



**COMBINATORIAL PROOFS OF SOME IDENTITIES FOR THE
FIBONACCI AND LUCAS NUMBERS**

Alex Eustis

Department of Mathematics, University of San Diego, La Jolla, California
akeustis@gmail.com

Mark Shattuck

Department of Mathematics, University of Tennessee, Knoxville, Tennessee
shattuck@math.utk.edu

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Abstract

We study the previously introduced bracketed tiling construction and obtain direct proofs of some identities for the Fibonacci and Lucas numbers. By adding a new type of tile we call a *superdomino* to this construction, we obtain combinatorial proofs of some formulas for the Fibonacci and Lucas polynomials, which we were unable to find in the literature. Special cases of these formulas occur in the text by Benjamin and Quinn, where the question of finding their combinatorial proofs is raised. In the process, we also show, via direct bijections, that the bracketed $(2n)$ -bracelets as well as the linear bracketed $(2n + 1)$ -tilings both number 5^n .

1. Introduction

Let F_n and L_n denote the Fibonacci and Lucas numbers defined, respectively, by $F_0 = 0$, $F_1 = 1$ with $F_n = F_{n-1} + F_{n-2}$ if $n \geq 2$ and by $L_0 = 2$, $L_1 = 1$ with $L_n = L_{n-1} + L_{n-2}$ if $n \geq 2$. As recently popularized by Benjamin and Quinn in their text [5], proofs which use tilings can explain a variety of identities involving Fibonacci and Lucas numbers and their relatives. See, for example, [1, 2, 4, 7]. In this paper, we provide tiling proofs of the following four identities where $m \geq 0$, $n \geq 1$:

$$5^n F_m^{2n+1} = \sum_{k=0}^n (-1)^{(m+1)k} \binom{2n+1}{k} F_{(2n+1-2k)m}, \quad (1)$$

$$5^n F_m^{2n} = (-1)^{(m+1)n} \binom{2n}{n} + \sum_{k=0}^{n-1} (-1)^{(m+1)k} \binom{2n}{k} L_{(2n-2k)m}, \quad (2)$$

$$F_{(2n+1)m} = 5^n F_m^{2n+1} + F_m \sum_{k=1}^n (-1)^{mk} \frac{2n+1}{k} \binom{2n-k}{k-1} (5F_m^2)^{n-k}, \quad (3)$$

and

$$L_{2nm} = 5^n F_m^{2n} + \sum_{k=1}^n (-1)^{mk} \frac{2n}{k} \binom{2n-1-k}{k-1} (5F_m^2)^{n-k}. \quad (4)$$

These four identities occur, respectively, as V80, V81, V83, and V84 on p. 145 of *Proofs that Really Count* [5], where Benjamin and Quinn raise the question of finding their combinatorial proofs. See Vajda [8] for algebraic proofs. In this paper, we continue our study of the bracketed tiling construction introduced in [6] and use it to provide direct arguments for some identities involving Fibonacci and Lucas numbers. Next, we show that there are 5^n bracketed $(2n)$ -bracelets as well as 5^n linear bracketed $(2n+1)$ -tilings via explicit bijections. As a consequence of our analysis, we obtain the requested proofs for (1) and (2) above. By adding a new kind of object we call a *superdomino* to the bracketed tiling construction, we are able to provide combinatorial proofs of some identities involving Fibonacci and Lucas polynomials which seem to be new from which formulas (3) and (4) above will follow as special cases.

2. Preliminaries

Consider a board of length n with cells labeled 1 to n . A tiling of this board (termed an n -tiling) is an arrangement of indistinguishable squares and indistinguishable dominos which cover it completely, where pieces do not overlap, a domino is a rectangular piece covering two cells, and a square is a piece covering a single cell. Let \mathfrak{F}_n denote the set of all (linear) n -tilings. When the board is circular, meaning that a domino may wrap around from cell n back to cell 1, we denote the set of all n -tilings by \mathfrak{L}_n . Members of \mathfrak{L}_n are also called *bracelets*. It is clear that $\mathfrak{F}_n \subseteq \mathfrak{L}_n$. Recall that

$$|\mathfrak{F}_n| = F_{n+1}, \quad n \geq 1,$$

and

$$|\mathfrak{L}_n| = L_n, \quad n \geq 1.$$

(If we let $\mathfrak{F}_0 = \{\emptyset\}$, the “empty tiling,” and \mathfrak{L}_0 consist of two empty tilings of opposite orientation, then these relations hold for $n = 0$ as well.)

Now assign the weight x to every square in a tiling and the weight y to every domino. Given $T \in \mathfrak{F}_n$ (or \mathfrak{L}_n), define the weight $\omega(T)$ of the tiling to be the product of the weights of its tiles. The Fibonacci and Lucas polynomials (see, e.g.,

[5]) are given, respectively, as

$$F_n(x, y) := \sum_{T \in \mathfrak{F}_n} \omega(T)$$

and

$$L_n(x, y) := \sum_{T \in \mathfrak{L}_n} \omega(T).$$

As an example, when $n = 3$, we have $\mathfrak{F}_3 = \{sss, slr, lrs\}$ so that $F_3(x, y) = x^3 + 2xy$, and $\mathfrak{L}_3 = \{sss, slr, lrs, rsl\}$ so that $L_3(x, y) = x^3 + 3xy$, where s is a square and l and r are the left and right halves of a domino. The $F_n(x, y)$ and $L_n(x, y)$ both satisfy a two-term recurrence of the form

$$a_{n+2} = xa_{n+1} + ya_n, \quad n \geq 1,$$

upon considering whether a tiling starts with a square or a domino. By defining $F_0(x, y) = 1$ and $L_0(x, y) = 2$, the recurrence holds for $n = 0$ as well. Note that when $x = y = 1$, all tilings have unit weight, which implies that $F_n(1, 1) = |\mathfrak{F}_n| = F_{n+1}$ and $L_n(1, 1) = |\mathfrak{L}_n| = L_n$.

The Fibonacci and Lucas polynomials have the well-known explicit formulas

$$F_n(x, y) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} y^k x^{n-2k} \binom{n-k}{k}, \quad n \geq 0, \tag{5}$$

and

$$L_n(x, y) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} y^k x^{n-2k} \frac{n}{n-k} \binom{n-k}{k}, \quad n \geq 1. \tag{6}$$

The Fibonacci and Lucas polynomials are sometimes defined as polynomials of one variable only, but can always be completed to the version presented here, for fixing the number of squares in a square-and-domino n -tiling fixes the number of dominos as well.

We will need the values of $F_n(x, y)$ and $L_n(x, y)$ when $x = L_m$ and $y = (-1)^{m+1}$. The following relations are equivalent to special cases of a more general result which was established in [3] and given both algebraic and combinatorial proofs. They may also be proven directly using Binet formulas.

Lemma 1. *If $m, n \geq 1$, then*

$$F_n(L_m, (-1)^{m+1}) = \frac{F_{(n+1)m}}{F_m} \tag{7}$$

and

$$L_n(L_m, (-1)^{m+1}) = L_{nm}. \tag{8}$$

We will also need the following, which is Identity 53 from [5]:

$$L_m^2 + 4(-1)^{m+1} = 5F_m^2, \quad m \geq 0. \tag{9}$$

(This has a direct combinatorial proof.)

3. Bracketed Tilings

3.1. Definitions and Properties

Introducing an object we call a *bracket* into the square-and-domino tilings described in the prior section yields a new construction we call a *bracketed tiling*. See [6] for further details. In this section, we review definitions.

A *bracket* is an object that occupies a single cell, like a square. They come in two varieties, which we denote by $<$ and $>$, and must be placed according to the following criterion:

Every group of consecutive brackets must be properly paired and nested in a manner identical to parentheses. Such groups may occur even between the left and right halves of a domino.

Bracketed tilings of length n with k bracket pairs, where $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$, may be formed as follows. First select the k positions to be occupied by the $<$ on a board of length n . This uniquely determines the positions of the $>$ since they must follow the $<$ without gaps. For once you have selected the slots to be occupied by left brackets, you fill in the gaps in such a way so that the first right bracket goes in the first available slot to the right of the first left bracket, the second right bracket goes in the first now available slot to the right of the second left bracket, and so on. If one runs out of spaces in which to place right brackets, then continue searching for spaces from left to right at the beginning of the tiling. Once they are placed, the left and right brackets are paired in a manner identical to parentheses. Below are two examples when $n = 8$ and $k = 3$:

$$_ _ _ < < _ < _ _ _ \longrightarrow _ _ _ < < > < > >$$

$$_ _ _ < _ _ < < _ _ \longrightarrow > _ < > _ < < >$$

Note that in the second example, the $<$ in the sixth slot is paired with the $>$ in the first slot. We will say that a bracket pair $<>$ *wraps around* if the $>$ occurs to the left of the corresponding $<$. Once the positions of the k pairs of brackets are determined, cover the remaining $n - 2k$ cells with a tiling of squares and dominos, where the left and the right halves of a domino may be separated by a group of consecutive brackets. This subtiling of squares and dominos may either be a linear tiling or a bracelet containing a wraparound domino.

One might wonder, *a priori*, whether the bracket pairs $\langle \rangle$ are uniquely determined once the brackets have been placed as described above. To see that they are, regard a group of $2i$ consecutive brackets, starting with the leftmost \langle (and wrapping around, if necessary), as a lattice path from $(0, 0)$ to $(2i, 0)$ in which each \langle stands for a $(1, 1)$ upstep and each \rangle stands for a $(1, -1)$ downstep. Given any \langle , the position of the matching bracket \rangle within this group can be gotten by drawing a horizontal line to the right starting from the upstep position (corresponding to the \langle) and noting the (downstep) position where this line first intersects the lattice path again.

We now differentiate two types of bracketed tilings. A *straight bracketed n -tiling* consists of k bracket pairs for some k , none of which wrap around, and whose subtiling of squares and dominos belongs to \mathfrak{F}_{n-2k} . A *bracketed n -bracelet* consists of k bracket pairs, some of which may wrap around, and whose subtiling of squares and dominos belongs to \mathfrak{L}_{n-2k} (except when n is even and $k = \frac{n}{2}$, in which case there is just one possibility for the subtiling, not two). Let \mathfrak{BF}_n denote the set of straight bracketed n -tilings and let \mathfrak{BL}_n denote the bracketed n -bracelets. It is clear that $\mathfrak{BF}_n \subseteq \mathfrak{BL}_n$. Below are some examples:

$$\{l \langle \langle \rangle \rangle \langle \rangle r, slr \langle \rangle \langle \rangle s, \langle \langle \rangle \langle \rangle \rangle \rangle\} \subset \mathfrak{BF}_8,$$

$$\{rl \langle \rangle \langle \rangle \langle \rangle \rangle, \rangle rsslrl \langle, \langle \rangle \rangle \langle \langle \rangle \rangle \rangle\} \subset \mathfrak{BL}_8 - \mathfrak{BF}_8.$$

The following are not bracketed tilings at all:

$$\langle \rangle \langle, \langle lr \rangle, \langle s \rangle, ll \langle \rangle rr, slr \rangle \langle \rangle \langle \rangle.$$

As a convention, we will assume when $n = 0$ that both \mathfrak{BF}_n and \mathfrak{BL}_n consist of only the empty tiling and thus have cardinality one.

We extend the weights defined above for \mathfrak{F}_n and \mathfrak{L}_n to \mathfrak{BF}_n and \mathfrak{BL}_n by assigning every bracket pair the weight y and defining the weight of a bracketed tiling to be the product of the weights of all of its tiles and brackets. Define the polynomials $F_n^*(x, y)$ and $L_n^*(x, y)$ by

$$F_n^*(x, y) := \sum_{T \in \mathfrak{BF}_n} \omega(T)$$

and

$$L_n^*(x, y) := \sum_{T \in \mathfrak{BL}_n} \omega(T).$$

When $n = 3$, for example, we have $\mathfrak{BF}_3 = \{sss, slr, lrs, s \langle \rangle, \langle \rangle s\}$ so that $F_3^*(x, y) = x^3 + 4xy$ and $\mathfrak{BL}_3 = \{sss, slr, lrs, rsl, s \langle \rangle, \langle \rangle s, \rangle s \langle\}$ so that $L_3^*(x, y) = x^3 + 6xy$. By convention, $F_0^*(x, y) = L_0^*(x, y) = 1$.

We conclude this section by recalling two formulas for $L_{2n}^*(x, y)$ from [6] which we will need later. The proofs are included here for completeness. Similar formulas hold in the odd case. Recall the notation $[m] := \{1, 2, \dots, m\}$ if $m \geq 1$, with $[0] := \emptyset$.

Proposition 2. *If $n \geq 0$, then*

$$L_{2n}^*(x, y) = y^n \binom{2n}{n} + \sum_{k=0}^{n-1} y^k \binom{2n}{k} L_{2n-2k}(x, y) \tag{10}$$

and

$$L_{2n}^*(x, y) = \sum_{k=0}^n y^k \binom{2n+1}{k} F_{2n-2k}(x, y). \tag{11}$$

Proof. For (10), first select k cells of a $(2n)$ -board to be occupied by a $<$ in $\binom{2n}{k}$ ways, which uniquely determines the positions of the $>$, and then cover the remaining $2n - 2k$ cells with a (weighted) member of \mathfrak{L}_{2n-2k} . The cases when $k = n$ and $k < n$ must be differentiated. The k bracket pairs contribute y^k .

For (11), first choose an arbitrary subset S of $[2n + 1]$ of size k , where $0 \leq k \leq n$. If $2n + 1 \notin S$, then place a $<$ in the positions on an $(2n)$ -board corresponding to the elements of the subset, which uniquely determines the $>$. If $2n + 1 \in S$, place a $<$ in the $k - 1$ positions on an $(2n)$ -board corresponding to the elements of $S - \{2n + 1\}$, which determines the $>$; then add the left half of a domino to the rightmost cell not occupied by a bracket and the right half of a domino to the leftmost unoccupied cell. In either case, fill in the remaining $2n - 2k$ cells with a (weighted) member of \mathfrak{F}_{2n-2k} . If $n = 4$, for example, the bracketed tilings $> sl <> rs <$ and $>> rssl <<$ would be members of \mathfrak{BL}_8 which correspond, respectively, to the subsets $S = \{4, 8\}$ and $S = \{7, 8, 9\}$ of [9]. \square

3.2. Some Identities

In this section, we derive some relations involving Fibonacci and Lucas polynomials using combinatorial arguments by placing a restriction on the positions to be occupied by the bracket pairs. Throughout this section, the sign of a bracketed tiling will be defined as $(-1)^k$, where k denotes the number of bracket pairs. The results of this section will not be used in the sequel. Our first identity is a formula for the Fibonacci polynomial involving an alternating sum.

Proposition 3. *If $n \geq 0$, then*

$$x^p F_q(x, y) = \sum_{k=0}^p (-1)^k y^k \binom{p}{k} F_{n-2k}(x, y), \tag{12}$$

where $p = \lfloor \frac{n}{2} \rfloor$ and $q = \lfloor \frac{n+1}{2} \rfloor$.

Proof. We establish only the odd case, which, by (5), is equivalent to showing

$$\sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} y^k x^{2n+1-2k} \binom{n+1-k}{k} = \sum_{k=0}^n (-1)^k y^k \binom{n}{k} F_{2n+1-2k}(x, y), \quad n \geq 0, \tag{13}$$

upon replacing n with $2n + 1$. The even case is similar. Note that the left side of (13) is $x^n F_{n+1}(x, y) = x^p F_q(x, y)$. For the right side, we consider the total signed weight of all members of $\mathfrak{B}\mathfrak{F}_{2n+1}$ in which bracket pairs can only occupy consecutive cells $2i - 1$ and $2i$ for some i . Upon choosing k members of the set $\{1, 3, \dots, 2n - 1\}$ to be occupied by a $<$, we see that the weight of all such members of $\mathfrak{B}\mathfrak{F}_{2n+1}$ is given by the sum $\sum_{k=0}^n (-1)^k y^k \binom{n}{k} F_{2n+1-2k}(x, y)$.

We now define a sign-reversing, weight-preserving involution on this subset of $\mathfrak{B}\mathfrak{F}_{2n+1}$ as follows. First identify the leftmost position t occupied by either the left half of a domino or by a $<$. If t is odd, then cells t and $t + 1$ are covered by either a domino or by a bracket pair $<>$, in which case we switch to the other option. If t is even, then cell t must be covered by the left half l of a domino and there are the three possibilities: (i) $t + 1$ is covered by r and either $t = 2n$ or $t + 2$ is covered by s ; (ii) $t + 1$ is covered by r and $t + 2$ is covered by l , which is followed by exactly $i \geq 0$ bracket pairs and then r ; or (iii) $t + 1$ and $t + 2$ are covered by $<>$, followed by $i \geq 0$ additional bracket pairs and then r . If t is even, then replace the subtiling $lrl <> \dots <> r$ of length $2i + 4$ covering cells t through $t + 2i + 3$ in case (ii) above with the subtiling $l <><> \dots <> r$ covering the same cells in case (iii) and vice-versa.

The set of survivors of the involution above are those tilings containing no bracket pairs and where the set of numbers covered by the left halves of dominos comprise a subset of $\{2, 4, \dots, 2n\}$ with no consecutive members allowed. Since such subsets with exactly k elements are synonymous with compositions of the form $x_1 + x_2 + \dots + x_{k+1} = n - k$, where $x_1 \geq 0$, $x_{k+1} \geq 0$, and $x_i \geq 1$ if $2 \leq i \leq k$, they number $\binom{(n-2k+1)+k}{k} = \binom{n+1-k}{k}$. Thus, the set of survivors has total weight given by the sum $\sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} y^k x^{2n+1-2k} \binom{n+1-k}{k}$. Equating this with the above expression for the total weight yields (13), as desired. \square

Taking $x = L_m$ and $y = (-1)^{m+1}$ in (12), and applying (7), yields the following relation.

Corollary 4. *If $m, n \geq 0$, then*

$$L_m^p F_{(q+1)m} = \sum_{k=0}^p (-1)^{mk} \binom{p}{k} F_{(n+1-2k)m}, \tag{14}$$

where $p = \lfloor \frac{n}{2} \rfloor$ and $q = \lfloor \frac{n+1}{2} \rfloor$.

Taking $m = 1$ in (14) gives the following identity:

$$F_{q+1} = \sum_{k=0}^p (-1)^k \binom{p}{k} F_{n+1-2k}. \tag{15}$$

We now prove a formula comparable to (12) for Lucas polynomials.

Proposition 5. *If $n \geq 0$, then*

$$x^p L_q(x, y) = \sum_{k=0}^p (-1)^k y^k \binom{p}{k} L_{n-2k}(x, y), \tag{16}$$

where $p = \lfloor \frac{n}{2} \rfloor$ and $q = \lfloor \frac{n+1}{2} \rfloor$.

Proof. We prove only the odd case, rewritten as

$$x^n L_{n+1}(x, y) = \sum_{k=0}^n (-1)^k y^k \binom{n}{k} L_{2n+1-2k}(x, y), \quad n \geq 0, \tag{17}$$

upon replacing n with $2n + 1$. Similar reasoning applies to the even case. The $n = 0$ and $n = 1$ cases of (17) are easily verified, so we may assume $n \geq 2$. Let $S \subseteq \mathfrak{B}\mathfrak{L}_{2n+1}$ comprise those tilings in which a bracket pair must occupy positions $2i - 1$ and $2i$ for some i . Then the right side of (17) gives the total signed weight of all members of S . We define a sign-reversing involution on S whose survivors have weight $x^n L_{n+1}(x, y)$. To do so, first apply the mapping used for (13) above to the subset T of S comprised of linear tilings, the survivors of which have weight $x^n F_{n+1}(x, y)$. Since $L_{n+1}(x, y) = F_{n+1}(x, y) + yF_{n-1}(x, y)$, to prove (17), it suffices to define a sign-reversing involution of $S - T$ whose set of survivors has weight $x^n y F_{n-1}(x, y)$.

First observe that if $\lambda \in S - T$, then the numbers $2n+1, 1, 2, \dots, 2t+1$ are covered by l , followed by t bracket pairs $\langle \rangle$, and then r , for some $t, 0 \leq t \leq n - 1$. Let λ' be the linear subtiling of λ which covers cells $2t + 2$ through $2n$. Upon applying the involution used for (13) above (slightly modified), we may assume that λ' contains no bracket pairs and that the left halves of any dominos within λ' cover only even numbers, with no two dominos directly adjacent.

We define an involution in this case as follows. Suppose that the number $2n - 2$ is covered by the left half of a domino in λ (which implies $t \leq n - 2$). We pick up this domino and change it to a bracket pair $\langle \rangle$, leaving a gap of two cells (at $2n - 2$ and $2n - 1$). We then insert this bracket pair into the tiling so that it covers cells $2t + 1$ and $2t + 2$, sliding the right half of the wraparound domino which previously covered $2t + 1$ (and any pieces coming after it) two cells to the right (so as to fill in the gap). Note that if cells $2n - 2$ and $2n - 1$ are not covered by a domino, then they must be covered by squares or by $> r$ (in the latter case, we must have $t = n - 1$), which implies that the above operation can be reversed. Neither the operation nor its inverse is defined in the case when $t = 0$ and cells $2n - 2$ and $2n - 1$ are covered by squares.

Thus, the set of survivors of the involution consists of those members of $S - T$ containing no bracket pairs in which (i) the left half of any non-wraparound domino covers an even number; (ii) no two non-wraparound dominos are directly adjacent; and (iii) cells $2n - 2$ and $2n - 1$ are covered by squares. Reasoning as in the proof

of Proposition 3 above, we see that this set has weight $x^2y \cdot x^{n-2}F_{n-1}(x, y) = x^n y F_{n-1}(x, y)$, which completes the proof of (17). \square

Setting $x = L_m$ and $y = (-1)^{m+1}$ in (16), and using (8), yields the following result.

Corollary 6. *If $m, n \geq 0$, then*

$$L_m^p L_{qm} = \sum_{k=0}^p (-1)^{mk} \binom{p}{k} L_{(n-2k)m}, \tag{18}$$

where $p = \lfloor \frac{n}{2} \rfloor$ and $q = \lfloor \frac{n+1}{2} \rfloor$.

Taking $m = 1$ in (18) yields the following identity:

$$L_q = \sum_{k=0}^p (-1)^k \binom{p}{k} L_{n-2k}. \tag{19}$$

4. Proofs of (1) Through (4)

4.1. Proofs of (1) and (2)

In this section, we provide combinatorial proofs of formulas (1) and (2), see Proposition 10 below. They will follow from a more general result, namely $L_{2n}^*(x, y) = (x^2 + 4y)^n$, which we establish below. Note that by the definition of $L_{2n}^*(x, y)$, this implies $|\mathfrak{BL}_{2n}| = 5^n$, upon taking $x = y = 1$. In addition, we show that $F_{2n+1}^*(x, y) = x(x^2 + 4y)^n$, which implies $|\mathfrak{BF}_{2n+1}| = 5^n$ as well. On the other hand, there do not appear to be comparable formulas for the polynomials $L_{2n+1}^*(x, y)$ and $F_{2n}^*(x, y)$.

As a first step, we count members of \mathfrak{BL}_{2n} containing no squares.

Lemma 7. *If $n \geq 0$, then the total weight of all members of \mathfrak{BL}_{2n} not containing a square is equal to $(4y)^n$.*

Proof. Let P be the set of all words of length $2n$ in the alphabet $\{<, >\}$, where $n \geq 1$. Let $\mathfrak{BL}'_{2n} \subseteq \mathfrak{BL}_{2n}$ consist of those tilings containing no squares. Within a member of \mathfrak{BL}_{2n} , we will call a domino *in-phase* if its left half l covers an odd-numbered cell and *out-of-phase* if l covers an even-numbered cell. No members of \mathfrak{BL}'_{2n} can contain both kinds of domino. We define a mapping f between \mathfrak{BL}'_{2n} and P by replacing each in-phase domino with $<<$ and each out-of-phase domino with $>>$.

The inverse may be found by first counting the number of $<$ versus the number of $>$ within a member of P . If they are equal in number, then we already have

a member of $\mathfrak{B}\mathcal{L}'_{2n}$ and we are finished. Otherwise, erase the set that is in the majority (replace them with blanks to be filled in). Now give each remaining bracket a mate, as described at the beginning of the second section, and fill in the remaining cells with in-phase dominos (if $<$ were erased) or out-of-phase dominos (if $>$ were erased). Note that the brackets in the minority could not have come from dominos originally; hence, they determine the positions of the bracket pairs. Thus, the mapping f is a bijection, which implies $\mathfrak{B}\mathcal{L}'_{2n}$ has cardinality 2^{2n} and total weight $2^{2n}y^n = (4y)^n$. \square

Lemma 8. *The total weight of all members of $\mathfrak{B}\mathfrak{F}_{2n}$ with a square on cell 1 and containing exactly one other square is equal to $x^2(4y)^{n-1}$.*

Proof. We'll show that each of the following sets contains 4^{n-1} members:

- X : The set of all words having n letters in the alphabet $\{ss, <<, <>, ><, >>\}$, where the first letter is ss and no other letter is ss ;
- X_2 : The set of all lattice paths in the plane starting from the origin and having $2(n - 1)$ steps, each of which is either a $(1, 1)$ upstep or a $(1, -1)$ downstep;
- X_3 : The members of X_2 above in which all path minima occur after an even number of steps and in which exactly one path minimum is marked;
- X_4 : The set of pairs (S, T) , where S and T are square free tilings belonging to $\mathfrak{B}\mathfrak{F}_{2j}$ and $\mathfrak{B}\mathfrak{F}_{2k}$, respectively, for some $j \geq 0$ and $k \geq 0$ such that $k + j = n - 1$;
- $\mathfrak{B}\mathfrak{F}_{2n}^\circ$: The members of $\mathfrak{B}\mathfrak{F}_{2n}$ having a square on cell 1 and containing exactly one other square.

Since it is obvious that $|X| = 4^{n-1}$, we need only show

$$|X| = |X_2| = |X_3| = |X_4| = |\mathfrak{B}\mathfrak{F}_{2n}^\circ|.$$

That $|X| = |X_2|$ is trivial; simply ignore the initial ss and transform $<$ into upsteps and $>$ into downsteps. So is the equality $|X_4| = |\mathfrak{B}\mathfrak{F}_{2n}^\circ|$; just insert a square between S and T , and another at the beginning which becomes cell 1.

($|X_2| = |X_3|$): Consider the following map from X_2 to X_3 . Given $P \in X_2$, if the minima are even, mark the last one. If they are odd, let P_1 be the sub-path from the first minimum to the last one. Now invert the downstep and upstep which, respectively, precede and follow P_1 ; this causes P_1 to be "raised" 2 units. Place the mark on the newly created minimum just to the left of P_1 (which is even).

There is also a newly created minimum to the right of P_1 , and there can be none in between. Therefore we can invert this map as follows: given any $Q \in X_3$, if the mark appears on the last minimum, simply remove it. Otherwise, let Q_1 be the sub-path from the marked minimum to the next one, and "lower" Q_1 by inverting its first and last steps.

($|X_3| = |X_4|$): For this step, it will be convenient to view each member of X_3 as a pair of words (Q_1, Q_2) , both of which have an even number of letters in the alphabet $\{<, >\}$ and whose combined length is $2(n - 1)$. Now consider applying the bracket-to-domino transformation f^{-1} from the previous lemma to Q_2 . Because the lattice path corresponding to Q_2 never drops below the zero level, we can successfully erase the $<$ and mate off the $>$ without having to use wraparound. Fill in the remaining blanks with in-phase dominos as usual. Apply the obvious dual transformation to Q_1 , and we have a pair $(S, T) \in X_4$. This map can be inverted because of the fact that the transformation f applied to a straight tiling (rather than a bracelet) always creates a path where the origin is a minimum.

Composing the above mappings yields an explicit bijection $g : X \mapsto \mathfrak{B}\mathfrak{L}_{2n}^o$, which implies $\mathfrak{B}\mathfrak{L}_{2n}^o$ has total weight $4^{n-1} \cdot x^2 y^{n-1} = x^2(4y)^{n-1}$. \square

We may now establish the main result of the section.

Theorem 9. *If $n \geq 0$, then*

$$L_{2n}^*(x, y) = (x^2 + 4y)^n. \tag{20}$$

Proof. Let R be the set of words of length n in the alphabet $\{ss, \langle\langle, \langle\rangle, \rangle\langle, \rangle\rangle\}$, where each letter ss is assigned the weight x^2 and each of the four other letters is assigned the weight y . We define a weight-preserving bijection $\lambda \mapsto \lambda'$ between the sets R and $\mathfrak{B}\mathfrak{L}_{2n}$ when $n \geq 1$ as follows. If $\lambda \in R$ does not have an occurrence of ss , then apply the transformation f^{-1} used in the proof of Lemma 7 above to obtain a member $\lambda' \in \mathfrak{B}\mathfrak{L}'_{2n} \subseteq \mathfrak{B}\mathfrak{L}_{2n}$. If $\lambda \in R$ does contain an occurrence of ss , then decompose λ as $w_1ssw_2ssw_3 \cdots ssw_j$, where $j \geq 2$ and each w_i is a (possibly empty) word in the alphabet $\{\langle\langle, \langle\rangle, \rangle\langle, \rangle\rangle\}$. In this case, let λ' be the member of $\mathfrak{B}\mathfrak{L}_{2n} - \mathfrak{B}\mathfrak{L}'_{2n}$ obtained by applying the mapping g from the proof of Lemma 8, independently, to each group of consecutive letters ssw_i , $2 \leq i \leq j - 1$, as well as to the group ssw_jw_1 (wrapping around, if necessary). For example, we have

$$\begin{aligned} \lambda &= \rangle\langle\langle ss \rangle ss \rangle\rangle ss \rangle\rangle \\ &\mapsto \rangle s \langle \rangle ss \langle \rangle s \rangle \rangle ss \rangle \rangle \langle \\ &\mapsto \rangle s \langle \rangle ss \langle \rangle s l r s s l r \langle = \lambda'. \end{aligned}$$

\square

Formulas (1) and (2) are now direct consequences.

Proposition 10. *Formula (20) implies formulas (1) and (2).*

Proof. Substituting $x = L_m$ and $y = (-1)^{m+1}$ into (11) and (10), and applying (7) and (8) to the right sides and (20) and (9) to the left sides, yields (1) and (2), respectively. \square

We close this section with a formula for $F_{2n+1}^*(x, y)$ similar to (20) above.

Theorem 11. *If $n \geq 0$, then*

$$F_{2n+1}^*(x, y) = x(x^2 + 4y)^n. \tag{21}$$

Proof. Given $\lambda \in \mathfrak{BF}_{2n+1}$, insert a square at the beginning (which becomes cell 1). Decompose the tiling $s\lambda$ as $w_1w_2 \cdots w_r$, where each w_i is a member of $\mathfrak{BF}_{2n_i}^o$ for some $n_i \geq 1$ with $n_1 + n_2 + \cdots + n_r = n + 1$ (where \mathfrak{BF}_s^o is as in the proof of Lemma 8). Independently apply the mapping g^{-1} implicit in the proof of Lemma 8 to each subtiling w_i to obtain the word $g^{-1}(w_1)g^{-1}(w_2) \cdots g^{-1}(w_r)$. This operation defines a weight-preserving bijection between the members of \mathfrak{BF}_{2n+2} starting with a square and the words in the alphabet $\{ss, \ll, \langle \rangle, \gg, \rangle \rangle\}$ of length $n + 1$ starting with ss , which implies $x\mathfrak{BF}_{2n+1}^*(x, y) = x^2(x^2 + 4y)^n$. \square

4.2. Proofs of (3) and (4)

In this section, we establish formulas for the Fibonacci and Lucas polynomials which we were unable to find in the literature from which (3) and (4) will follow as special cases (see Proposition 15 below) by adding a certain feature to the bracketed tiling construction described above. A *superdomino* is an object which occupies two adjacent cells (on either a linear or circular board) and having assigned weight y . They will be the innermost object in the nesting hierarchy, which now reads

$$\textit{superdominos} < \textit{brackets} < \textit{squares and dominos}.$$

Each object can only nest inside other objects that are higher on the list—with the exception of brackets, which are also self-nesting. Consequently, we may create a “bracketed superdomino tiling” by first choosing the positions for the superdominos and then filling in the remaining positions with a bracketed tiling. In the three propositions which follow, the sign of a bracketed superdomino tiling will be defined as $(-1)^k$, where k denotes the number of superdominos.

Proposition 12. *If $n \geq 0$, then*

$$F_{2n+1}(x, y) = x \sum_{k=0}^n (-1)^k y^k \binom{2n+1-k}{k} (x^2 + 4y)^{n-k}. \tag{22}$$

Proof. The sum on the right side gives the total signed weight of all linear bracketed superdomino tilings of length $2n+1$, where a superdomino is not allowed to cover the first and last cells and where a member of $\mathfrak{BF}_{2n+1-2k}$ is used to cover the remaining cells once the positions for the k superdominos have been determined (starting with the lowest numbered cell not covered by a superdomino). Note that there are $\binom{2n+1-k}{k}$ choices regarding the positions of the k superdominos and $x(x^2 + 4y)^{n-k}$ choices regarding the remaining positions to be filled in, by Theorem 11.

Define an *innermost tile* to be either a superdomino or a bracket pair covering two adjacent cells and the *first innermost tile* to be the one whose first half covers the cell of lowest number. Define a sign-changing, weight-preserving involution by replacing the first innermost tile with its opposite. The survivors of this involution are the tilings of length $2n + 1$ comprised solely of squares and dominos, which all have positive sign and hence have total weight $F_{2n+1}(x, y)$. \square

Proposition 13. *If $n \geq 0$, then*

$$F_{2n}(x, y) = (x^2 + 4y)^n + \sum_{k=1}^n (-1)^k y^k \frac{2n+1}{k} \binom{2n-k}{k-1} (x^2 + 4y)^{n-k}. \quad (23)$$

Proof. Consider now the set of linear bracketed superdomino tilings of length $2n + 1$ as in the prior proof except that we may circle a superdomino covering cells 1 and 2. If $1 \leq k \leq n$, then there are $\frac{2n+1}{k} \binom{2n-k}{k-1}$ ways in which to position k superdominos within a tiling of length $2n + 1$. This is apparent once we observe that

$$\frac{2n+1}{k} \binom{2n-k}{k-1} = \binom{2n-k}{k-1} + \binom{2n+1-k}{k},$$

where $\binom{2n-k}{k-1}$ and $\binom{2n+1-k}{k}$ are the total number of possibilities with and without a circled superdomino. By Theorem 11, note that x times the right side of (23) then gives the total signed weight of all possible tilings.

If a superdomino tiling does not start with a circled superdomino, then apply the involution used in the prior proof. If a tiling does start with a circled superdomino, then apply this involution to only the final $2n - 1$ cells, ignoring the initial two cells. The set of survivors of this extended involution consists of those superdomino tilings having no brackets and having no superdominos with the possible exception of a circled superdomino on cells 1 and 2. Within this set of survivors, those tilings starting with a domino pair off with those starting with a superdomino. This leaves only the square-and-domino tilings of length $2n + 1$ starting with a square, which have weight $x F_{2n}(x, y)$. Equating this with the prior expression for the total weight and canceling x yields (23). \square

Proposition 14. *If $n \geq 1$, then*

$$L_{2n}(x, y) = (x^2 + 4y)^n + \sum_{k=1}^n (-1)^k y^k \frac{2n}{k} \binom{2n-1-k}{k-1} (x^2 + 4y)^{n-k}. \quad (24)$$

Proof. The sum on the right side of (24) gives the total signed weight of all circular bracketed superdomino tilings of length $2n$, where a superdomino is allowed to cover the first and last cells and where a member of $\mathfrak{B}\mathfrak{L}_{2n-2k}$ is used to cover the remaining cells once the positions for the k superdominos have been determined. Note that if $1 \leq k \leq n$, then there are $\frac{2n}{k} \binom{2n-1-k}{k-1}$ choices for the positions of k

superdominos and $(x^2 + 4y)^{n-k}$ choices for the remaining positions to be filled in, by Theorem 9. The involution used in the proof of (22) again applies, the survivors now being the members of \mathfrak{L}_{2n} . \square

Formulas (3) and (4) are now easy consequences.

Proposition 15. *Formulas (23) and (24) imply formulas (3) and (4).*

Proof. Substituting $x = L_m$ and $y = (-1)^{m+1}$ into (23) and (24), and applying (9) to the right sides and applying (7) and (8) to the left sides, yields (3) and (4), respectively. \square

Still other identities follow directly from the above propositions.

Remarks Taking $x = y = 1$ in the above propositions, we obtain combinatorial proofs for identities such as

$$F_{2n} = \sum_{k=0}^{n-1} (-1)^k \binom{2n-1-k}{k} 5^{n-1-k}, \quad n \geq 1. \tag{25}$$

Taking $x = 1$ and $y = -1$ in (22), for example, and using the well-known fact (see, e.g., Identity 172 of [5]) that

$$F_n(1, -1) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} = \begin{cases} (-1)^n, & \text{if } n \equiv 0 \pmod{3}; \\ (-1)^{n+1}, & \text{if } n \equiv 1 \pmod{3}; \\ 0, & \text{if } n \equiv 2 \pmod{3}, \end{cases} \tag{26}$$

we get direct proofs of identities like

$$\sum_{k=0}^n (-3)^{n-k} \binom{2n+1-k}{k} = \begin{cases} 1, & \text{if } n \equiv 0 \pmod{3}; \\ -1, & \text{if } n \equiv 1 \pmod{3}; \\ 0, & \text{if } n \equiv 2 \pmod{3}. \end{cases} \tag{27}$$

Substituting $x = L_m$ and $y = (-1)^{m+1}$ into (22), and applying (7) and (9), yields the following formula for $F_{(2n+2)m}$ which seems to be new:

$$F_{(2n+2)m} = F_m L_m \sum_{k=0}^n (-1)^{mk} \binom{2n+1-k}{k} (5F_m^2)^{n-k}, \quad m, n \geq 0. \tag{28}$$

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