



 ON MULTIPLICATIVE SIDON SETS

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Abstract

Fix integers $b > a \geq 1$ with $g := \gcd(a, b)$. A set $S \subseteq \mathbb{N}$ is $\{a, b\}$ -multiplicative if $ax \neq by$ for all $x, y \in S$. For all n , we determine an $\{a, b\}$ -multiplicative set with maximum cardinality in $[n]$, and conclude that the maximum density of an $\{a, b\}$ -multiplicative set is $\frac{b}{b+g}$. For $A, B \subseteq \mathbb{N}$, a set $S \subseteq \mathbb{N}$ is $\{A, B\}$ -multiplicative if for all $a \in A$ and $b \in B$ and $x, y \in S$, the only solutions to $ax = by$ have $a = b$ and $x = y$. For $1 < a < b < c$ and a, b, c coprime, we give a $\mathcal{O}(1)$ time algorithm to approximate the maximum density of an $\{\{a\}, \{b, c\}\}$ -multiplicative set to arbitrary given precision.

1. Introduction

Erdős [3], Erdős [4], Erdős [5] defined a set $S \subseteq \mathbb{N}$ to be *multiplicative Sidon*² if $ab = cd$ implies $\{a, b\} = \{c, d\}$ for all $a, b, c, d \in S$; see [9, 10, 11]. In a similar direction, Wang [14] defined a set $S \subseteq \mathbb{N}$ to be *double-free* if $x \neq 2y$ for all $x, y \in S$, and proved that the maximum density of a double-free set is $\frac{2}{3}$; see [1] for related results. Here $\mathbb{N} := \{1, 2, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $[n] := \{1, 2, \dots, n\}$, and the *density* of $S \subseteq \mathbb{N}$ is

$$\lim_{n \rightarrow \infty} \frac{|S \cap [n]|}{n}.$$

Motivated by some questions in graph colouring, Pór and Wood [8] generalised the notion of double-free sets as follows. For $k \in \mathbb{N}$, a set $S \subseteq \mathbb{N}$ is *k-multiplicative*

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²Additive Sidon sets have been more widely studied; see the classical papers [6, 12, 13] and the survey by O'Bryant [7].

(Sidon) if $ax = by$ implies $a = b$ and $x = y$ for all $a, b \in [k]$ and $x, y \in S$. Pór and Wood [8] proved that the maximum density of a k -multiplicative set is $\Theta(\frac{1}{\log k})$.

Here we study the following alternative generalization of double-free sets. For distinct $a, b \in \mathbb{N}$, a set $S \subseteq \mathbb{N}$ is $\{a, b\}$ -multiplicative if $ax \neq by$ for all $x, y \in S$. Our first result is to determine the maximum density of an $\{a, b\}$ -multiplicative set. Assume that $a < b$ throughout.

Say $x \in \mathbb{N}$ is an i -th subpower of b if $x = b^i y$ for some $y \not\equiv 0 \pmod{b}$. If x is an i -th subpower of b for some even/odd i then x is an even/odd subpower of b . The following table gives the even subpowers of $b \in \{2, 3, 4\}$ and the corresponding entry in *The On-Line Encyclopedia of Integer Sequences*.

$b = 2$	1, 3, 4, 5, 7, 9, 11, 12, 13, 15, 16, 17, 19, 20, 21, 23, ...	[A003159]
$b = 3$	1, 2, 4, 5, 7, 8, 9, 10, 11, 13, 14, 16, 17, 18, 19, 20, 22, ...	[A007417]
$b = 4$	1, 2, 3, 5, 6, 7, 9, 10, 11, 13, 14, 15, 16, 17, 18, 19, 21, ...	[A171948]

We prove the following result:

Theorem 1. *Fix integers $b > a \geq 1$. Let $g := \gcd(a, b)$. Then for every integer $n \in \mathbb{N}$, the even subpowers of $\frac{b}{g}$ in $[n]$ are an $\{a, b\}$ -multiplicative set in $[n]$ with maximum cardinality. And the even subpowers of $\frac{b}{g}$ are an $\{a, b\}$ -multiplicative set with density $\frac{b}{b+g}$, which is maximum.*

Note that if $g = a$ then $b \geq 2g$ and $b + g \leq \frac{3}{2}b$, and if $g < a$ then $a \geq 2g$ and $b + g \leq b + a < \frac{3}{2}b$. In both cases the density bound $\frac{b}{b+g}$ in Theorem 1 is at least $\frac{2}{3}$, which is the bound obtained by Wang [14] for the $a = 1$ and $b = 2$ case.

We propose a further generalization of double-free sets. Let $A, B \subseteq \mathbb{N}$. Say $S \subseteq \mathbb{N}$ is $\{A, B\}$ -multiplicative if $ax = by$ implies $a = b$ and $x = y$ for all $a \in A$ and $b \in B$, and $x, y \in S$. One case is easily dealt with. If $B := \{b\}$ and b is coprime to each element of A , and there is some element $a \in A$ such that $a < b$, then, by the reasoning above, the even subpowers of b form an $\{A, B\}$ -multiplicative set of (maximum) density $\frac{b}{b+1}$.

The simplest nontrivial case (not covered by Theorem 1) is $\{A, B\}$ -multiplicativity for $A = \{a\}$, $B = \{b, c\}$, $1 < a < b < c$, with a, b, c pairwise coprime. We have the following theorem:

Theorem 2. *Consider $a, b, c \in \mathbb{N}$ pairwise coprime, with $1 < a < b < c$. For all fixed $\epsilon > 0$, there is a $\mathcal{O}(1)$ time algorithm that computes the maximum density of an $\{\{a\}, \{b, c\}\}$ -multiplicative set to within ϵ .*

2. Proof of Theorem 1

First suppose that $\gcd(a, b) = 1$. Let T be the set of even subpowers of b . We now prove that T is an $\{a, b\}$ -multiplicative set with maximum density. In fact, for all $[n]$, we prove that $T_n := T \cap [n]$ has maximum cardinality out of all $\{a, b\}$ -multiplicative sets contained in $[n]$.

The key to our proof is to model the problem using a directed graph. Let G be the directed graph with $V(G) := [n]$ where $(x, y) \in E(G)$ whenever $bx = ay$ (implying $x < y$). Thus $S \subseteq [n]$ is $\{a, b\}$ -multiplicative if and only if S is an independent set in G . If (x, y, z) is a directed path in G , then $x = \frac{a}{b}y$ and $z = \frac{b}{a}y$. Thus each vertex y has indegree and outdegree at most 1. Since $(x, y) \in E(G)$ implies $x < y$, G contains no directed cycles. Thus G is a collection of disjoint directed paths. Hence a maximum independent set in G is obtained by taking all the vertices at even distance from the source vertices³, where a vertex y is a source (indegree 0) if and only if $\frac{a}{b}y$ is not an integer; that is, if $y \not\equiv 0 \pmod{b}$.

We now prove that the vertices at distance d from a source vertex are precisely the d -th subpowers of b . We proceed by induction on $d \geq 0$. Each vertex y of G has an incoming edge if and only if $\frac{a}{b}y \in \mathbb{N}$, which occurs if and only if $y \equiv 0 \pmod{b}$ since $\gcd(a, b) = 1$. Thus the source vertices of G are precisely the 0-th subpowers of b . This proves the $d = 0$ case of the induction hypothesis. Now consider a vertex y at distance d from a source vertex. Thus $y = \frac{b}{a}x$ for some vertex x at distance $d - 1$ from a source vertex. By induction, x is a $(d - 1)$ -th subpower of b . That is, $x = b^{d-1}z$ for some $z \not\equiv 0 \pmod{b}$. Thus $y = b^d \frac{z}{a}$, which, since $\gcd(a, b) = 1$, implies that $\frac{z}{a}$ is an integer. Hence $\frac{z}{a} \not\equiv 0 \pmod{b}$ and y is a d -th subpower of b , as claimed.

This proves that the even subpowers of b form a maximum independent set in G . That is, T_n is an $\{a, b\}$ -multiplicative set of maximum cardinality in $[n]$. To illustrate this proof, the following table shows two examples of the graph G with $b = 3$. Observe that the i -th row consists of the i -th subpowers of 3 regardless of a .

$a = 1$ and $b = 3$								$a = 2$ and $b = 3$												
1	2	4	5	7	8	10	11	...	1	2	4	5	7	8	10	11	13	14	16	...
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓				↓	↓			↓	↓	
3	6	12	15	21	24	30	33	...	3	6				12	15			21	24	...
↓	↓	↓	↓	↓	↓	↓	↓	↓			↓			↓					↓	
9	18	36	45	63	72	90	99	...	9					18					36	...
↓	↓	↓	↓	↓	↓	↓	↓	↓							↓				↓	
27	48	108	135	189	216	270	297	...						27					48	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮												⋮
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮												⋮

³Note that this is not necessarily the only maximum independent set—for a path component with odd length, we may take the vertices at odd distance from the source of this path. This observation readily leads to a characterization of all maximum independent sets in G , and thus of all $\{a, b\}$ -multiplicative sets in $[n]$ with maximum cardinality. Details omitted.

We now bound $|T_n|$ from above. Observe that

$$T_n = \left\{ b^{2i}y : 0 \leq i \leq \frac{1}{2} \log_b n, 1 \leq y \leq \frac{n}{b^{2i}}, y \not\equiv 0 \pmod{b} \right\} .$$

Thus

$$\begin{aligned} |T_n| &\leq \sum_{i=0}^{\lfloor (\log_b n)/2 \rfloor} \left\lceil \frac{b-1}{b} \frac{n}{b^{2i}} \right\rceil \\ &\leq 1 + \frac{1}{2}(\log_b n) + \frac{(b-1)n}{b} \sum_{i \geq 0} \frac{1}{b^{2i}} \\ &\leq 1 + \frac{1}{2}(\log_b n) + \frac{(b-1)n}{b} \frac{b^2}{b^2-1} \\ &= 1 + \frac{1}{2}(\log_b n) + \frac{b}{b+1} n . \end{aligned}$$

We now bound $|T_n|$ from below. Observe that

$$T_n = [n] \setminus \left\{ b^{2i+1}y : 0 \leq i \leq \frac{1}{2}((\log_b n) - 1), 1 \leq y \leq \frac{n}{b^{2i+1}}, y \not\equiv 0 \pmod{b} \right\} .$$

Thus

$$\begin{aligned} |T_n| &\geq n - \sum_{i=0}^{\lfloor ((\log_b n)-1)/2 \rfloor} \left\lceil \frac{b-1}{b} \frac{n}{b^{2i+1}} \right\rceil \\ &\geq n - \frac{1}{2}((\log_b n) + 1) - \frac{(b-1)n}{b^2} \sum_{i \geq 0} \frac{1}{b^{2i}} \\ &\geq n - \frac{1}{2}((\log_b n) + 1) - \frac{(b-1)n}{b^2} \frac{b^2}{b^2-1} \\ &= n - \frac{1}{2}((\log_b n) + 1) - \frac{n}{b+1} \\ &= \frac{b}{b+1} n - \frac{1}{2}((\log_b n) + 1) . \end{aligned}$$

These upper and lower bounds on $|T_n|$ imply that

$$|T_n| = \frac{b}{b+1} n + \Theta(\log_b n) .$$

Hence the density of T is $\frac{b}{b+1}$, and because T_n is optimal for each n , no $\{a, b\}$ -multiplicative set has density greater than $\frac{b}{b+1}$.

We now drop the assumption that $\gcd(a, b) = 1$. Let $g := \gcd(a, b)$. Since $ax = by$ if and only if $\frac{a}{g}x = \frac{b}{g}y$, a set S is $\{a, b\}$ -multiplicative if and only if S is $\{\frac{a}{g}, \frac{b}{g}\}$ -multiplicative. Since $\frac{b/g}{b/g+1} = \frac{b}{b+g}$, the theorem is proved.

3. Proof of Theorem 2

Fix $A = \{a\}$ and $B = \{b, c\}$, where $1 < a < b < c$, and a, b, c are pairwise coprime. For convenience, we use the infinite graph G with vertex set \mathbb{N} and edge set

$$E(G) = \{\{x, y\} : bx = ay \text{ or } cx = ay, \text{ and } x, y \in \mathbb{N}\}.$$

Let G_n denote the subgraph of G induced by the vertex set $[n]$. Let δ be the maximum density of an $\{\{a\}, \{b, c\}\}$ -multiplicative set. Then

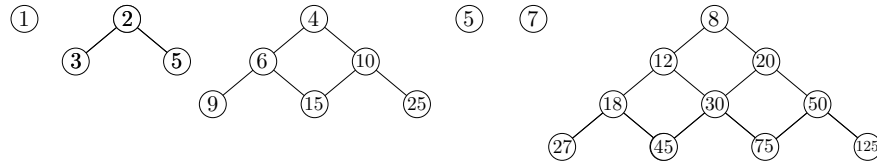
$$\delta = \lim_{n \rightarrow \infty} \frac{\alpha(G_n)}{n},$$

where $\alpha(G_n)$ is the size of a maximum independent set in G_n .

The infinite graph G has components $C_{p,q}$ with vertex set

$$V(C_{p,q}) = \{a^{p-x-y}b^xc^yq : x, y \in \mathbb{N}_0\}$$

for all $p \in \mathbb{N}_0, q \in \mathbb{N}$, and q not divisible by a, b , or c . Note that each $C_{p,q}$ is finite. Define p as the *height* of the component, and subsets of constant $x + y$ as *rows*. Note that the maximum and minimum vertices in $C_{p,q}$ are $c^p q$ and $a^p q$ respectively. The first few components of G for $a = 2, b = 3$, and $c = 5$ are shown below:



For a, b, c as above and fixed $\epsilon > 0$, let d be a non-negative integer $d \in \mathbb{N}_0$, to be specified later. Basically, d is a cutoff height which allows us to partition the components of G_n into three types, for any given $n \in \mathbb{N}$. The first are *complete* components $C_{p,q}$ where $n > c^p q$. The second are *small* incomplete components $S_{p,q}$ where $p \leq d$ and $a^p q \leq n < c^p q$. The third are *large* incomplete components $L_{p,q}$ with $p > d$ and $a^p q \leq n < c^p q$.

Let $\alpha_T(G_n)$ denote the size of a maximum independent set in the components of type T in G_n , for $T \in \{C, S, L\}$. We clearly have

$$\alpha(G_n) = \alpha_C(G_n) + \alpha_S(G_n) + \alpha_L(G_n).$$

Thus,

$$\delta = \lim_{n \rightarrow \infty} \frac{\alpha_C(G_n)}{n} + \lim_{n \rightarrow \infty} \frac{\alpha_S(G_n)}{n} + \lim_{n \rightarrow \infty} \frac{\alpha_L(G_n)}{n} = \delta_C + \delta_S + \delta_L$$

where

$$\delta_T = \lim_{n \rightarrow \infty} \frac{\alpha_T(G_n)}{n}.$$

Below we show that these limits exist, and we determine δ_C and δ_S explicitly. Then we show that, for any $\epsilon > 0$, we can choose d so that $\delta_L < \epsilon$. Hence, we can calculate δ to arbitrary precision.

3.1. Complete Components

We require the following lemma about independent sets in grid-like graphs by Cas-saigne and Zimmerman [2].

Lemma 1. *Define a graph H by $V(H) := \mathbb{N}_0 \times \mathbb{N}_0$ and*

$$E(H) := \{\{\mathbf{v}, \mathbf{w}\} : \mathbf{v}, \mathbf{w} \in V(H), |v_1 - w_1| + |v_2 - w_2| = 1\}.$$

Suppose that F is a finite subgraph of H such that $(x, y) \in V(F)$ implies $(x - 1, y) \in V(F)$ unless $x = 0$, and $(x, y - 1) \in V(F)$ unless $y = 0$. Then one of the sets

$$\begin{aligned} O &:= \{(x, y) \in V(F) : x + y \text{ is odd}\} \text{ or} \\ E &:= \{(x, y) \in V(F) : x + y \text{ is even}\} \end{aligned}$$

is a maximum independent set in F .

Now, consider a complete component $C_{p,1}$ of G_n . Note that every complete component $C_{p,q}$ of height q is isomorphic to $C_{p,1}$, and can be obtained by multiplying each vertex by q . Thus, we call $C_{p,q}$ a q -copy of $C_{p,1}$. In general, we use this terminology for isomorphic components of any type obtained by multiplying each vertex by q .

Observe that we can apply Lemma 1 to $C_{p,1}$, since it is isomorphic to a subgraph of H with the required properties. Define a function $\varphi : V(C_{p,1}) \rightarrow \mathbb{N}_0 \times \mathbb{N}_0$ by

$$\varphi(a^{p-x-y}b^xc^y) = (x, y).$$

If $a^{p-x-y}b^xc^y$ is adjacent to $a^{p-x'-y'}b^{x'}c^{y'}$, then $|x - x'| + |y - y'| = 1$ since they must differ by a factor of b/a or c/a . Thus, since φ is injective, it defines an isomorphism from $C_{p,1}$ to a subgraph of H . Assume $a^{p-x-y}b^xc^y \in V(C_{p,1})$. Then $a^{p-x-y+1}b^{x-1}c^y \in V(C_{p,1})$ unless $x = 0$, and similarly $a^{p-x-y+1}b^xc^{y-1} \in V(C_{p,1})$ unless $y = 0$. Under φ , these are clearly equivalent to the conditions required for Lemma 1.

Hence, by Lemma 1 and the definition of φ , a maximum independent set in $C_{p,1}$ is given by choosing all rows with $x + y$ even, or all rows with $x + y$ odd. In fact, it is clear that a maximum independent set is obtained by choosing the bottom row first, then alternating between remaining rows. Thus, if $p = 2i - 1$, then $\alpha(C_{p,1}) = i(i + 1)$. If $p = 2i$, then $\alpha(C_{p,1}) = (i + 1)^2$. Since the largest vertex in such a component is c^p , we must have $p \leq \log_c n$ for the component $C_{p,1}$ to be complete. Hence, the maximum height of a complete component is $\lfloor \log_c n \rfloor$.

Now we multiply by the number of components of height p that are complete. For a given p , we require $1 \leq q \leq nc^{-p}$. Since a, b, c are pairwise coprime, the density of numbers not divisible by a, b , or c is

$$\frac{(a-1)(b-1)(c-1)}{abc},$$

the number of components of height p in G_n is

$$\frac{(a-1)(b-1)(c-1)n}{c^p abc} + o(n).$$

Let $M(n) = \frac{1}{2} \lfloor \log_c n \rfloor$. The total number of vertices in a maximum independent set in complete components is therefore

$$\alpha_C(G_n) = \frac{(a-1)(b-1)(c-1)n}{abc} \sum_{i=0}^{M(n)} \left[\frac{i(i+1)}{c^{2i-1}} + \frac{(i+1)^2}{c^{2i}} \right] + o(n).$$

Thus, the density contribution is

$$\begin{aligned} \delta_C &= \lim_{n \rightarrow \infty} \frac{\alpha_C(G_n)}{n} = \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \frac{(a-1)(b-1)(c-1)n}{abc} \sum_{i=0}^{M(n)} \left[\frac{i(i+1)}{c^{2i-1}} + \frac{(i+1)^2}{c^{2i}} \right] \\ &= \frac{(a-1)(b-1)(c-1)}{abc} \sum_{i=0}^{\infty} \left[\frac{i(i+1)}{c^{2i-1}} + \frac{(i+1)^2}{c^{2i}} \right] \\ &= \frac{(a-1)(b-1)(c-1)}{abc} \cdot \frac{c^4}{(c-1)^3(c+1)} \\ &= \frac{(a-1)(b-1)c^3}{ab(c-1)^2(c+1)}. \end{aligned}$$

3.2. Small Incomplete Components

Now we consider the small incomplete components. Let $C_{p,1}[r]$ be the subgraph of $C_{p,1}$ induced by $[r]$. Define

$$f(p, r) := \alpha(C_{p,1}[r])$$

for $r \in \mathbb{N}$. We can calculate all f for $p \leq d$ in $\mathcal{O}(c^d)$ time with a computer, again using Lemma 1. (In fact, these components have bounded size, so any exponential time maximum independent set algorithm runs in $\mathcal{O}(1)$ time.) Note that $C_{p,q}[n]$ is a q -copy of $C_{p,1}[\lfloor n/q \rfloor]$, and therefore $\alpha(C_{p,q}[n]) = f(p, \lfloor n/q \rfloor)$. So we can find the size of maximum independent sets in the small components using the f 's.

More precisely, given $p \leq d$ and n , for how many values of q is $C_{p,q}[n]$ a q -copy of $C_{p,1}[r]$, where $r = \lfloor n/q \rfloor$? First note that

$$\frac{n}{r+1} < q \leq \frac{n}{r}.$$

Thus, there are

$$\frac{(a-1)(b-1)(c-1)n}{abc} \left(\frac{1}{r} - \frac{1}{r+1} \right) + o(n) = \frac{(a-1)(b-1)(c-1)n}{abcr(r+1)} + o(n)$$

q -copies of $C_{p,1}[r]$. The only restriction on r is that $a^p \leq r \leq c^p - 1$. Hence, the size of a maximum independent set in components of type S is

$$\sum_{p=0}^d \sum_{r=a^p}^{c^p-1} \frac{(a-1)(b-1)(c-1)n}{abcr(r+1)} f(p,r) + o(n) .$$

As $n \rightarrow \infty$, the density contribution of small components is therefore

$$\begin{aligned} \delta_S &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{p=0}^d \sum_{r=a^p}^{c^p-1} \frac{(a-1)(b-1)(c-1)n}{abcr(r+1)} f(p,r) \\ &= \sum_{p=0}^d \sum_{r=a^p}^{c^p-1} \frac{(a-1)(b-1)(c-1)}{abcr(r+1)} f(p,r). \end{aligned}$$

Since a , b and c are constants and d will be chosen so that it is bounded by a function of a , b and c (see the next section for details), δ_S can be computed in $\mathcal{O}(1)$ time.

3.3. Large Incomplete Components

Finally, we show that we can choose d so that the density of a maximum independent set in components of type L is less than ϵ . For large components,

$$p > d \text{ and } a^p q \leq n < c^p q .$$

The latter implies $c^{-p}n < q \leq a^{-p}n$. From the density of q , the number of large incomplete components $L_{p,q}$ for a given $p > d$ is

$$\frac{(a-1)(b-1)(c-1)n}{abc} \left(\frac{1}{a^p} - \frac{1}{c^p} \right) + o(n) .$$

Since there are less than p^2 vertices in a component of height p ,

$$\begin{aligned} \alpha_L(G_n) &\leq \sum_{p=d}^{\infty} p^2 \cdot \frac{(a-1)(b-1)(c-1)n}{abc} \left(\frac{1}{a^p} - \frac{1}{c^p} \right) \\ &\leq \frac{(a-1)(b-1)(c-1)n}{abc} \sum_{p=d}^{\infty} \frac{p^2}{a^p} \\ &= \frac{(a-1)(b-1)(c-1)n}{abc} \cdot \frac{a^{1-d}((a-1)^2 d^2 + 2(a-1)d + a + 1)}{(a-1)^3} \end{aligned}$$

$$\leq \frac{(b-1)(c-1)n}{bc} \cdot a^{-d/2}$$

where the last inequality holds for $d \geq 22$. Define $\beta := (b-1)(c-1)/bc$. Hence,

$$\delta_L = \lim_{n \rightarrow \infty} \frac{\alpha_L(G_n)}{n} \leq \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \beta n \cdot a^{-d/2} = \beta a^{-d/2} .$$

So, to obtain a precision of ϵ in the approximation $\delta \approx \delta_C + \delta_S$, we pick

$$d = \max\{2 \log_a(\beta/\epsilon), 22\}$$

which is a function of a, b, c , and ϵ . This completes the proof of Theorem 2.

The following table gives approximate values of δ for small a, b , and c :

a	b	c	δ
2	3	5	0.7292
2	3	7	0.7407
2	5	7	0.8235
2	5	9	0.8187
2	7	9	0.8709
3	4	5	0.7093
3	4	7	0.7934
3	5	7	0.8239
3	5	8	0.8212
3	7	8	0.8727

These results were obtained by incrementing d and looking for convergence to 4 decimal places. We also approximated δ_S using a naive algorithm (based on Lemma 1) for large n . Numerical convergence occurred at values of d slightly lower than the bound given above.

References

- [1] JEAN-PAUL ALLOUCHE, ANDRÉ ARNOLD, JEAN BERSTEL, SREČKO BRLEK, WILLIAM JOCKUSCH, SIMON PLOUFFE, AND BRUCE E. SAGAN. A relative of the Thue-Morse sequence. *Discrete Math.*, 139(1–3):455–461, 1995. doi:10.1016/0012-365X(93)00147-W.
- [2] JULIEN CASSAIGNE AND PAUL ZIMMERMAN. Numerical evaluation of the strongly triple-free set constant, 1996. <http://iml.univ-mrs.fr/~cassaigne/publis/cz.ps.gz>.
- [3] PAUL ERDŐS. On sequences of integers no one of which divides the product of two others and some related problems. *Izvestiya Naustno-Issl. Inst. Mat. i Meh. Tomsk*, 2:74–82, 1938. http://www.renyi.hu/~p_erdos/1938-07.pdf.
- [4] PAUL ERDŐS. On some applications of graph theory to number theoretic problems. *Publ. Ramanujan Inst. No.*, 1:131–136, 1968/1969. http://www.renyi.hu/~p_erdos/1968-09.pdf.

- [5] PAUL ERDŐS. Some applications of graph theory to number theory. In *The Many Facets of Graph Theory (Proc. Conf., Western Mich. Univ., Kalamazoo, Mich., 1968)*, pp. 77–82. Springer, Berlin, 1969. http://www.renyi.hu/~p_erdos/1969-14.pdf.
- [6] PAUL ERDŐS AND PÁL TURÁN. On a problem of Sidon in additive number theory, and on some related problems. *J. London Math. Soc.*, 16:212–215, 1941. doi:10.1112/jlms/s1-16.4.212.
- [7] KEVIN O'BRYANT. A complete annotated bibliography of work related to Sidon sequences. *Electron. J. Combin.*, DS11, 2004. <http://www.combinatorics.org/ojs/index.php/eljc/article/view/ds11>.
- [8] ATTILA PÓR AND DAVID R. WOOD. Colourings of the Cartesian product of graphs and multiplicative Sidon sets. *Combinatorica*, 29(4):449–466, 2009. doi:10.1007/s00493-009-2257-0.
- [9] IMRE Z. RUZSA. Erdős and the integers. *J. Number Theory*, 79(1):115–163, 1999. doi:10.1006/jnth.1999.2395.
- [10] IMRE Z. RUZSA. Additive and multiplicative Sidon sets. *Acta Math. Hungar.*, 112(4):345–354, 2006. doi:10.1007/s10474-006-0102-0.
- [11] ANDRÁS SÁRKÖZY. Unsolved problems in number theory. *Period. Math. Hungar.*, 42(1-2):17–35, 2001. doi:10.1023/A:1015236305093.
- [12] SIMON SIDON. Ein Satz über trigonometrische Polynome und seine Anwendung in der Theorie der Fourier-Reihen. *Math. Ann.*, 106:536–539, 1932. MR: 1512772.
- [13] JAMES SINGER. A theorem in finite projective geometry and some applications to number theory. *Trans. Amer. Math. Soc.*, 43(3):377–385, 1938. doi:10.2307/1990067.
- [14] EDWARD T. H. WANG. On double-free sets of integers. *Ars Combin.*, 28:97–100, 1989. MR: 1039135.