □

This also establishes the first equality in (1.2) because

$$\#\widetilde{\mathcal{F}}_{Q,\ell}=\#\mathcal{F}_Q-\#\mathcal{F}_Q^{(\ell)}\sim\left(\frac{1}{2\zeta(2)}-\frac{C(\ell)}{2\ell}\right)Q^2.$$

Letting $\xi>0$ and $Q,\ell\in\mathbb{N}$ with $\ell\geqslant 2$, we set out to asymptotically estimate the number $N_Q^{(\ell)}(\xi)$ as $Q\to\infty$. Now if $\gamma=\frac{a}{q}$ and $\gamma'=\frac{a'}{q'}$ are consecutive elements in $\mathcal{F}_Q$ and $\gamma'\in\mathcal{F}_Q^{(\ell)}$, then $1=a'q-aq'\equiv -aq' \pmod{\ell}$, which implies that $(a,\ell)=1$, and thus $\gamma\notin\mathcal{F}_Q^{(\ell)}$. Similarly, if $\gamma\in\mathcal{F}_Q^{(\ell)}$, then $\gamma'\notin\mathcal{F}_Q^{(\ell)}$; and so no two consecutive elements of $\mathcal{F}_Q$ belong simultaneously to $\mathcal{F}_Q^{(\ell)}$. This means that if $\gamma<\gamma'$ are consecutive elements in $\widetilde{\mathcal{F}}_{Q,\ell}$, then two cases can occur:
Case 1. γ and γ' are consecutive elements in $\mathcal{F}_Q$ and $\gamma,\gamma'\notin\mathcal{F}_Q^{(\ell)}$. In this case the number of gaps in consecutive fractions of length $\leqslant\frac{\xi}{Q^2}$ is equal to $\mathcal{N}_1(Q,\xi)=N_Q(\xi)-M_1(Q,\xi)-M_2(Q,\xi)$, where $M_1(Q,\xi)$ is the number of pairs $(\gamma,\gamma')\in\mathcal{G}_Q(\xi)$ with $\gamma'\in\mathcal{F}_Q^{(\ell)}$, and $M_2(Q,\xi)$ is the number of pairs $(\gamma,\gamma')\in\mathcal{G}_Q(\xi)$ with $\gamma\in\mathcal{F}_Q^{(\ell)}$.

The number $N_Q(\xi)$ is estimated employing the well-known fact that $\gamma<\gamma'$ are consecutive elements in $\mathcal{F}_Q$ if and only if $q,q'\in\{1,\dots,Q\}$, $q+q'>Q$, and $a'q-aq'=1$. Furthermore, $\frac{a'}{q'}-\frac{a}{q}=\frac{1}{qq'}$, and so $\frac{a'}{q'}-\frac{a}{q}\leqslant\frac{\xi}{Q^2}$ if and only if $qq'\geqslant\frac{Q^2}{\xi}$. This establishes the equality

$$\begin{aligned}N_Q(\xi)&=\#\left\{(q,q')\in\mathbb{N}^2: q,q'\leqslant Q, q+q'>Q, (q,q')=1, qq'\geqslant\frac{Q^2}{\xi}\right\}\\&=\sum_{q'=1}^{Q}\sum_{\substack{q\in I_Q(q')\\(q,q')=1}}1,\end{aligned}\tag{2.1}$$

where $I_Q(q')=Q\cdot\left[\eta_Q(q'),1\right]$ and $\eta_Q(q')=\max\left\{1-\frac{q'-1}{Q},\frac{Q}{\xi q'}\right\}$.

Standard Möbius summation provides

$$N_Q(\xi) = \sum_{q'=1}^{Q} \left(\frac{\varphi(q')}{q'} |I_Q(q')| + O(\sigma_0(q')) \right) = \sum_{q'=1}^{Q} \frac{\varphi(q')}{q'} |I_Q(q')| + O(Q \log Q)$$
$$= \frac{A(\xi)}{\zeta(2)} Q^2 + O(Q \log Q),$$

where

$$A(\xi) = \left| \left\{ (x,y) \in (0,1]^2 : x + y > 1, xy \geqslant \frac{1}{\xi} \right\} \right|$$
$$= \begin{cases} 0 & \text{if } 0 < \xi \leqslant 1 \\ 1 - \frac{\log \xi + 1}{\xi} & \text{if } 1 \leqslant \xi \leqslant 4 \\ 1 - \frac{1}{\xi} - \frac{1}{2}\sqrt{1 - \frac{4}{\xi}} + \frac{2}{\xi} \log\left(\frac{1+\sqrt{1-4/\xi}}{2} \right) & \text{if } \xi \geqslant 4. \end{cases} \tag{2.2}$$

Next, we estimate $M_1(Q,\xi)$. Clearly $M_1(Q,\xi) = 0$ if $\xi \in (0,1]$, and so assume $\xi > 1$. If $\frac{a'}{q'} \in \mathcal{F}_Q^{(\ell)}$, then $(a',q') = 1$ and $\ell \mid a'$. Since $(a',q') = 1$, we have $(\ell, q') = 1$. Therefore, we have to count all pairs of integers $(q,q') \in (0,Q]^2$ with $q + q' > Q$, $(q,q') = 1$, $qq' \geqslant \frac{Q^2}{\xi}$, in which $(\ell, q') = 1$, and there is an $a' \in \{1, \ldots, q'\}$ such that $a'q \equiv 1 \pmod{q'}$ and $\ell \mid a'$. As a result, after also letting $k = \frac{a'}{\ell}$, $\ell\bar{\ell} \equiv 1 \pmod{q'}$, $M_1(Q,\xi)$ can be expressed as

$$M_1(Q,\xi) = \sum_{\substack{q'=1 \\ (\ell,q')=1}}^{Q} \sum_{\substack{q \in I_Q(q') \\ (q,q')=1}} \sum_{\substack{a'=1 \\ a'q \equiv 1 \ (\mathrm{mod}\ q') \\ \ell \mid a'}}^{q'} 1 = \sum_{\substack{q'=1 \\ (\ell,q')=1}}^{Q} \sum_{\substack{q \in I_Q(q') \\ (q,q')=1}} \sum_{\substack{k \in (0,q'/\ell] \\ kq \equiv \bar{\ell} \ (\mathrm{mod}\ q')}} 1. \tag{2.3}$$

Now by (2.3) and (A.4), for any $\delta > 0$,

$$M_1(Q,\xi) = \sum_{\substack{q'=1 \\ (\ell,q')=1}}^{Q} \left(\frac{\varphi(q')}{q'^2} \iint_{I_Q(q') \times [0,q'/\ell]} dx\, dy + O_\delta\big(q'^{1/2+\delta}\big) \right)$$
$$= \frac{1}{\ell} \sum_{\substack{q'=1 \\ (\ell,q')=1}}^{Q} \frac{\varphi(q')}{q'} |I_Q(q')| + O_{\ell,\delta}\big(Q^{3/2+\delta}\big).$$

Then using (A.2), we have

$$\frac{1}{\ell} \sum_{\substack{q'=1 \\ (\ell,q')=1}}^{Q} \frac{\varphi(q')}{q'} |I_Q(q')| = \frac{C(\ell)}{\ell} \int_0^Q |I_Q(q')|\, dq' + O_\ell(Q \log Q)$$
$$= \frac{C(\ell)}{\ell} A(\xi) Q^2 + O_\ell(Q \log Q).$$

This proves $M_1(Q,\xi) \sim \frac{C(\ell)}{\ell} A(\xi) Q^2$ if $\xi > 1$. The formula for $M_2(Q,\xi)$ is analogous and we infer

$$\begin{aligned} \mathcal{N}_1(Q,\xi) &= N_Q(\xi) - M_1(Q,\xi) - M_2(Q,\xi) \\ &= \left(\frac{1}{\zeta(2)} - \frac{2C(\ell)}{\ell} \right) A(\xi) Q^2 + O_{\ell,\delta}\big(Q^{3/2+\delta}\big). \end{aligned} \tag{2.4}$$

Case 2. There is exactly one fraction in $\mathcal{F}_Q$ between γ and γ' that belongs to $\mathcal{F}_Q^{(\ell)}$. It is more convenient to change γ' to γ'', so we shall consider triples $\gamma < \gamma' < \gamma''$ of elements in $\mathcal{F}_Q$ with $\gamma' \in \mathcal{F}_Q^{(\ell)}$ and with $\gamma'' - \gamma \leqslant \frac{\xi}{Q^2}$. The equalities

$$\frac{a''+a}{a'} = \frac{q''+q}{q'} = K \quad \text{and} \quad \gamma'' - \gamma = \frac{K}{qq''}, \tag{2.5}$$

involving the number

$$K = \nu_2(\gamma) = \left[\frac{Q+q}{q'} \right],$$

called the *index* of the Farey fraction $\gamma = \frac{a}{q} \in \mathcal{F}_Q$, will be useful here. In particular, the inequality $\gamma'' - \gamma \leqslant \frac{\xi}{Q^2}$ enforces $K \leqslant \xi$. Consider the set $J_{Q,K,\xi}(q')$ of elements $q \in (Q-q', Q] \cap \big[Kq' - Q, (K+1)q' - Q\big)$ that satisfy $\frac{K}{q(Kq'-q)} \leqslant \frac{\xi}{Q^2}$. This set is either empty, an interval, or the union of two intervals. The number $\mathcal{N}_2(Q,\xi)$ of gaps of consecutive elements in $\widetilde{\mathcal{F}}_{Q,\ell}$ of length $\leqslant \frac{\xi}{Q^2}$ that arise in this case can now be expressed, with k and $\overline{\ell}$ as in (2.3), as

$$\begin{aligned} \mathcal{N}_2(Q,\xi) &= \sum_{1 \leqslant K \leqslant \xi} \sum_{q' \leqslant Q} \sum_{\substack{q \in J_{Q,K,\xi}(q') \\ (q,q')=1}} \sum_{\substack{a'=1 \\ a'q \equiv 1 \ (\mathrm{mod}\ q') \\ \ell \mid a'}}^{q'} 1 \\ &= \sum_{1 \leqslant K \leqslant \xi} \sum_{\substack{q' \leqslant Q \\ (\ell,q')=1}} \sum_{\substack{q \in J_{Q,K,\xi}(q') \\ k \in (0, q'/\ell] \\ kq \equiv \overline{\ell} \ (\mathrm{mod}\ q')}} 1. \end{aligned} \tag{2.6}$$

We will employ elementary properties of the area preserving invertible transformation $T : \mathcal{T} \to \mathcal{T}$ defined [4] by

$$T(x,y) = \big(y, \kappa(x,y)y - x\big), \qquad (x,y) \in \mathcal{T}, \qquad \text{where} \tag{2.7}$$

$$\mathcal{T} = \{(x,y) \in (0,1]^2 : x+y > 1\} \quad \text{and} \quad \kappa(x,y) = \left[\frac{1+x}{y} \right].$$

An important connection with Farey fractions is given by the equality

$$T\left(\frac{q_i}{Q}, \frac{q_{i+1}}{Q} \right) = \left(\frac{q_{i+1}}{Q}, \frac{q_{i+2}}{Q} \right). \tag{2.8}$$

For each $K \in \mathbb{N}$ consider the subset $\mathcal{T}_K = \{(x,y) \in \mathcal{T} : \kappa(x,y) = K\}$ of $\mathcal{T}$, described by the inequalities $0 < x, y \leqslant 1$, $x + y > 1$, and $Ky - 1 \leqslant x < (K+1)y - 1$.

Denote $V_{Q,K,\xi}(q') = |J_{Q,K,\xi}(q')|$, so $V_{Q,K,\xi}(Qu) = QW_{K,\xi}(u)$, where

$$W_{K,\xi}(u) = \big|\{v : (v,u) \in \mathcal{T}_K\} \cap \{v : K \leqslant \xi v(Ku - v)\}\big|.$$

Similar arguments as in the proof of (2.4) lead to

$$\begin{aligned}\mathcal{N}_2(Q,\xi) &= \frac{C(\ell)}{\ell}\, Q^2 \sum_{K \leqslant \xi} \int_0^1 W_{K,\xi}(u)\, du + O_{\ell,\delta,\xi}(Q^{3/2+\delta}) \\ &= \frac{C(\ell)}{\ell} Q^2 \sum_{K \leqslant \xi} A_K(\xi) + O_{\ell,\delta,\xi}(Q^{3/2+\delta}),\end{aligned}$$

uniformly in ξ on compact subsets of $[0,\infty)$, where

$$A_K(\xi) = \operatorname{Area}\big(\Omega_K(\xi)\big), \qquad \Omega_K(\xi) = \left\{(v,u) \in \mathcal{T}_K : u \geqslant f_{K,\xi}(v) := \frac{v}{K} + \frac{1}{\xi v}\right\}.$$

Summarizing, we have shown

$$N_Q^{(\ell)}(\xi) = G_\ell(\xi) Q^2 + O_{\ell,\xi,\delta}(Q^{3/2+\delta}) \qquad (\text{as } Q \to \infty),$$

where

$$G_\ell(\xi) = \left(\frac{1}{\zeta(2)} - \frac{2C(\ell)}{\ell}\right) A(\xi) + \frac{C(\ell)}{\ell} \sum_{K \leqslant \xi} A_K(\xi). \tag{2.9}$$

Taking also into account Lemma 1 we conclude that the gap limiting measure of $(\widetilde{\mathcal{F}}_{Q,\ell})$ exists and its distribution function is given by

$$\widetilde{F}_\ell(\xi) = \int_0^\xi d\tilde{\nu}_\ell = \frac{1}{\widetilde{K}_\ell} G_\ell\left(\frac{\xi}{\widetilde{K}_\ell}\right).$$

2.1. Explicit Expressions of $A_K(\xi)$

2.1.1. $K = 1$

$\mathcal{T}_1$ is the triangle with vertices $(0,1)$, $(1,1)$, and $\left(\frac{1}{3}, \frac{2}{3}\right)$. When $\xi \leqslant 4$ we have $f_{1,\xi}(v) \geqslant 1$ for every $v > 0$, so $A_1(\xi) = 0$. When $\xi > 4$ we have

$$A_1(\xi) = \int_{u_1}^{u_2} \left(1 - \max\left\{f_{1,\xi}(v), 1 - v, \frac{v+1}{2}\right\}\right) dv,$$

where $u_{1,2} = \frac{1}{2}\big(1 \pm \sqrt{1 - \frac{4}{\xi}}\big)$, $0 < u_1 < u_2 < 1$, are the solutions of $f_{1,\xi}(v) = 1$. When $4 < \xi \leqslant 8$ we have $f_{1,\xi}(v) \geqslant \max\left\{1 - v, \frac{1+v}{2}\right\}$, so $A_1(\xi)$ is the area of the region defined by $v \in [u_1, u_2]$ and $u \in [f_{1,\xi}(v), 1]$. When $\xi \geqslant 8$ let $v_{1,2} =$

$\frac{1}{4}\big(1 \pm \sqrt{1-\frac{8}{\xi}}\big)$, $v_1 < v_2$, denote the solutions of $f_{1,\xi}(v) = 1-v$ and by $w_{1,2} := 2v_{1,2}$ the solutions of $f_{1,\xi}(w) = \frac{w+1}{2}$. If $8 \leqslant \xi \leqslant 9$, then $0 < u_1 < v_1 \leqslant v_2 \leqslant \frac{1}{3} \leqslant w_1 \leqslant w_2 < u_2 < 1$. In this case $A_1(\xi)$ is the area of the region described by $v \in [u_1, v_1] \cup [v_2, w_1] \cup [w_2, u_2]$ and $u \in [f_{1,\xi}(v), 1]$, $v \in [v_1, v_2]$ and $u \in [1-v, 1]$, or $v \in [w_1, w_2]$ and $u \in \big[\frac{1+v}{2}, 1\big]$. Finally, if $\xi > 9$, then $0 < u_1 < v_1 < w_1 < \frac{1}{3} < v_2 < w_2 < u_2 < 1$, and $A_1(\xi)$ is the area of the region described by $v \in [u_1, v_1] \cup [w_2, u_2]$ and $u \in [f_{1,\xi}(v), 1]$, or $v \in \big[v_1, \frac{1}{3}\big]$ and $u \in [1-v, 1]$, or $v \in \big[\frac{1}{3}, w_2\big]$ and $u \in \big[\frac{1+v}{2}, 1\big]$. A plain calculation gives

$$A_1(\xi) = \begin{cases} 0 & \text{if } 0 < \xi \leqslant 4 \\ \frac{1}{2}\sqrt{1-\frac{4}{\xi}} - \frac{1}{\xi}\ln\big(\frac{u_2}{u_1}\big) & \text{if } 4 \leqslant \xi \leqslant 8 \\ \frac{1}{2}\sqrt{1-\frac{4}{\xi}} - \frac{1}{\xi}\ln\big(\frac{u_2}{u_1}\big) - \frac{1}{2}\sqrt{1-\frac{8}{\xi}} + \frac{2}{\xi}\ln\big(\frac{v_2}{v_1}\big) & \text{if } 8 \leqslant \xi \leqslant 9 \\ \frac{1}{2}\sqrt{1-\frac{4}{\xi}} - \frac{1}{\xi}\ln\big(\frac{u_2}{u_1}\big) - \frac{1}{4}\sqrt{1-\frac{8}{\xi}} - \frac{1}{12} + \frac{1}{\xi}\ln\big(\frac{2v_2}{v_1}\big) & \text{if } \xi \geqslant 9. \end{cases}$$

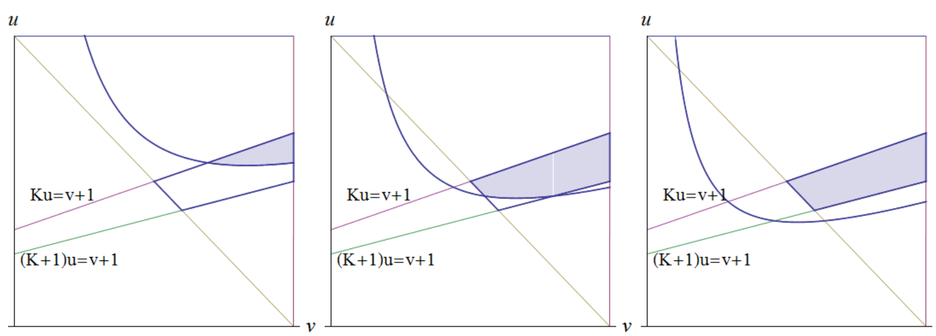

Figure 1: The intersection between the quadrilateral $\mathcal{T}_K$ and the curve $u = f_{K,\xi}(v)$ when $K < \xi < \frac{K(K+1)}{K-1}$, $\frac{K(K+1)}{K-1} \leqslant \xi < \frac{(K+2)^2}{K}$, and respectively $\xi \geqslant \frac{(K+2)^2}{K}$

2.1.2. $K \geqslant 2$

Note that $f_{K,\xi}(1) = f_{K,\xi}\big(\frac{K}{\xi}\big) = \frac{1}{K} + \frac{1}{\xi}$. The situation is described by Figure 1. The solution of $f_{K,\xi}(v) = \frac{v+1}{K}$ is $v = \frac{K}{\xi}$, so the curve $u = f_{K,\xi}(v)$ intersects the upper edge of $\mathcal{T}_K$ if and only if $K < \xi < \frac{K(K+1)}{K-1}$, in which case it does not intersect the two lower edges of $\mathcal{T}_K$ and

$$A_K(\xi) = \int_{K/\xi}^{1} \left(\frac{v+1}{K} - f_{K,\xi}(v)\right) dv = \int_{K/\xi}^{1} \left(\frac{1}{K} - \frac{1}{\xi v}\right) dv.$$

The solution of $f_{K,\xi}\big(\frac{K}{K+2}\big) > \frac{2}{K+2}$ is $\xi < \frac{(K+2)^2}{K}$. This shows that when $\frac{K(K+1)}{K-1} \leqslant \xi < \frac{(K+2)^2}{K}$ the graph of $u = f_{K,\xi}(u)$ intersects the segment $u = 1-v$, $v \in$

$\left[\frac{K-1}{K+1}, \frac{K}{K+2}\right]$, exactly when $v = v_K = \frac{K}{2(K+1)}\left(1+\sqrt{1-\frac{4}{\xi}(1+\frac{1}{K})}\right)$, and the segment $u = \frac{v+1}{K+1}$, $v \in \left[\frac{K}{K+2}, 1\right]$, exactly at $v = w_K = \frac{K}{2}\left(1-\sqrt{1-\frac{4}{\xi}\left(1+\frac{1}{K}\right)}\right)$, so in this case

$$A_K(\xi) = \text{Area}(\mathcal{T}_K) - \int_{v_K}^{w_K} f_{K,\xi}(v)\,dv + \int_{v_K}^{K/(K+2)} (1-v)\,dv + \int_{K/(K+2)}^{w_K} \frac{v+1}{K+1}\,dv.$$

Finally, when $\xi > \frac{(K+2)^2}{K}$, the graph of $u = f_{K,\xi}(v)$ does not intersect any of the edges of $\mathcal{T}_K$ and

$$A_K(\xi) = \text{Area}(\mathcal{T}_K).$$

In summary, a quick calculation leads to

$$A_K(\xi) = \begin{cases} 0 & \text{if } 0 \leqslant \xi \leqslant K \\ \frac{1}{K} - \frac{1}{\xi} - \frac{1}{\xi}\ln\left(\frac{\xi}{K}\right) & \text{if } K \leqslant \xi \leqslant \frac{K(K+1)}{K-1} \\ \frac{K^3+8}{2K(K+1)(K+2)} - \frac{1}{\xi}\ln\left(\frac{w_K}{v_K}\right) - \frac{v_K}{2} + \frac{w_K}{2(K+1)} & \text{if } \frac{K(K+1)}{K-1} \leqslant \xi \leqslant \frac{(K+2)^2}{K} \\ \frac{4}{K(K+1)(K+2)} & \text{if } \xi \geqslant \frac{(K+2)^2}{K}. \end{cases}$$

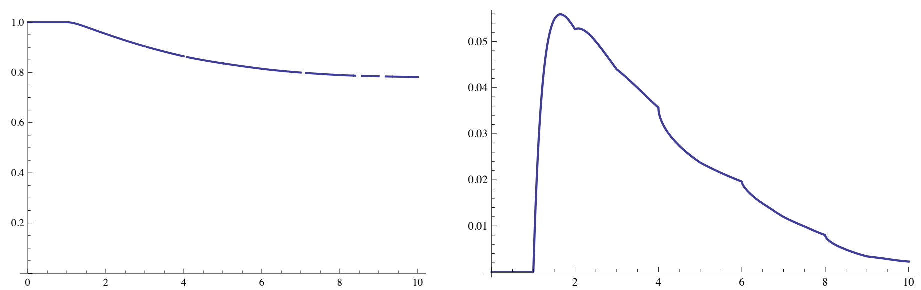

Figure 2: The repartition function $1 - G_3(\xi)$ and the density $-G_3'(\xi)$

3. Consecutive Elements in $\mathcal{F}_Q$ with Denominator Relatively Prime to d

In this section we comment on the first two steps in the proof of (1.1) from [2].

3.1. Upper Bounds on the Number of Consecutive Farey Fractions Whose Denominators Are Not Relatively Prime to d

One of the key steps in the proof of (1.1) in [2] is to show that for any Q and any d, any string of consecutive elements in $\mathcal{F}_Q$ of length $4d^3$ contains at least one element whose denominator is coprime with d. Next we provide two arguments which show that the upper bound $L(d)$ should actually be much smaller than $4d^3$.

Lemma 2. *If $\omega(d) \leqslant \min\{p \in \mathcal{P} : p \mid d\}$, then $L(d) \leqslant 4\omega(d)^3$.*

Proof. We first revisit the proof of the first part of Step (i) in the proof of Theorem 1 in [2] (pp. 210–211). Suppose Q and $i_1 < i_2$ are chosen such that, for every $j \in [i_1, i_2]$,

$$\max\{q_{i_1}, q_{i_2}\} \leqslant q_j \quad \text{and} \quad (q_j, d) > 1.$$

Then $(q_{i_1}, q_{i_2}) = 1$ and

$$\{q_j : i_1 < j < i_2\} \subset \{mq_{i_1} + nq_{i_2} : m, n \in \mathbb{N}, (m, n) = 1, mq_{i_1} + nq_{i_2} \leqslant Q\}. \quad (3.1)$$

Let $d_1 = p_1^{\alpha_1} \cdots p_\omega^{\alpha_\omega}$, with $p_1 < \cdots < p_\omega$ primes, be the largest divisor of d which is coprime to q_{i_1}. Then $\omega < \omega(d) \leqslant \min\{p \in \mathcal{P} : p \mid d\} \leqslant p_1$. Fix some integer L with $\omega + 1 \leqslant L \leqslant p_1$. We claim that there exists $m_1 \in \mathbb{N}$, $m_1 \leqslant L$ such that $(m_1 q_{i_1} + q_{i_2}, d_1) = 1$. If not, then $(\ell q_{i_1} + q_{i_2}, d_1) > 1$ for all $\ell \in \{1, \ldots, L\}$. Since $L > \omega$, the Pigeonhole Principle shows that there exist $i_0 \in \{1, \ldots, \omega\}$ and $1 \leqslant \ell < \ell' \leqslant L$ such that $p_{i_0} \mid (\ell q_{i_1} + q_{i_2})$ and $p_{i_0} \mid (\ell' q_{i_1} + q_{i_2})$, and so $p_{i_0} \mid (\ell' - \ell) q_{i_1}$. But $(p_{i_0}, q_{i_1}) = 1$, hence $L > \ell' - \ell \geqslant p_{i_0} \geqslant p_1$, which contradicts $L \leqslant p_1$.

So if $(m_1 q_{i_1} + q_{i_2}, d) > 1$, then there exists p prime with $p \mid q_{i_1}$ and $p \mid (m_1 q_{i_1} + q_{i_2})$, thus contradicting $(q_{i_1}, q_{i_2}) = 1$. Hence $(m_1 q_{i_1} + q_{i_2}, d) = 1$, which in turn yields $Q \leqslant m_1 q_{i_1} + q_{i_2} \leqslant L q_{i_1} + q_{i_2}$. In a similar way one has $Q \leqslant q_{i_1} + L q_{i_2}$, thus (3.1) leads to

$$\{q_j : i_1 < j < i_2\} \subset \{mq_{i_1} + nq_{i_2} : 1 \leqslant m, n \leqslant L\},$$

and in particular $i_2 - i_1 \leqslant L^2$.

The second part of the proof proceeds ad litteram as in the proof of Step (i) [2, pp. 211–212] replacing d there by L. □

When d is the product of two prime powers the bound above can be lowered. In this case we show that $L(d) \leqslant 5$, which is sharp for $d = 6$ because $\frac{1}{4} < \frac{1}{3} < \frac{1}{2} < \frac{2}{3} < \frac{3}{4}$ are consecutive in $\mathcal{F}_4$. Our proof employs elementary properties of the transformation T from (2.7). In particular (2.8) and the following inclusions will be useful in the proof of Lemma 3:

$$\begin{aligned} &T\mathcal{T}_k \subseteq \mathcal{T}_1 \ \text{ if } k \geqslant 5, && T(\mathcal{T}_3 \cup \mathcal{T}_4) \subseteq \mathcal{T}_1 \cup \mathcal{T}_2, \\ &T\mathcal{T}_2 \subseteq \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{T}_4, && T(T\mathcal{T}_3 \cap \mathcal{T}_2) \subseteq \mathcal{T}_1 \cup \mathcal{T}_2. \end{aligned}$$

Lemma 3. *If $d = p^\alpha q^\beta$, then for each $i \in \{0, \ldots, \#\mathcal{F}_Q - 5\}$ there exists $j \in \{0, \ldots, 5\}$ such that $(q_{i+j}, d) = 1$, and so $L(d) \leqslant 5$.*

Proof. We have $q_{i+2} = Kq_{i+1} - q_i$, $q_{i+3} = K'q_{i+2} - q_{i+1}$, $q_{i+4} = K''q_{i+3} - q_{i+2}$, $q_{i+5} = K'''q_{i+4} - q_{i+3}$, where $K = \kappa\big(\frac{q_i}{Q}, \frac{q_{i+1}}{Q}\big)$, $K' = \kappa\big(\frac{q_{i+1}}{Q}, \frac{q_{i+2}}{Q}\big)$, $K'' = \kappa\big(\frac{q_{i+2}}{Q}, \frac{q_{i+3}}{Q}\big)$, and $K''' = \kappa\big(\frac{q_{i+3}}{Q}, \frac{q_{i+4}}{Q}\big)$. Suppose that $(q_i, d), \ldots, (q_{i+5}, d) > 1$. Then either $p \mid (q_i, q_{i+2}, q_{i+4})$ and $q \mid (q_{i+1}, q_{i+3}, q_{i+5})$, or vice versa.

Without loss of generality we can work in the first case. The equality $q_{i+2}+q_i = Kq_{i+1}$ and $p \nmid q_{i+1}$ yield $p \mid K$. Similarly we have $q \mid K'$. Assume first that $K \geqslant 5$. Since $\left(\frac{q_i}{Q}, \frac{q_{i+1}}{Q}\right) \in \mathcal{T}_K$ and $T\mathcal{T}_K \subseteq \mathcal{T}_1$ we must have $K' = 1$, which contradicts $q \geqslant 2$. In particular $p \geqslant 5$ cannot occur.

When $p = 3$ and $K = 3$, from $T\mathcal{T}_3 \subseteq \mathcal{T}_1 \cup \mathcal{T}_2$ it follows that $K' \in \{1, 2\}$. Since $q \mid K'$, we infer $q = 2$. The region $T\mathcal{T}_3 \cap \mathcal{T}_2$ is the quadrilateral with vertices at $\left(\frac{1}{2}, \frac{1}{2}\right)$, $\left(\frac{2}{5}, \frac{3}{5}\right)$, $\left(\frac{3}{5}, \frac{4}{5}\right)$, and $\left(\frac{3}{7}, \frac{5}{7}\right)$, being further mapped by T into a subset of $\mathcal{T}_1 \cup \mathcal{T}_2$ whence $K'' \in \{1, 2\}$. Again $K'' = 1$ leads to an immediate contradiction, while $K'' = 2$ yields $q_{i+2}+q_{i+4} = 2q_{i+3}$, showing that $p = 2$, another contradiction.

When $p = 2$ and $K < 5$, we have $K \in \{2, 4\}$. Assume first $K = 2$. As $T\mathcal{T}_2 \subseteq \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{T}_4$ and $K' \neq 1$ it remains that $K' \in \{2, 3, 4\}$. Since $q \geqslant 3$ divides K', we infer $q = 3$. Furthermore, $T\mathcal{T}_3 \subseteq \mathcal{T}_1 \cup \mathcal{T}_2$ and $K'' \neq 1$ yield $K'' = 2$. Employing again $T(T\mathcal{T}_3 \cap \mathcal{T}_2) \subseteq \mathcal{T}_1 \cup \mathcal{T}_2$, we infer $K''' = 2$, and so $q_{i+3}+q_{i+5} = 2q_{i+4}$. This is again a contradiction, because 3 divides $q_{i+3}+q_{i+5}$ and cannot divide $2q_{i+4}$. Finally, assume $K = 4$, so $K' \in \{1, 2\}$, which is not possible because $q \geqslant 3$ divides K'. □

Note that if (p_n) is the sequence of primes, then none of the denominators of the fractions in $\mathcal{F}_{p_n} \setminus \{1\}$ are relatively prime to $\prod_{i=1}^{n} p_i$. This gives the lower bound $\#\mathcal{F}_{p_n} - 1$ on the size of the largest string of consecutive fractions in $\mathcal{F}_Q \setminus \mathcal{F}_{Q,d}$ for some $Q, d \in \mathbb{N}$ with $\omega(d) = n$. Since $p_n \sim n \log n$ as $n \to \infty$ and $\#\mathcal{F}_Q \sim \frac{3}{\pi^2} Q^2$ as $Q \to \infty$, there exists $A > 0$ such that $\#\mathcal{F}_{p_n} - 1 \geqslant A(n \log n)^2$. Thus any upper bound on $L(d)$ involving only $\omega(d)$ must be greater than $A(\omega(d) \log \omega(d))^2$.

3.2. The Index and the Continuant

The second step in the proof of (1.1) in [2] relies on [2, Lemma 1], which is actually exactly Remark 2.6 in [6] (see also [4, Lemma 5]), and on a result relating the ℓ-index of a Farey fraction and the continuant of regular continued fractions. The ℓ-index of $\gamma_i = \frac{a_i}{q_i} \in \mathcal{F}_Q$ is the positive integer $\nu_\ell(\gamma_i) = a_{i+\ell-1}q_{i-1} - a_{i-1}q_{i+\ell-1}$ where $\frac{a_{i+k}}{q_{i+k}}$ denotes the k^{th} successor of γ_i in $\mathcal{F}_Q$. The (regular continued fraction) *continuants* are defined as usual by $K_0(\cdot) = 1$, $K_1(x_1) = 1$, and

$$K_\ell(x_1, \ldots, x_\ell) = x_\ell K_{\ell-1}(x_1, \ldots, x_{\ell-1}) + K_{\ell-2}(x_1, \ldots, x_{\ell-2}) \quad \text{if } \ell \geqslant 2.$$

In [10] the identity

$$\nu_\ell(\gamma_i) = \epsilon_\ell K_{\ell-1}\big(-\nu_2(\gamma_i), \nu_2(\gamma_{i+1}), \ldots, (-1)^{\ell-1}\nu_2(\gamma_{i+\ell-2})\big) \tag{3.2}$$

was proved, with $\epsilon_\ell = 1$ if $\ell \in \{0, 1\} \pmod 4$ and $\epsilon_\ell = -1$ if $\ell \in \{2, 3\} \pmod 4$.

We give a very short proof of (3.2). We define the *Farey continuants* K_ℓ^F by $K_0^F(\cdot) = 1$, $K_1^F(x_1) = x_1$, and

$$K_\ell^F(x_1, \ldots, x_\ell) = x_\ell K_{\ell-1}^F(x_1, \ldots, x_{\ell-1}) - K_{\ell-2}^F(x_1, \ldots, x_{\ell-2}) \quad \text{if } \ell \geqslant 2.$$

The defining equalities for K_ℓ and K_ℓ^F plainly yield, for all $\ell \geqslant 2$,

$$\begin{pmatrix} x_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} x_\ell & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} K_\ell(x_1,\ldots,x_\ell) & K_{\ell-1}(x_1,\ldots,x_{\ell-1}) \\ K_{\ell-1}(x_2,\ldots,x_\ell) & K_{\ell-2}(x_2,\ldots,x_{\ell-1}) \end{pmatrix}, \tag{3.3}$$

$$\begin{pmatrix} x_1 & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} x_\ell & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} K_\ell^F(x_1,\ldots,x_\ell) & K_{\ell-1}^F(x_1,\ldots,x_{\ell-1}) \\ -K_{\ell-1}^F(x_2,\ldots,x_\ell) & -K_{\ell-2}^F(x_2,\ldots,x_{\ell-1}) \end{pmatrix}. \tag{3.4}$$

From (3.4) and the definition of $\nu_\ell(\gamma_i)$ we now infer

$$\nu_\ell(\gamma_i) = K_{\ell-1}^F\big(\nu_2(\gamma_i), \nu_2(\gamma_{i+1}), \ldots, \nu_2(\gamma_{i+\ell-2})\big). \tag{3.5}$$

The equality (3.2) follows immediately from (3.3), (3.4), (3.5) and

$$\begin{pmatrix} x & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y & 1 \\ -1 & 0 \end{pmatrix} = -\begin{pmatrix} -x & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y & 1 \\ 1 & 0 \end{pmatrix}.$$

4. The Gap Distribution of $\mathcal{F}_{Q,d}$

Letting $d \in \mathbb{N}$ and $\xi > 0$, we wish to asymptotically estimate the number of pairs of consecutive elements $\gamma < \gamma'$ in $\mathcal{F}_{Q,d}$ with $\gamma' - \gamma \leqslant \frac{\xi}{Q^2}$ as $Q \to \infty$. It is plain that

$$\#\mathcal{F}_{Q,d} = \sum_{\substack{q=1 \\ (q,d)=1}}^{Q} \varphi(q) = C(d) \int_0^Q q\, dq + O_d(Q \log Q) = \frac{C(d)}{2} Q^2 + O_d(Q \log Q),$$

showing the second equality in (1.2). Denote $N_Q = \#\mathcal{F}_Q$ and $\gamma_j = \frac{a_j}{q_j}$, so the number of pairs of fractions we wish to estimate is

$$\begin{aligned} N_d(Q,\xi) &= \sum_{\ell=1}^{L(d)} \# \left\{ i \in [1, N_Q] : \begin{array}{c} \frac{\nu_\ell(\gamma_i)}{q_{i-1} q_{i+\ell-1}} \leqslant \frac{\xi}{Q^2}, \quad (q_{i-1}, d) = (q_{i+\ell-1}, d) = 1 \\ (q_i, d) > 1, \ldots, (q_{i+\ell-2}, d) > 1 \end{array} \right\} \\ &= \sum_{\ell=1}^{L(d)} \sum_{k=1}^{[\xi]} \# \left\{ i \in [1, N_Q] : \begin{array}{c} \frac{k}{q_{i-1} q_{i+\ell-1}} \leqslant \frac{\xi}{Q^2}, \quad \nu_\ell(\gamma_i) = k \\ (q_{i-1}, d) = (q_{i+\ell-1}, d) = 1 \\ (q_i, d) > 1, \ldots, (q_{i+\ell-2}, d) > 1 \end{array} \right\}. \end{aligned}$$

It is shown in [4, 5] that given $i \in [1, N_Q]$ and $k, \ell \in \mathbb{N}$ with $\ell \geqslant 2$, if $\nu_\ell(\gamma_i) = k$, then the $(\ell - 1)$-tuple $\big(\nu_2(\gamma_i), \ldots, \nu_2(\gamma_{i+\ell-2})\big)$ can take on $n(k,\ell)$ values, where $n(k,\ell) \in \mathbb{N} \cup \{0\}$ depends only on k and ℓ and not on i or Q; and in [10], it is proven that $\nu_\ell(\gamma_i)$ can be determined if $\big(\nu_2(\gamma_i), \ldots, \nu_2(\gamma_{i+\ell-2})\big)$ is known (cf. identity (3.2) above). Therefore, letting $\{x(k,\ell,m)\}_{m=1}^{n(k,\ell)}$ be the $(\ell-1)$-tuples for which $\nu_\ell(\gamma_i) = k$ whenever $x(k,\ell,m) = \big(\nu_2(\gamma_i), \ldots, \nu_2(\gamma_{i+\ell-2})\big)$ for some $m \in \{1, \ldots, n(k,\ell)\}$, we have

$$N_d(Q,\xi) = \#\left\{ i \in [1,N_Q] : (q_{i-1},d) = (q_i,d) = 1, \quad q_{i-1}q_i \geqslant \frac{Q^2}{\xi} \right\}$$
$$+ \sum_{\ell=2}^{L(d)} \sum_{k=1}^{[\xi]} \sum_{m=1}^{n(k,\ell)} \# \left\{ i \in [1,N_Q] : \begin{array}{c} q_{i-1}q_{i+\ell-1} \geqslant \frac{kQ^2}{\xi},\ (q_{i-1},d) = (q_{i+\ell-1},d) = 1 \\ (q_i,d) > 1, \ldots, (q_{i+\ell-2},d) > 1 \\ x(k,\ell,m) = (\nu_2(\gamma_i), \ldots, \nu_2(\gamma_{i+\ell-2})) \end{array} \right\}.$$

Since $q_{j+1} = \nu_2(\gamma_j)q_j - q_{j-1}$ for $j \in [1, N_Q - 1]$, the residue classes of the denominators $q_{i-1}, \ldots, q_{i+\ell-1}$ can be determined once the residue classes of q_{i-1} and q_i, and the $(\ell-1)$-tuple $\big(\nu_2(\gamma_i), \ldots, \nu_2(\gamma_{i+\ell-2})\big)$ are known. Thus, there is a subset $\mathcal{A}_{k,\ell,m} \subseteq \{1,\ldots,d\}^2$ such that when $\big(\nu_2(\gamma_i), \ldots, \nu_2(\gamma_{i+\ell-2})\big) = x(k,\ell,m)$, we have $(q_{i-1},d) = (q_{i+\ell-1},d) = 1$ and $(q_{i+j-1},d) > 1$ for $1 \leqslant j < \ell$ if and only if $(q_{i-1},q_i) \pmod d \in \mathcal{A}_{k,\ell,m}$. (Note clearly that $(a,d) = 1$ for $(a,b) \in \mathcal{A}_{k,\ell,m}$.) Furthermore, if we let $x(k,\ell,m) = \big(x_1(k,\ell,m), \ldots, x_{\ell-1}(k,\ell,m)\big)$ and denote $\mathbb{Z}^2_{\text{vis}} = \{(a,b) \in \mathbb{Z}^2 : (a,b) = 1\}$, it is clear that $\big(\nu_2(\gamma_i), \ldots, \nu_2(\gamma_{i+\ell-2})\big) = x(k,\ell,m)$ if and only if

$$(q_{i-1},q_i) \in Q \cdot (\mathcal{T}_{x_1(k,\ell,m)} \cap T^{-1}\mathcal{T}_{x_2(k,\ell,m)} \cap \cdots \cap T^{-(\ell-2)}\mathcal{T}_{x_{\ell-1}(k,\ell,m)}) \cap \mathbb{Z}^2_{\text{vis}}.$$

Now if we let $\pi_1, \pi_2 : \mathbb{R}^2 \to \mathbb{R}$ be the canonical projections, then

$$\frac{q_{i-1}q_{i+\ell-1}}{Q^2} = \pi_1\left(\frac{q_{i-1}}{Q}, \frac{q_i}{Q}\right) \cdot (\pi_2 \circ T^{\ell-1})\left(\frac{q_{i-1}}{Q}, \frac{q_i}{Q}\right),$$

and so

$$q_{i-1}q_{i+\ell-1} \geqslant \frac{kQ^2}{\xi} \quad \Longleftrightarrow \quad (q_{i-1},q_i) \in Qg_\ell^{-1}\left[\frac{k}{\xi}, \infty\right),$$

where $g_\ell = \pi_1 \cdot (\pi_2 \circ T^{\ell-1})$. Now set $g_1(x,y) = xy$ and

$$\Omega_{k,\ell,m}(\xi) = \mathcal{T}_{x_1(k,\ell,m)} \cap T^{-1}\mathcal{T}_{x_2(k,\ell,m)} \cap \cdots \cap T^{-(\ell-2)}\mathcal{T}_{x_{\ell-1}(k,\ell,m)} \cap g_\ell^{-1}\left[\frac{k}{\xi}, \infty\right),$$
$$\Omega_1(\xi) = \mathcal{T} \cap g_1^{-1}\left[\frac{1}{\xi}, \infty\right), \qquad \mathcal{A}_1 = \{(a,b) : a,b \in [1,d], (a,d) = (b,d) = 1\}.$$

We then have

$$N_d(Q,\xi) = \sum_{(a,b)\in\mathcal{A}_1} \#Q\Omega_1(\xi) \cap \big((a,b) + d\mathbb{Z}^2\big) \cap \mathbb{Z}^2_{\text{vis}}$$
$$+ \sum_{\ell=2}^{L(d)} \sum_{k=1}^{[\xi]} \sum_{m=1}^{n(k,\ell)} \sum_{(a,b)\in\mathcal{A}_{k,\ell,m}} \#Q\Omega_{k,\ell,m}(\xi) \cap \big((a,b) + d\mathbb{Z}^2\big) \cap \mathbb{Z}^2_{\text{vis}},$$

where we have used the fact that if $(a,b) \in Q\mathcal{T} \cap \mathbb{Z}^2_{\text{vis}}$, then there is an i such that $a = q_{i-1}$ and $b = q_i$. One can prove in a similar manner to [2, Lemma 2] that for all bounded $\Omega \subseteq \mathbb{R}^2$ whose boundary can be covered by the images of finitely many Lipschitz functions from $[0,1]$ to $\mathbb{R}^2$, and for all $\mathcal{A} \subseteq \{1,\ldots,d\}^2$ in which $(a,d)=1$ for all $(a,b)\in\mathcal{A}$, we have

$$\sum_{(a,b)\in\mathcal{A}} \#Q\Omega\cap\big((a,b)+d\mathbb{Z}^2\big)\cap\mathbb{Z}^2_{\mathrm{vis}} = \frac{\mathrm{Area}(\Omega)\#\mathcal{A}}{\zeta(2)d^2}\prod_{\substack{p\in\mathcal{P}\\ p\mid d}}\left(1-\frac{1}{p^2}\right)^{-1} Q^2 + O_d(Q\log Q)$$

as $Q\to\infty$. It is easily seen that the boundaries of $\Omega_1(\xi)$ and $\Omega_{k,\ell,m}(\xi)$ can be covered by finitely many Lipschitz functions from $[0,1]$ to $\mathbb{R}^2$, and so we have

$$N_d(\xi,Q) = C_d(\xi)Q^2 + O_d(Q\log Q),$$

where

$$C_d(\xi) = \frac{1}{\zeta(2)d^2}\prod_{\substack{p\in\mathcal{P}\\ p\mid d}}\left(1-\frac{1}{p^2}\right)^{-1} \cdot\left(\varphi(d)^2\,\mathrm{Area}(\Omega_1(\xi)) + \sum_{\ell=2}^{L(d)}\sum_{k=1}^{[\xi]}\sum_{m=1}^{n(k,\ell)}\mathrm{Area}(\Omega_{k,\ell,m}(\xi))\#\mathcal{A}_{k,\ell,m}\right),$$

noting that $\#\mathcal{A}_1=\varphi(d)^2$.

The gap limiting measure of $(\mathcal{F}_{Q,d})_Q$ exists with distribution function given by

$$F_d(\xi) = \int_0^{\xi} d\nu_d = \frac{1}{K_d}C_d\left(\frac{\xi}{K_d}\right).$$

When d is a prime power this can be expressed more explicitly as in (2.9).

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A. Appendix

For the convenience of the reader we collect in this appendix the asymptotic formulas used in this paper.

Assuming that f is a C^1 function on the interval of integration in (A.1)-(A.3) and that I, J are intervals and $f \in C^1(I \times J)$ in (A.4), we have

$$\sum_{\substack{a<k\leqslant b\\(k,q)=1}} f(k) = \frac{\varphi(q)}{q} \int_a^b f(x)\,dx + O\Big(\sigma_0(q)\big(\|f\|_\infty + T_a^b f\big)\Big). \tag{A.1}$$

$$\sum_{\substack{1\leqslant k\leqslant N\\(k,q)=1}} \frac{\varphi(k)}{k} f(k) = C(\ell) \int_0^N f(x)\,dx + O_\ell\Big(\big(\|f\|_\infty + T_0^N f\big)\log N\Big). \tag{A.2}$$

$$\sum_{1\leqslant k\leqslant N} \frac{\varphi(\ell k)}{k} f(k) = \ell C(\ell) \int_0^N f(x)\,dx + O_{\ell,\delta}\Big(\big(\|f\|_\infty + T_0^N f\big)N^\delta\Big). \tag{A.3}$$

$$\begin{aligned}\sum_{\substack{a\in I, b\in J\\ ab\equiv h \pmod q\\(b,q)=1}} f(a,b) = &\frac{\varphi(q)}{q^2} \iint_{I\times J} f(x,y)\,dxdy + O_\delta\big(T^2\|f\|_\infty q^{1/2+\delta}(h,q)^{1/2}\big)\\ &+ O_\delta\Big(T\|\nabla f\|_\infty q^{3/2+\delta}(h,q)^{1/2} + \frac{1}{T}\|\nabla f\|_\infty |I|\cdot|J|\Big).\end{aligned} \tag{A.4}$$

Proofs can be found for instance in [3, Lemma 2.2], [5, Lemmas 2.1 and 2.2], and respectively in [7, Proposition A4].