



COMPOSITIONS CONSTRAINED BY GRAPH LAPLACIAN
MINORS

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Abstract

Motivated by examples of symmetrically constrained compositions, super convex partitions, and super convex compositions, we initiate the study of partitions and compositions constrained by graph Laplacian minors. We provide a complete description of the multivariate generating functions for such compositions in the case of trees. We answer a question due to Corteel, Savage, and Wilf regarding super convex compositions, which we describe as compositions constrained by Laplacian minors for cycles; we extend this solution to the study of compositions constrained by Laplacian minors of leafed cycles. Connections are established and conjectured between compositions constrained by Laplacian minors of leafed cycles of prime length and algebraic/combinatorial properties of reflexive simplices.

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1. Introduction

A *partition* of a positive integer n is a sequence $\lambda := (\lambda_1, \lambda_2, \dots, \lambda_k)$ of non-negative integers satisfying $\lambda_k \leq \lambda_{k-1} \leq \dots \leq \lambda_1$ such that $\sum_i \lambda_i = n$. We call each λ_i a *part* of λ and say that λ has k *parts*. A (*weak*) *composition* of n is a sequence $\lambda := (\lambda_1, \lambda_2, \dots, \lambda_k)$ of non-negative integers satisfying $\sum_i \lambda_i = n$. Beginning with [1] and through eleven subsequent papers, George Andrews and various coauthors successfully revived Percy MacMahon's technique of *Partition Analysis* (PA) from obscurity as a tool for computing generating functions for partitions and compositions. PA is particularly well-suited to the study of partitions and compositions constrained by a linear system of inequalities, i.e., of the form $A\lambda \geq b$ for some integral matrix A and integral vector b ; this is equivalent to the study of integer points in a rational polyhedral cone. Motivated in part by this renewed interest in PA, there has been growing interest in the application of polyhedral-geometric techniques to the study of integer partitions and compositions.

For example, in [21], Igor Pak considers several partition theory problems from the point of view of lattice-point enumeration, leading to the establishment of unique bijective correspondences. In [4], Matthias Beck, Ira Gessel, Sunyoung Lee, and Carla Savage consider the general case of determining integer-point transforms of symmetrically constrained families of compositions; their techniques include both unimodular decompositions of cones and the use of permutation statistics. In [11], Katie Bright and Carla Savage produce a bijection between certain “boxes of width t ” of lecture hall partitions and integer points in the cube $[0, t]^d$; the methods for defining this bijection are motivated via geometry and again involve permutation statistics. In [3], the first author, Matthias Beck, and Nguyen Le give geometric proofs of several of the results in the PA papers by George Andrews et al; these proofs often produce multivariate generating functions in cases where PA produced only univariate identities. Finally, motivated by [4] and [11], the first author and Matthias Beck use geometric techniques to establish new multivariate Euler-Mahonian distributions in [10].

This paper continues this line of investigation by initiating the explicit study of partitions and compositions constrained by graph Laplacian minors. Algebraic graph theory is a rich source of interesting constraint matrices, the first being the signed incidence matrix of a graph, defined as follows. Let $G = (\{0, 1, \dots, n\}, E)$ be a simple graph with $n + 1$ vertices and $|E| = m$ edges.

Definition 1.1. For any fixed edge orientation ε , the *signed vertex-edge incidence matrix* of G with respect to ε is the $(n + 1) \times m$ matrix ∂ , with rows indexed by vertices of G and columns indexed by edges of G , whose (v, e) -th entry is 1 if v is the positive end of e , -1 if v is the negative end of e , and 0 otherwise.

The study of compositions with constraint matrices given by ∂^T for various graphs was recently advanced in [13] by using five guidelines from MacMahon's

partition analysis along with the Omega software package. Additionally, such compositions often fall within the realm of the theory of P -partitions [24, Chapter 4]. Our goal is to extend this line of investigation by using as constraint matrices various minors of graph Laplacians, defined as follows.

Definition 1.2. The *vertex Laplacian* of G is $L := D - A$, where D is the $(n + 1) \times (n + 1)$ matrix indexed by the vertices of G such that the matrix entry $d_{i,j}$ is the degree of vertex i if $i = j$ and 0 otherwise and A is the 0/1-valued adjacency matrix of G indexed in the same way as D . Given a graph G , the i -th *Laplacian minor* of G is L_i , the Laplacian matrix L with row i and column i deleted.

The connection between the incidence matrix of a graph and its Laplacian is the well-known equation $L = \partial\partial^T$ [8, Chapter 6]. Kirchoff's Matrix-Tree Theorem [8] asserts that given a graph G on $\{0, \dots, n\}$ and given any i between 0 and n , $\det(L_i)$ is equal to the number of labeled spanning trees of G . This establishes a strong connection between Laplacian minors and enumerative properties of graphs.

There are several examples of partitions and compositions occurring in the literature on partition and composition enumeration that are constrained by graph Laplacian minors. The following three such examples provide further motivation for the study of these objects; note that these examples were not originally introduced in the context of graph Laplacian minors.

Example 1.3. Let K_{n+1} be the complete graph on $\{0, 1, \dots, n\}$. The compositions determined by $L_0\lambda \geq 0$ are one example of the class of *symmetrically constrained compositions*. A study of multivariate generating functions for symmetrically constrained compositions was given by Beck, Gessel, Lee, and Savage in [4].

Example 1.4. Let P_n be the path graph with vertices $\{0, 1, \dots, n - 1\}$. The partitions determined by $L_0\lambda \geq 0$ are named *super convex partitions* by Corteel, Savage, and Wilf in Example 3 of [12]; they were first studied using techniques from MacMahon's Partition Analysis by Andrews in [1], where they are named *partitions with mixed difference conditions*. The multivariate generating function for such partitions is described in [12].

Example 1.5. Let C_n be the cycle on vertices $\{0, 1, \dots, n - 1\}$. The compositions determined by $L_0\lambda \geq 0$ were introduced by Corteel, Savage, and Wilf in the concluding remarks of [12], where they are named *super convex compositions*. The determination of the multivariate generating function for super convex compositions was given as an open problem at the end of [12].

We now introduce our primary object of study.

Definition 1.6. Given a graph G with Laplacian L and a vertex i of G , the *compositions constrained by the i -th Laplacian minor of G* are those satisfying $L_i\lambda \geq 0$.

Our paper is structured as follows. In Section 2, we review techniques from integer point enumeration in polyhedral cones that will be useful for producing generating function identities. In Section 3, we investigate compositions constrained by Laplacian minors for trees. Theorem 3.1 gives a combinatorial interpretation for the entries of the matrix inverse for tree Laplacian minors, leading to Corollary 3.2, the main result of the section.

In Section 4, we provide a solution to the open problem of Corteel, Savage, and Wilf discussed in Example 1.5. However, as shown in the section, our solution is less clear than one might want. Thus, motivated by the underlying graphical structure of our compositions, in Section 5 we examine compositions constrained by Laplacian minors for leafed cycles, i.e., for cycles with an additional vertex forming a leaf appended. These compositions are closely related to the super convex compositions in Example 1.5. The addition of a leaf to the cycle causes a change in the structure of the resulting compositions, leading to Theorem 5.1. We also consider in Conjecture 5.5 a possible combinatorial interpretation for the compositions constrained by leafed cycles, and we prove special cases of this conjecture.

In Section 6, we consider generating functions for certain subfamilies of compositions constrained by Laplacian minors for leafed cycles of prime length. Surprisingly, these generating functions are Ehrhart series for a family of reflexive simplices, as shown in Theorem 6.3. After introducing Ehrhart series and reflexive polytopes and proving Theorem 6.3, we indicate some possible connections between our enumerative problem and the algebraic/combinatorial structure of these simplices.

We close in Section 7 with a conjecture regarding compositions constrained by Laplacian minors for leafed cycles of length 2^k .

2. Cones and Generating Functions

The problem of enumerating integer compositions and partitions satisfying a system of linear constraints $A\lambda \geq b$ is identical to that of enumerating integer points in the polytope or polyhedral cone defined by these constraints. For further background related to integer point enumeration in polyhedra, see [6].

Given a square matrix A with $\det(A) \neq 0$, the solution set

$$\mathcal{C} = \{x \in \mathbb{R}^n : Ax \geq 0\}$$

is called an *n-dimensional simplicial cone*; the reason for the use of “simplicial” comes from \mathcal{C} being an n -dimensional cone with n bounding hyperplanes, also called *facets*. The following lemma, whose proof is straightforward, indicates that simplicial cones can be described either by specifying these n facets or by specifying the n ray generators for the cone.

Lemma 2.1. *Let A be an invertible matrix in $\mathbb{R}^{n \times n}$. Then*

$$\{x \in \mathbb{R}^n : Ax \geq 0\} = \{A^{-1}y : y \geq 0\}$$

where each inequality is understood componentwise.

We encode the integer points in a cone \mathcal{C} using the *integer point transform* of \mathcal{C} given by the multivariate generating function

$$\sigma_{\mathcal{C}}(z_1, z_2, \dots, z_n) := \sum_{m \in \mathcal{C} \cap \mathbb{Z}^n} z_1^{m_1} z_2^{m_2} \dots z_n^{m_n}.$$

This function encodes the integer points in \mathcal{C} as a formal sum of monomials. Throughout this paper, we will typically need to specialize $\sigma_{\mathcal{C}}$ in one of two ways. If we set $z_1 = z_2 = \dots = z_n = q$, we obtain

$$\sigma_{\mathcal{C}}(q, q, \dots, q) = \sigma_{\mathcal{C}}(q) = \sum_{m \in \mathcal{C} \cap \mathbb{Z}^n} q^{|m|},$$

where $|m| = m_1 + m_2 + \dots + m_n$. Setting $z_1 = q$ and $z_2 = z_3 = \dots = z_n = 1$, we obtain

$$\sigma_{\mathcal{C}}(q, 1, \dots, 1) = \sigma_{\mathcal{C}}(q) = \sum_{m \in \mathcal{C} \cap \mathbb{Z}^n} q^{m_1}.$$

When A is an integer matrix having $\det(A) = 1$, the ray generating matrix A^{-1} for the n -dimensional simplicial cone $\mathcal{C} = \{x \in \mathbb{R}^n : Ax \geq 0\}$ has integer entries; hence, every integer point in \mathcal{C} can be expressed as $A^{-1}y$ for a unique integer vector $y \geq 0$. In this case, the integer point transform can be simplified as a product of geometric series as follows.

Lemma 2.2. *Let $A = (a_{i,j})$ be an integer matrix with $\det(A) = 1$ and inverse matrix $A^{-1} = (b_{i,j})$. Then*

$$\sigma_{\mathcal{C}}(z_1, z_2, \dots, z_n) := \frac{1}{\prod_{j=1}^n (1 - z_1^{b_{1,j}} z_2^{b_{2,j}} \dots z_n^{b_{n,j}})}$$

More generally, suppose that $\det(A) \neq 1$. In this case, $\sigma_{\mathcal{C}}(z_1, z_2, \dots, z_n)$ is still rational, but A^{-1} need not have integer entries. As a result, representations of integer points in \mathcal{C} are not necessarily obtained by using integral combinations of the minimal integral vectors on the generating rays of \mathcal{C} . We can work around this difficulty in the following manner.

Let v_1, v_2, \dots, v_n be the (integral) ray generators of \mathcal{C} . Define the (*half-open*) *fundamental parallelepiped* (FPP) to be

$$\Pi = \{\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n : 0 \leq \lambda_1, \lambda_2, \dots, \lambda_n < 1\}.$$

The cone \mathcal{C} can be tiled by integer translates of Π , but there may be integer points contained in Π that can only be obtained using strictly rational scaling factors of the ray generators of \mathcal{C} . This leads to our main tool for computing generating functions.

Lemma 2.3. *Using the preceding notation, if w_1, w_2, \dots, w_m are the integer points contained in Π , then the integer point transform of \mathcal{C} is given by*

$$\sigma_{\mathcal{C}}(z_1, z_2, \dots, z_n) := \frac{\sum_{i=1}^m z_1^{w_{i,1}} z_2^{w_{i,2}} \dots z_n^{w_{i,n}}}{\prod_{j=1}^n (1 - z_1^{v_{1,j}} z_2^{v_{2,j}} \dots z_n^{v_{n,j}})}. \tag{1}$$

3. Trees

We begin our study of compositions constrained by graph Laplacian minors with the special case of trees. Since a tree T has a unique spanning tree, namely itself, the Matrix-Tree Theorem implies that $\det(L_i) = 1$ for any vertex i . Thus, compositions constrained by tree Laplacian minors will have a generating function of the form

$$\sigma_{\mathcal{C}}(z_1, z_2, \dots, z_n) = \frac{1}{\prod_{j=1}^n (1 - z_1^{b_{1,j}} z_2^{b_{2,j}} \dots z_n^{b_{n,j}})}.$$

Since the $b_{k,j}$'s arise as the entries in the inverse matrix for L_i , we are interested in calculating this inverse.

Observe that without loss of generality we may consider only those vertices i which are leaves of T . This is due to the fact that taking the minor at a non-leaf vertex produces a matrix L_i which is a block matrix. Each individual block consists of a Laplacian minor from a tree formed by re-attaching vertex i to a connected component of $T \setminus \{i\}$, then taking the minor of the resulting tree at i . Thus, the generating function for compositions constrained by L_i is the product of those for the compositions constrained by the blocks of L_i .

3.1. Inverses of Laplacian Minors for Trees

To determine the denominator of $\sigma_{\mathcal{C}}$ for an arbitrary tree T minored at a leaf, we first find a combinatorial interpretation for the columns of the inverse of the Laplacian minor for T . Throughout this section, we assume that T is a tree on the vertex set $\{0, \dots, n\}$ with an arbitrary orientation of the edges of T . The following theorem is a special case of Corollary 4.3 in [26]; we include a simple proof for the sake of completeness.

Theorem 3.1. *Consider a tree T and its corresponding Laplacian matrix L . Minor at a leaf n of T and let $L_n^{-1} = (l_{i,j})$. Then entry $l_{i,j}$ is the distance from n to the path connecting the vertices i and j . Equivalently, $l_{i,j}$ is the length of the intersection of the path from n to i with the path from n to j .*

Theorem 3.1 has the following implication on the level of generating functions. Let $\sigma_{\mathcal{C}}(q)$ denote the specialized generating function $\sigma_{\mathcal{C}}(q, q, \dots, q)$.

Corollary 3.2. *The specialized generating function for a tree T minored at leaf n is given by*

$$\sigma_c(q) = \frac{1}{\prod_{i=0}^{n-1} (1 - q^{b_i})},$$

where b_i is the sum of the distances between n and the paths connecting vertex i with the other vertices of T .

For example, let G have vertex set $\{1, 2, \dots, 7\}$ and edges $\{1, 2\}$, $\{2, 3\}$, $\{3, 4\}$, $\{4, 5\}$, $\{3, 6\}$, and $\{6, 7\}$. Thus, G has three branches of length two emanating from 3. If $n = 7$, then $b_1 = 14 = 1 + 2 + 2 + 2 + 3 + 4$, where for example the distance from 7 to the path connecting 1 to 4 contributes a summand of two, since the path from 1 to 4 is $(1, 2, 3, 4)$ and the path from 7 to this path is $(7, 6, 3)$.

To prove Theorem 3.1, we return to the fact that $L = \partial\partial^T$ and proceed combinatorially. We say the n -th subminor of ∂ (resp. of ∂^T) imposed by a vertex n of a tree is the square matrix in which the n -th row, (resp. column) is deleted. We denote the n -th subminor of ∂ as ∂_n , with the understanding that ∂_n^T is $(\partial_n)^T$. It is easy to see that since $L = \partial\partial^T$, we have $L_n = \partial_n\partial_n^T$; thus, $L_n^{-1} = (\partial_n^T)^{-1}\partial_n^{-1}$.

Proposition 3.3. *Let $G_n = (g_{e,j})$, $0 \leq j \leq n - 1$, where $g_{e,j}$ is determined as follows:*

- 1 if edge e is on the path between j and n and e points away from n ;
- -1 if edge e is on the path between j and n and e points towards n ;
- 0 if edge e is not on the path between j and n .

Then $G_n\partial_n = I$, hence $\partial_n^{-1} = G_n$.

Proof. Set $G_n\partial_n = M = (m_{i,j})$ and first consider $m_{i,i}$. Entries of row i of ∂_n are non-zero for each edge, i.e., column, adjacent to i . The path connecting vertices i and n overlaps with exactly one of the edges adjacent to i , which is reflected in G_n . By construction, the entries in ∂_n and G_n corresponding to this vertex/edge pairing have the same sign. So $m_{i,i} = 1$ for all i .

Now consider $m_{i,j}$ with $i \neq j$. The path from j to n overlaps with either no edges or exactly two edges connected to i . If there is no overlap, then the product of row i in G_n and column j in ∂_n is zero. If there is overlap, suppose both edges point either toward or away from n . Then the entries in G_n have the same sign, but those in ∂_n have different signs, thus summing to zero. A similar canceling occurs if the edges point in different directions. So, $m_{i,j} = 0$ if $i \neq j$, implying $M = I$. Therefore, $G_n = \partial_n^{-1}$. \square

Proof of Theorem 3.1. Since $L_n^{-1} = (\partial_n^T)^{-1}\partial_n^{-1}$, we may multiply columns i and j in ∂_n^{-1} component-wise to find $l_{i,j}$. Consider an arbitrary edge of the graph. If the

edge belongs to only one of the paths from i to n or from j to n , then there will be a zero corresponding to that edge in either column i or j , contributing nothing to entry $l_{i,j}$. Otherwise we obtain a 1, since the sign of a row for any edge is constant. Thus, summing the component-wise product will record the length of the path from n to the path between i and j . \square

Theorem 3.1 provides some insight into the form of the generating function for compositions constrained by an arbitrary tree. However, determining the lengths of paths between vertices in trees is not an easy exercise in general. In the next subsection, we analyze a special case.

3.2. k -ary Trees

We next provide an explicit formula for a specialized generating function for compositions constrained by Laplacian minors of k -ary trees. Recall that a k -ary tree is a tree formed from a root node by adding k leaves to the root, then adding k leaves to each of these vertices, etc., stopping after a finite number of iterations of this process. To study the compositions constrained by a Laplacian minor of a k -ary tree T , we add a leaf labeled 0 to the root of T and minor at this new vertex. We define the *level* of a vertex as its distance to vertex 0 and a *subtree* of a vertex as the tree which includes the vertex and all of its children (we assume all edges are oriented away from vertex 0). Recall that $[n]_k := 1 + k + k^2 + k^3 + \dots + k^{n-1}$.

Theorem 3.4. *For a k -ary tree with n levels,*

$$\sigma_c(q) = \frac{1}{\prod_{j=1}^n \left(1 - q^{j[n-j+1]_k + \sum_{i=1}^{j-1} (j-i)k^{n-(j-i)}} \right)^{k^{j-1}}}.$$

Proof. Suppose a k -ary tree T is given and consider a column of L_0^{-1} corresponding to a vertex $v \in T$. This column sum will be the sum of the distances between 0 and all distinct paths in the tree with v as an endpoint. By the symmetry of k -ary trees, we choose an arbitrary vertex v at level $j \in [n]$, since the corresponding column sum will be the same as any other vertex on that level.

We must show that $j[n-j+1]_k + \sum_{i=1}^{j-1} (j-i)k^{n-(j-i)}$ is the column sum corresponding to any vertex at level j . For any vertex c that is a descendent of v (and including v), the distance between 0 and the path from v to c will be j . There are

$$1 + k + k^2 + \dots + k^{n-j} = \frac{k^{n-j+1} - 1}{k - 1} = [n-j+1]_k$$

such vertices and each of these vertices contributes j to the column sum, yielding our first summand above.

Consider the path between 0 and v , $P = \{v = w_0, w_1, w_2, \dots, w_{j-1}, 0\}$. For each $i \in [j - 1]$, define S_i to be the set of vertices which are in the subtree with w_i as the root but not in the subtree with w_{i-1} as the root; note that the vertex w_i is on level $j - i$ of T . There are $k^0 + k^1 + \dots + k^{n-(j-i)}$ vertices in the subtree with w_i as a root, and $k^0 + k^1 + \dots + k^{n-(j-i)-1}$ vertices in the subtree of w_{i-1} , yielding

$$k^0 + k^1 + \dots + k^{n-(j-i)} - (k^0 + k^1 + \dots + k^{n-(j-i)-1}) = k^{n-(j-i)}$$

vertices in S_i . Since the path connecting v and s passes through w_i , which is the closest point on this path to 0, the shortest distance between 0 and the path connecting v and any vertex $s \in S_i$ is $j - i$. Hence, the column sum corresponding to an element in S_i will be $(j - i)(k^{n-(j-i)})$, yielding $\sum_{i=1}^{j-1} (j - i)(k^{n-(j-i)})$ as the total column sum for a vertex not a descendent of v , which is our second desired summand. The exponents in the denominator of our generating function follow after noting that there are k^{j-1} vertices on the j -th level. \square

When $k = 2$, the formulas for our exponents simplify nicely, as follows.

Corollary 3.5. *For a binary tree with n levels,*

$$\sigma_C(q) = \frac{1}{\prod_{j=1}^n \left(1 - q^{2^{n-j+1}}(2^j - 1) - j\right) 2^{j-1}}$$

We omit the proof of Corollary 3.5, as it results from a tedious calculation where the key step is the use of $\sum_{i=1}^{j-1} (j - i)2^i = \sum_{i=1}^{j-1} (2 + 2^2 + \dots + 2^i)$.

4. n -Cycles

In this section we consider the open problem due to Corteel, Savage, and Wilf given in Example 1.5. Recall that the Laplacian minor for a general n -cycle on vertices $1, 2, \dots, n$, labelled cyclically and minored at n , has the following form when rows (resp. columns) are indexed $1, 2, \dots, n - 1$ from top to bottom (resp. left to right).

$$L_{n,cyc} := \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}.$$

Define

$$\mathcal{K}_n := \{\lambda \in \mathbb{R}^{n-1} : L_{n,cyc}\lambda \geq 0\}$$

and

$$\sigma_{\mathcal{K}_n}(z_1, \dots, z_{n-1}) := \sum_{\lambda \in \mathcal{K}_n \cap \mathbb{Z}^{n-1}} z_1^{\lambda_1} \cdots z_{n-1}^{\lambda_{n-1}}.$$

Our goal is to study the integer point transform for \mathcal{K}_n by enumerating the integer points in a fundamental parallelepiped for \mathcal{K}_n and applying Lemma 2.3; note that a fundamental parallelepiped is determined by a choice of integral ray generators, which we must first identify. By Lemma 2.1, the columns of $L_{n,cyc}^{-1}$ provide (non-integral) ray generators of \mathcal{K}_n , with entries given as follows.

Proposition 4.1. *For an n -cycle and $1 \leq i, j \leq n - 1$, $L_{n,cyc}^{-1} = (b_{i,j})$ where*

$$b_{i,j} = \begin{cases} \frac{i(n-j)}{n} & \text{if } i \leq j \\ \frac{j(n-i)}{n} & \text{if } i > j \end{cases}.$$

Proof. Define the matrix $B = (b_{i,j})$ using the b -values given above. Let $B \cdot L_{n,cyc} = (a_{i,j})$; since $L_{n,cyc}$ and B are symmetric matrices, it follows that $a_{i,j} = a_{j,i}$. Entry $a_{i,j}$ is given by $2b_{i,j} - b_{i,x} - b_{i,y}$ where x and y are the neighbors of j in the cycle, where we drop the b -terms for x or y equal to n . It is straightforward to show that $BL_{n,cyc} = I$ through a case-by-case analysis.

As a representative example, for $i < j$ we have

$$\begin{aligned} a_{i,j} &= 2b_{i,j} - b_{i,j-1} - b_{n,j+1} \\ &= 2 \left(\frac{i(n-j)}{n} \right) - \left(\frac{i(n-j+1)}{n} \right) - \left(\frac{i(n-j-1)}{n} \right) \\ &= 0. \end{aligned}$$

□

We next wish to classify the integer points in the fundamental parallelepiped Π_n , as defined prior to Lemma 2.3, for the ray generators of \mathcal{K}_n given by the columns of $nL_{n,cyc}^{-1}$. Observe first that every integer point $\lambda \in \mathcal{K}_n$, hence every integer point in $\Pi_n \subset \mathcal{K}_n$, can be expressed in the form $\lambda = L_{n,cyc}^{-1}c = nL_{n,cyc}^{-1}(\frac{c}{n})$ for an integer vector c ; this follows from the fact that $L_{n,cyc}\lambda = c$ is an integer vector for all such λ . By definition, every integer point in Π_n is a linear combination of the columns of $nL_{n,cyc}^{-1}$ with coefficients in $[0, 1)$. Thus, if $\lambda \in \Pi_n$, it follows that

$$\lambda = (nL_{n,cyc}^{-1}) \left(\frac{c}{n} \right)$$

where the entries in c are necessarily taken from $\{0, 1, 2, \dots, n - 1\}$.

Our next two propositions identify and use an underlying group-theoretic structure for the columns of $nL_{n,cyc}^{-1}$ to classify the integer points in Π_n .

Proposition 4.2. *Let $M := nL_{n,cyc}^{-1} \pmod n$, i.e., reduce all entries in $nL_{n,cyc}^{-1}$ modulo n . Then*

$$M = \begin{pmatrix} | & | & \cdots & | & | \\ v_1 & v_2 & \cdots & v_{n-2} & v_{n-1} \\ | & | & \cdots & | & | \end{pmatrix}$$

where $v_1 = (n - 1, n - 2, \dots, 2, 1)^T \in \mathbb{Z}_n^{n-1}$ and $v_k = kv_1 \in \mathbb{Z}_n^{n-1}$.

Proof. Let $M = (m_{i,j})$. Since $m_{i,j}$ is equal to either $i(n - j)$ or $j(n - i)$, it follows that $m_{i,j} \equiv -ij \pmod n$. Consider column v_1 . We find

$$\begin{aligned} v_1 &= (-1 \cdot 1, -2 \cdot 1, \dots, -(n - 1) \cdot 1)^T \pmod n, \\ &= (n - 1, n - 2, \dots, 1)^T \in \mathbb{Z}_n^{n-1}. \end{aligned}$$

Evaluating an arbitrary column v_k , $1 \leq k \leq n - 1$, we have

$$\begin{aligned} v_k &= (-1 \cdot k, -2 \cdot k, \dots, -(n - 1) \cdot k)^T \pmod n, \\ &= k \cdot v_1 \in \mathbb{Z}_n^{n-1}. \end{aligned}$$

□

Proposition 4.3. *If the point $x = (x_1, x_2, \dots, x_{n-1}) \in \Pi_n$ is of the form $L_{n,cyc}^{-1}c$ for some vector c with entries taken from $\{0, 1, 2, \dots, n - 1\}$, then x is an integer point if and only if x_1 is an integer.*

Proof. If x is an integer vector, then x_1 is certainly an integer. For the converse, suppose that x_1 is an integer. By the symmetry of $nL_{n,cyc}^{-1}$, v_1 is the first row of $nL_{n,cyc}^{-1} \pmod n$, where v_1 is as defined in Proposition 4.2. Since x_1 is an integer, it must be that $v_1 \cdot c \equiv \sum_{j=1}^n v_{1,j}c_j \equiv 0 \pmod n$. Since $v_k = kv_1$,

$$\sum_{j=1}^n v_{k,j}c_j \equiv \sum_{j=1}^n kv_{1,j}c_j \equiv k \sum_{j=1}^n v_{1,j}c_j \equiv 0 \pmod n.$$

Therefore, x_k is also an integer.

□

The integer points in Π_n are therefore parametrized by the set of solutions to the system

$$\sum_{j=1}^{n-1} (n - j)c_j \equiv 0 \pmod n$$

where $c_j \in \{0, 1, 2, \dots, n - 1\}$ (which has also appeared in the study of ‘‘Hermite reciprocity’’ given in [15]). Equivalently, these points are parametrized by the set of partitions of multiples of n into positive parts not exceeding $n - 1$, with no more than $n - 1$ of each part. Our next theorem results from combining the preceding discussion with Lemma 2.3.

Theorem 4.4. Let S_n denote the set of solutions to the system $\sum_{j=1}^{n-1} (n-j)c_j \equiv 0 \pmod{n}$ where $c_j \in \{0, 1, 2, \dots, n-1\}$. Let $l_{i,j}$ denote the i, j -th entry of $nL_{n,cyc}^{-1}$. Then

$$\sigma_{\mathcal{K}_n}(z_1, \dots, z_n) = \frac{\sum_{c \in S_n} \left(z_1^{\sum_{j=1}^n l_{1,j}c_j} z_2^{\sum_{j=1}^n l_{2,j}c_j} \dots z_n^{\sum_{j=1}^n l_{n,j}c_j} \right)}{\prod_{j=1}^n (1 - z_1^{l_{1,j}} z_2^{l_{2,j}} \dots z_n^{l_{n,j}})}.$$

While Theorem 4.4 and its specializations resolve the question of Corteel, Savage, and Wilf, this resolution is not as clean as one might like. In the next section, we show how a slight modification of the graph underlying this problem leads to much more simply stated and elegant, but still closely related, univariate generating function identities.

5. Leafed n -Cycles

In this section we examine the case of compositions constrained by n -cycles augmented with a leaf; we will refer to such a graph as a *leafed n -cycle*. Throughout this section, we label the leaf vertex as n and label the vertices of the n -cycle cyclically as $0, 1, \dots, n-1$, where n is adjacent to 0 as depicted in Figure 1.

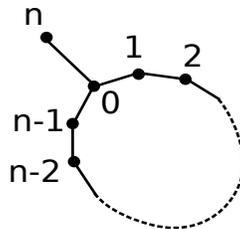


Figure 1: A leafed n -cycle

For a leafed n -cycle, we denote by L_n the minor of the Laplacian taken at vertex n and observe that it has the form

$$L_n = \begin{pmatrix} 3 & -1 & 0 & 0 & 0 & \dots & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}$$

where the columns (resp. rows) are labeled from top to bottom (resp. left to right) by $0, 1, 2, \dots, n - 1$. Define

$$\mathcal{C}_n := \{\lambda \in \mathbb{R}^n : L_n \lambda \geq 0\} \quad \text{and} \quad \sigma_{\mathcal{C}_n}(q) := \sum_{\lambda \in \mathcal{C}_n \cap \mathbb{Z}^n} q^{\lambda_0}.$$

Note that we will focus on the first coordinate of the integer points in \mathcal{C}_n .

5.1. A Generating Function Identity

Our main result regarding leafed cycles is the following identity.

Theorem 5.1. *Let S_n denote the set of solutions to the system*

$$0 \cdot c_0 + \sum_{j=1}^{n-1} (n - j)c_j \equiv 0 \pmod{n},$$

where $c_j \in \{0, 1, 2, \dots, n - 1\}$. For $c \in S_n$, define $\phi(c) := \sum_{j=0}^{n-1} c_j$. Then

$$\sigma_{\mathcal{C}_n}(q) = \frac{\sum_{c \in S_n} q^{\phi(c)}}{(1 - q^n)^n}.$$

The sum $\sum_{c \in S_n} q^{\phi(c)}$ is the ordinary generating function for the “number of parts” statistic on the set of all partitions of multiples of n into parts not exceeding $n - 1$, with no more than $n - 1$ of each part, where we may include 0 as a part. The similarity between Theorem 5.1 and Theorem 4.4 is immediate. However, by switching from n -cycles to leafed n -cycles, a more elegant univariate identity is obtained than those obtained by straightforward specializations of Theorem 4.4. We remark that our proof of Theorem 5.1 can easily be extended to produce a multivariate statement analogous to Theorem 4.4; we omit this, since it is the specialization above that we find most interesting. As in the previous section, our goal is to prove Theorem 5.1 through a careful study of the integer points in a fundamental parallelepiped for \mathcal{C}_n and an application of Lemma 2.3.

We first explicitly describe the entries in L_n^{-1} , which are closely related to the entries of $L_{n,cyc}^{-1}$.

Proposition 5.2. *For a leafed n -cycle and $0 \leq i, j \leq n - 1$, $L_n^{-1} = (b_{i,j})$ where*

$$b_{i,j} = \begin{cases} \frac{i(n-j)}{n} + 1 & \text{if } i \leq j \\ \frac{j(n-i)}{n} + 1 & \text{if } i > j \end{cases}.$$

Proof. This is a straightforward adaptation of the proof of Proposition 4.1. □

Every entry in the top row of L_n^{-1} is equal to 1; it is this property of the inverse that allows us to effectively study the generating function recording only the first entry of each composition in \mathcal{C}_n .

As in the previous section, we next wish to classify the integer points in the fundamental parallelepiped Π_n , which now denotes the fundamental parallelepiped for \mathcal{C}_n , for the ray generators of \mathcal{C}_n given by the columns of nL_n^{-1} . As in the situation for \mathcal{K}_n , every integer point $\lambda \in \mathcal{C}_n$, hence every integer point in $\Pi_n \subset \mathcal{C}_n$, can be expressed in the form $\lambda = L_n^{-1}c = nL_n^{-1}(\frac{c}{n})$ for an integer vector c . Thus, for the same reasons as given in the last section, if $\lambda \in \Pi_n$, then $\lambda = (nL_n^{-1}) (\frac{c}{n})$ where the entries in c are necessarily taken from $\{0, 1, 2, \dots, n - 1\}$.

The next two propositions are analogues of Propositions 4.2 and 4.3.

Proposition 5.3. *Let the matrix $M = nL_n^{-1} \pmod n$. Then*

$$M = \begin{pmatrix} | & | & | & \cdots & | & | \\ v_0 & v_1 & v_2 & \cdots & v_{n-2} & v_{n-1} \\ | & | & | & \cdots & | & | \end{pmatrix}$$

where $v_1 = (0, n - 1, n - 2, \dots, 2, 1)^T \in \mathbb{Z}_n^n$ and $v_k = kv_1 \in \mathbb{Z}_n^n$.

Proof. This proof is a straightforward extension of that for Proposition 4.2. □

Proposition 5.4. *If the point $x = (x_0, x_1, \dots, x_{n-1}) \in \Pi_n$ is of the form $L_n^{-1}c$ for some vector c with entries taken from $\{0, 1, 2, \dots, n - 1\}$, then x is an integer point if and only if x_1 is an integer.*

Proof. The proof is almost identical to that for Proposition 4.3. □

Proof of Theorem 5.1. By Proposition 5.3 and 5.4, the integer points in Π_n are parametrized by the set of solutions to the system

$$0 \cdot c_0 + \sum_{j=1}^{n-1} (n - j)c_j \equiv 0 \pmod n,$$

where $c_j \in \{0, 1, 2, \dots, n - 1\}$. Equivalently, these points are parametrized by the set of partitions of multiples of n into non-negative parts not exceeding $n - 1$, with no more than $n - 1$ of each part.

The key observation needed to apply Lemma 2.3 and complete our proof is that $\lambda_0 = \sum_i c_i$ for any point λ such that $L_n\lambda = c \in \Pi_n$. This is a consequence of Proposition 5.2, which shows that the first row (and column) of L_n^{-1} are vectors of all ones. □

5.2. Cyclically Distinct Compositions

We define two compositions, each having n parts, to be *cyclically distinct* if one cannot be obtained from the other via a cyclic shift of the parts. For example, $(0, 3, 1)$ is not cyclically distinct from $(1, 0, 3)$ or $(3, 1, 0)$, since the latter two arise

as cyclic shifts of the entire of the first, while $(0, 3, 1)$ and $(1, 3, 0)$ are cyclically distinct. This subsection is devoted to the following conjecture, which provides a combinatorial interpretation for the coefficients of $\sigma_{\mathcal{C}_n}(q)$.

Conjecture 5.5. *The number of integer points $\lambda \in \mathcal{C}_n$ with $\lambda_0 = m$ is equal to the number of cyclically distinct classes of compositions of m into n parts.*

Conjecture 5.5 is based on experimental evidence for small values of n ; further, we used LattE [18] to check the conjecture for all prime values of n up to $n = 17$. An obvious way to attack Conjecture 5.5 is to attempt to prove that for any $\lambda \in \mathcal{C}_n$ with $\lambda_0 = m$, we have that $\lambda = L_n^{-1}c$ for an integer vector c whose entries form a weak composition of m that is cyclically distinct from all other such c vectors forming elements of \mathcal{C}_n . Unfortunately, this does not work out, as the following example demonstrates.

Example 5.6. The weak compositions of 3 into 3 parts are given by the following families F_1, F_2, F_3 , and F_4 ; each family is the orbit of a single composition under the cyclic shift operation.

$$\begin{aligned} F_1 : & (3, 0, 0) \quad (0, 3, 0) \quad (0, 0, 3) \\ F_2 : & (2, 1, 0) \quad (0, 2, 1) \quad (1, 0, 2) \\ F_3 : & (1, 2, 0) \quad (0, 1, 2) \quad (2, 0, 1) \\ F_4 : & (1, 1, 1) \end{aligned}$$

The four compositions which satisfy $L_3^{-1}c \in \mathcal{C}_3$ are the elements of F_1 and F_4 , which correspond to the compositions $(3, 3, 3)$, $(3, 5, 4)$, $(3, 4, 5)$, and $(3, 4, 4)$ in \mathcal{C}_3 . Thus, while this shows that Conjecture 5.5 holds in this case, it is not a consequence of the c -vectors being themselves cyclically distinct.

While this obvious approach to Conjecture 5.5 fails in certain cases, it is successful in others, as seen in the following theorem.

Theorem 5.7. *Let n be prime and fix m such that $\gcd(m, n) = 1$. The number of integer points in \mathcal{C}_n with first coordinate m is equal to the number of cyclically distinct classes of compositions of m into n parts.*

Proof. We must show that for every composition c of m with n parts, exactly one of the cyclic shifts of c will produce an integer point when L_n^{-1} is applied. Begin with a composition $c = (c_0, c_1, \dots, c_{n-1})$ of m such that $L_n^{-1}c \in \mathcal{C}_n \cap \mathbb{Z}^n$. Then for all j ,

$$\sum_{i=0}^{n-1} c_i v_{j,i} \equiv 0 \pmod{n}$$

where $v_j = (v_{j,0}, v_{j,1}, \dots, v_{j,n-1})$ is the j th row of $nL_n^{-1} \pmod{n}$. Now consider another cyclic ordering c' of c where we shift the position of each entry by k , so

$c'_k = (c_k, c_{k+1}, \dots, c_{k+n-1})$ with the indices taken mod n . Then the j th coordinate of $L_n^{-1}c'$ is given by

$$\sum_{i=0}^{n-1} c_{i+k} v_{j,i} = \sum_{i=0}^{n-1} c_i v_{j,i-k}.$$

By properties of v_j given in Proposition 5.3, shifting the i th coordinate of v_j by k is equivalent to subtracting the row v_k from the row v_j component-wise. So,

$$\sum_{i=0}^{n-1} c_i v_{j,i-k} \equiv \sum_{i=0}^{n-1} c_i (v_{j,i} - v_{j,k}) \equiv \sum_{i=0}^{n-1} c_i v_{j,i} - \sum_{i=0}^{n-1} c_i v_{j,k} \equiv 0 - v_{j,k} \sum_{i=0}^{n-1} c_i.$$

Since $\sum_{i=0}^{n-1} c_i \not\equiv 0 \pmod n$ by assumption, and since $v_{j,k} \sum_{i=0}^{n-1} c_i \not\equiv 0 \pmod n$ for $j, k \not\equiv 0$ due to n being prime, we have that

$$\sum_{i=0}^{n-1} c_i v_{j,i-k} \not\equiv 0.$$

Thus, the non-trivial cyclic shifts of c do not correspond to integer points in \mathcal{C}_n . \square

It is straightforward to show that for the case where n is prime and $n|m$, for any composition c of m into n parts that satisfies $L_n^{-1}c \in \mathcal{C}_n \cap \mathbb{Z}^n$ all the cyclic shifts of c also satisfy this condition. Thus, the remaining challenge regarding Conjecture 5.5 in the case where n is prime is to show that this union of orbits of compositions under the cycle action contains as many compositions as there are orbits of the cycle action. While we have not been successful in proving this, the study of these compositions reveals a beautiful underlying geometric structure for our cone, which we discuss in the following section.

6. Reflexive Polytopes and Leafed Cycles of Prime Length

For leafed cycles of prime length, the remaining cases of Conjecture 5.5 turn out to be equivalent to the problem of enumerating points in integer dilates of an integer polytope, which is the subject of Ehrhart theory. The polytope in question is a simplex, and we will prove that it has the additional structure of being reflexive, a property we now introduce.

6.1. Reflexive Polytopes

An n -dimensional *polytope* P in \mathbb{R}^n may equivalently [25, Chapter 1] be described as the convex hull of at least $n + 1$ points in \mathbb{R}^n or as the intersection of at least $n + 1$ linear halfspaces in \mathbb{R}^n . When $P = \text{conv}(V)$ and V is the minimal set such

that this holds, then we say the points in V are the *vertices* of P . If $P = \text{conv}(V)$ and $|V| = n + 1$, then we say P is a *simplex*. We say P is *integral* if the vertices of P are contained in \mathbb{Z}^n .

For $P \subset \mathbb{R}^n$ an integral polytope of dimension n , and for $t \in \mathbb{Z}_{>0}$, set $tP := \{tp : p \in P\}$ and $L_P(t) := |\mathbb{Z}^n \cap tP|$, i.e., the number of integer points in tP is $L_P(t)$. Define the *Ehrhart series* for P to be $\text{Ehr}_P(x) := 1 + \sum_{t \geq 1} L_P(t)x^t$. A fundamental theorem due to E. Ehrhart [14] states that for an n -dimensional integral polytope P in \mathbb{R}^n , there exist complex values h_j^* so that

$$\text{Ehr}_P(x) = \frac{\sum_{j=0}^n h_j^* x^j}{(1-x)^{n+1}} \quad \text{and} \quad \sum_j h_j^* \neq 0.$$

A stronger result, originally due to R. Stanley in [23], is that the h_j^* 's are actually non-negative integers; Stanley's proof of this used commutative algebra, though several combinatorial and geometric proofs have since appeared. We call the coefficient vector $h_P^* := (h_0^*, \dots, h_d^*)$ in the numerator of the rational generating function for $\text{Ehr}_P(x)$ the *h-star vector* for P . The volume of P can be recovered as $(\sum_j h_j^*)/n!$.

Obtaining a general understanding of the structure of h^* -vectors of integral polytopes is currently of great interest. Recent activity, e.g., [2, 5, 7, 9, 17, 16, 20, 22], has focused on the class of reflexive polytopes, which we now introduce as they will be needed in Section 6. Given an n -dimensional polytope P , the *polar* or *dual* polytope to P is

$$P^\Delta = \{x \in \mathbb{R}^n : x \cdot p \leq 1 \text{ for all } p \in P\}.$$

Let P° denote the topological interior of P .

Definition 6.1. An n -dimensional polytope P is *reflexive* if $0 \in P^\circ$ and both P and P^Δ are integral.

Reflexive polytopes have many rich properties, as seen in the following lemma.

Lemma 6.2. [2, 17] P is reflexive if and only if P is an integer polytope with $0 \in P^\circ$ that satisfies one of the following (equivalent) conditions:

1. P^Δ is integral.
2. $L_{P^\circ}(t+1) = L_P(t)$ for all $t \in \mathbb{Z}_{\geq 0}$, i.e., all lattice points in \mathbb{R}^n sit on the boundary of some non-negative integral dilate of P .
3. $h_i^* = h_{n-i}^*$ for all i , where h_i^* is the i^{th} coefficient in the numerator of the Ehrhart series for P .

Reflexive polytopes are simultaneously a very large class of integral polytopes and a very small one, in the following sense. Due to a theorem of Lagarias and Ziegler [19], there are only finitely many reflexive polytopes (up to unimodular equivalence) in each dimension. On the other hand, Haase and Melnikov [16] proved that every integral polytope is a face of some reflexive polytope. Because of this “large yet small” tension, combined with their importance in other areas of mathematics as noted before, whenever a reflexive polytope arises unexpectedly it is a surprise, an indication that something interesting is happening.

6.2. Reflexive Slices of Laplacian Minor Cones

Our main result in this subsection is Theorem 6.3, which asserts that for prime n values, reflexive polytopes arise as slices of the cone constrained by Laplacian minors for leafed n -cycles. Note that when n is not prime, this construction does not always yield reflexive simplices; this has been verified experimentally with LattE [18] and Lemma 6.2. However, exactly when the construction described below produces reflexive polytopes is at this time not clear.

Let p be an odd prime and let lCP_p be the simplex obtained by intersecting the cone constrained by a leafed p -cycle with the hyperplane $\lambda_1 = p$ in \mathbb{R}^p .

Theorem 6.3. *lCP_p is reflexive (after translation by an integral vector).*

Proof. By elementary results about polar polytopes found in [25, Chapter 2], P is a reflexive polytope if and only if P is integral, contains the origin in its interior, and has a half-space description of the form $\{x \in \mathbb{R}^d : Ax \geq -\mathbf{1}\}$ where A is an integral matrix and $\mathbf{1}$ denotes the vector of all ones.

We first observe that lCP_p is clearly an integer polytope, as its vertices are the columns of pL_p^{-1} , which are integral. A halfspace description of lCP_p is given by $\lambda_0 = p$ and

$$\begin{bmatrix} 3 & -1 & 0 & \cdots & -1 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 2 \end{bmatrix} \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{p-1} \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Let w_i be the i th column of L_p^{-1} , and observe that $L_p(\sum_i w_i) = L_p(\sum_i L_p^{-1}(e_i)) = \mathbf{1}$. Thus, $\sum_i w_i$ is an interior integer point of our simplex lCP_p ; we want to change coordinates so that $\sum_i w_i$ is translated to the origin. We therefore consider solutions to our halfspace description of the form $\lambda + \sum_i w_i$. It follows that the description of our translated lCP_p does not use a λ_0 -coordinate (since $\lambda_0 + p = p$ implies $\lambda_0 = 0$),

and is given by

$$\begin{bmatrix} -1 & 0 & \cdots & -1 \\ 2 & -1 & \cdots & 0 \\ -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{p-1} \end{bmatrix} \geq \begin{bmatrix} -1 \\ -1 \\ \vdots \\ -1 \end{bmatrix}.$$

Since lCP_p is an integral simplex, and $\sum_i w_i$ is an integer vector by which we translated lCP_p , our translated lCP_p is still integral. Thus lCP_p is reflexive, and our proof is complete. \square

While the proof above is elementary, it ties the study of compositions constrained by graph Laplacian minors into an interesting circle of questions regarding reflexive polytopes. The only remaining case of Conjecture 5.5 is that the number of integer points in $m \cdot lCP_p$ is equal to the number of compositions of mp with p parts, up to cyclic equivalence, which is now asking for a combinatorial interpretation for the Ehrhart series of a reflexive simplex. Because of reflexivity, the generating function for this series yields a rational function with a symmetric numerator, which yields a functional relation on the Ehrhart polynomial for this simplex.

The study of reflexive polytopes goes hand in hand with the study of several other interesting classes of polytopes; one such example are normal polytopes, where an integer polytope P is *normal* if every integer point in the m -th dilate of P is a sum of exactly m integer points in P . In our attempts to prove the prime case of Conjecture 5.5, we noticed that our techniques (though unsuccessful at providing a proof) provided evidence suggesting that lCP_p is normal. Normality is implied by the presence of a unimodular triangulation for an integral polytope; while we are not yet certain of the existence of such a triangulation for lCP_p or of the normality of lCP_p , it would not surprise us if such a triangulation can be found for all prime p .

Our final remark regarding lCP_p , reflexivity, and normality, regards the unimodality of the h^* vector of lCP_p . It is a major open question (see [22] and the references therein) whether all normal reflexive polytopes have unimodal h^* -vectors. For $p \leq 7$, we found using LattE [18] that the h^* -vector for lCP_p is unimodal. We again suspect that this holds in general.

7. Leafed Cycles of Length a Power of Two

We conclude our paper by considering leafed cycles for non-prime values of n ; specifically, we study when $n = 2^k$ for some k . Using LattE [18], one observes that the cone constrained by a leafed 8-cycle does not have a reflexive 8-th slice (we suspect

this to be the case for all non-prime values of n). Nevertheless, the integer point transform of this cone exhibits some interesting “near-symmetry” in the numerator for small powers of 2. Based on experimental evidence, we offer the following conjecture.

Conjecture 7.1. *Let \mathcal{C}_{2^k} be the cone constrained by the Laplacian minor of a leafed n -cycle where $n = 2^k$ for some integer $k \geq 2$. Then the generating function has the form*

$$\sigma_{\mathcal{C}_{2^k}}(q, 1, 1, \dots, 1) = \frac{f(q)}{(1 - q^{2^k}) \prod_{i=0}^{k-1} (1 - q^{2^{k-(i+1)}})^{2^i}}$$

where $f(q)$ has the following property. Let (a_0, \dots, a_j) denote the coefficient list of $f(q)$. If we append a 0 to the end of this coefficient list, then take the difference between the appended coefficient list and its reverse, we obtain the coefficient list of the polynomial

$$\sum_{i=0}^{n-2} \binom{n-2}{i} (-1)^i q^{ni}.$$

The near-symmetry of the numerator polynomial indicates that there should be some interesting structure to the finite group obtained by quotienting the semigroup of integer points in \mathcal{C}_{2^k} by the semigroup generated by a specific choice of integral ray generators for the cone.

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