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Coefficient ConditionsFor Certain Univalent Functions

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Abstract

Some subclasses of analytic functions $f(z)$ in the open unit disk $\mathbb{U}$ are introduced. In the present paper, Some interesting sufficient conditions, including coefficient inequalities related close-to-convex functions $f(z)$ of order α with respect to a fixed starlike function $g(z)$ and strongly starlike functions $f(z)$ of order μ in $\mathbb{U}$, are discussed. Several special cases and consequences of these coefficient inequalities are also pointed out.

Keywords: *Coefficient inequality, analytic function, univalent function, close-to-convex function, spiral-like function, strongly starlike function.*

1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Furthermore, let $\mathcal{P}$ be the class of functions $p(z)$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \tag{2}$$

which are analytic in $\mathbb{U}$.
If $f(z) \in \mathcal{A}$ satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (z \in \mathbb{U}), \tag{3}$$

then $f(z)$ is said to be starlike in $\mathbb{U}$.
We denote by $\mathcal{S}^*$ the class of all functions $f(z)$ which are starlike in $\mathbb{U}$. Also, $f(z) \in \mathcal{A}$ is said to be convex in $\mathbb{U}$ if it satisfies

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \qquad (z \in \mathbb{U}). \tag{4}$$

We denote by $\mathcal{K}$ the subclass of $\mathcal{A}$ consisting of all convex functions $f(z)$ in $\mathbb{U}$. We begin with the definitions for the subclasses $\mathcal{T}(\alpha)$ and $\mathcal{U}(\alpha)$ of $\mathcal{A}$.

Definition 1.1 *A function $f(z) \in \mathcal{A}$ belongs to $\mathcal{T}(\alpha)$ if and only if it satisfies*

$$\mathrm{Re}\frac{f(z)}{z} > \alpha \qquad (z \in \mathbb{U}) \tag{5}$$

for some α $(0 \leqq \alpha < 1)$.

Definition 1.2 *A function $f(z) \in \mathcal{A}$ is in the class $\mathcal{U}(\alpha)$ if and only if it satisfies*

$$\mathrm{Re} f'(z) > \alpha \qquad (z \in \mathbb{U}) \tag{6}$$

for some α $(0 \leqq \alpha < 1)$.

For the proof of our results, we need the following lemma.

Lemma 1.3 (see, [1], [3]) *A function $p(z) \in \mathcal{P}$ satisfies* Re $p(z) > 0$ $(z \in \mathbb{U})$ *if and only if*

$$p(z) \neq \frac{x-1}{x+1} \qquad (z \in \mathbb{U})$$

for all $|x| = 1$.

By observation of this, many relations concerning the various subclasses of $\mathcal{A}$, for example, the class of starlike, convex or λ-spiral-like functions were studied (cf. [1], [2], [3]). Our results are motivated by these investigation. In this paper, we discuss the sufficient conditions for the known or new classes involving the above.

Lemma 1.4 *A function $f(z) \in \mathcal{A}$ is in $\mathcal{T}(\alpha)$ if and only if*

$$1 + \sum_{n=2}^{\infty} A_n z^{n-1} \neq 0 \tag{7}$$

where

$$A_n = \frac{x+1}{2(1-\alpha)} a_n \quad (n \geqq 2).$$

Proof. Putting $p(z) = \dfrac{\frac{f(z)}{z} - \alpha}{1-\alpha}$ for $f(z) \in \mathcal{T}(\alpha)$, we obtain that $p(z) \in \mathcal{P}$, and Re $p(z) > 0$. Using Lemma 1.3, we have that

$$\frac{\frac{f(z)}{z} - \alpha}{1-\alpha} \neq \frac{x-1}{x+1} \qquad (z \in \mathbb{U})$$

for all $|x| = 1$. Then, we need not consider Lemma 1.3 for $z = 0$, because it follows that

$$p(0) = 1 \neq \frac{x-1}{x+1}$$

for all $|x| = 1$. This implies that

$$(x+1)f(z) + (1 - 2\alpha - x)z \neq 0. \tag{8}$$

It follows that (8) is equivalent to

$$(x+1)\left(z + \sum_{n=2}^{\infty} a_n z^n\right) + (1 - 2\alpha - x)z \neq 0$$

or

$$2(1-\alpha)z\left\{1 + \sum_{n=2}^{\infty} \frac{x+1}{2(1-\alpha)} a_n z^{n-1}\right\} \neq 0. \tag{9}$$

Dividing the both sides of (9) by $2(1-\alpha)z$ $(z \neq 0)$, we know that

$$1 + \sum_{n=2}^{\infty} \frac{x+1}{2(1-\alpha)} a_n z^{n-1} \neq 0.$$

This completes the proof of lemma. □

2 Coefficient conditions for functions in the classes $\mathcal{T}(\alpha)$ and $\mathcal{CC}_\lambda(\alpha; g(z))$

Our result for $f(z)$ to be in $\mathcal{T}(\alpha)$ is contained in

Theorem 2.1 *If $f(z) \in \mathcal{A}$ satisfies the following condition*

$$\sum_{n=2}^{\infty} \left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} (-1)^{k-j} \binom{\beta}{k-j} a_j \right\} \binom{\gamma}{n-k} \right| \leqq 1 - \alpha$$

for some α $(0 \leqq \alpha < 1)$, $\beta \in \mathbb{R}$ and $\gamma \in \mathbb{R}$, then $f(z) \in \mathcal{T}(\alpha)$.

Proof. Note that

$$(1-z)^{\beta} \neq 0,\ (1+z)^{\gamma} \neq 0\ (\beta,\ \gamma \in \mathbb{R};\ z \in \mathbb{U}).$$

Hence if the following expression

$$\left(1+\sum_{n=2}^{\infty} A_n z^{n-1}\right)(1-z)^{\beta}(1+z)^{\gamma} \neq 0 \tag{10}$$

holds true, then we have

$$1+\sum_{n=2}^{\infty} A_n z^{n-1} \neq 0,$$

which is the relation (7) of Lemma 1.4. We know that (10) is equivalent to

$$1+\sum_{n=2}^{\infty}\left[\sum_{k=1}^{n}\left\{\sum_{j=1}^{k} A_j(-1)^{k-j}\begin{pmatrix}\beta\\ k-j\end{pmatrix}\right\}\begin{pmatrix}\gamma\\ n-k\end{pmatrix}\right] z^{n-1} \neq 0.$$

Therefore, if $f(z) \in \mathcal{A}$ satisfies

$$\sum_{n=2}^{\infty}\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k} A_j(-1)^{k-j}\begin{pmatrix}\beta\\ k-j\end{pmatrix}\right\}\begin{pmatrix}\gamma\\ n-k\end{pmatrix}\right| \leqq 1$$

with $A_n = \dfrac{x+1}{2(1-\alpha)}a_n$, that is, that

$$\begin{aligned}
&\frac{1}{2(1-\alpha)}\sum_{n=2}^{\infty}\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}(x+1)a_j(-1)^{k-j}\begin{pmatrix}\beta\\ k-j\end{pmatrix}\right\}\begin{pmatrix}\gamma\\ n-k\end{pmatrix}\right| \\
\leqq\ & \frac{1}{2(1-\alpha)}\sum_{n=2}^{\infty}\left[\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}(-1)^{k-j}\begin{pmatrix}\beta\\ k-j\end{pmatrix}a_j\right\}\begin{pmatrix}\gamma\\ n-k\end{pmatrix}\right|\right. \\
&\qquad\left. +|x|\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}(-1)^{k-j}\begin{pmatrix}\beta\\ k-j\end{pmatrix}a_j\right\}\begin{pmatrix}\gamma\\ n-k\end{pmatrix}\right|\right] \\
=\ & \frac{1}{1-\alpha}\sum_{n=2}^{\infty}\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}(-1)^{k-j}\begin{pmatrix}\beta\\ k-j\end{pmatrix}a_j\right\}\begin{pmatrix}\gamma\\ n-k\end{pmatrix}\right| \\
\leqq\ & 1,
\end{aligned}$$

then $f(z) \in \mathcal{T}(\alpha)$. This completes the proof of Theorem 2.1. □

Putting $\beta = \gamma = 0$ in Theorem 2.1, we see the following corollary.

Corollary 2.2 *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} |a_n| \leqq 1 - \alpha$$

for some α $(0 \leqq \alpha < 1)$, then $f(z) \in \mathcal{T}(\alpha)$.

Next, we derive the coefficient condition for $f(z)$ to be in the class $\mathcal{U}(\alpha)$.

Theorem 2.3 *If $f(z) \in \mathcal{A}$ satisfies the following condition*

$$\sum_{n=2}^{\infty} \left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} (-1)^{k-j} \binom{\beta}{k-j} j a_j \right\} \binom{\gamma}{n-k} \right| \leqq 1 - \alpha$$

for some α $(0 \leqq \alpha < 1)$, $\beta \in \mathbb{R}$, and $\gamma \in \mathbb{R}$, then $f(z) \in \mathcal{U}(\alpha)$.

Proof. Since $f(z) \in \mathcal{U}(\alpha) \Leftrightarrow zf'(z) \in \mathcal{T}(\alpha)$ and

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \ , \ zf'(z) = z + \sum_{n=2}^{\infty} n a_n z^n,$$

replacing a_j of Theorem 2.1 with ja_j, we prove the theorem. □

Taking $\beta = \gamma = 0$ in Theorem 2.3, we obtain

Corollary 2.4 *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} n|a_n| \leqq 1 - \alpha$$

for some α $(0 \leqq \alpha < 1)$, then $f(z) \in \mathcal{U}(\alpha)$.

Definition 2.5 (see, for details, [2]) *If $f(z) \in \mathcal{A}$ satisfies*

$$\text{Re } e^{i\lambda} \left(\frac{zf'(z)}{g(z)} - \alpha \right) > 0 \qquad (z \in \mathbb{U}) \tag{11}$$

for some α $(0 \leqq \alpha < 1)$, λ $(-\frac{\pi}{2} < \lambda < \frac{\pi}{2})$ *and starlike function* $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, *then* $f(z)$ *is said to be close-to-convex of order* α *with respect to a fixed starlike function* $g(z)$, *and let* $\mathcal{CC}_\lambda(\alpha; g(z))$ *denote the class of functions* $f(z)$ *which are close-to-convex of order* α *with respect to a fixed starlike function* $g(z)$.

In particular, when $g(z) = z \in \mathcal{S}^*$ *and* $\lambda = 0$, *we see that*

$$\mathcal{CC}_0(\alpha; z) \equiv \mathcal{U}(\alpha).$$

Remark 2.6 Replacing $g(z)$ by $f(z)$ in (11), we say that $f(z)$ is said to be λ-spiral of order α in $\mathbb{U}$, and write $\mathcal{SP}(\lambda, \alpha)$ defined by

$$\mathcal{SP}(\lambda, \alpha) = \left\{ f(z) \in \mathcal{A} \ : \ \operatorname{Re} e^{i\lambda} \left(\frac{zf'(z)}{f(z)} - \alpha \right) > 0 \right\}.$$

Lemma 2.7 *A function* $f(z) \in \mathcal{A}$ *is in* $\mathcal{CC}_\lambda(\alpha; g(z))$ *if and only if*

$$1 + \sum_{n=2}^{\infty} B_n z^{n-1} \neq 0 \tag{12}$$

where

$$B_n = \frac{na_n + (2(1-\alpha)e^{-i\lambda}\cos\lambda - 1)b_n + x(na_n - b_n)}{2(1-\alpha)e^{-i\lambda}\cos\lambda}.$$

Proof. Letting $p(z) = \dfrac{e^{i\lambda}\left(\frac{zf'(z)}{g(z)} - \alpha\right) - i(1-\alpha)\sin\lambda}{(1-\alpha)\cos\lambda}$, we see that $p(z) \in \mathcal{P}$ and $\operatorname{Re} p(z) > 0$ $(z \in \mathbb{U})$. It follows from Lemma 1.1 that

$$\frac{e^{i\lambda}\left(\frac{zf'(z)}{g(z)} - \alpha\right) - i(1-\alpha)\sin\lambda}{(1-\alpha)\cos\lambda} \neq \frac{x-1}{x+1} \qquad (z \in \mathbb{U}) \tag{13}$$

for all $|x| = 1$. Then, we need not consider Lemma 1.1 for $z = 0$, because it follows that

$$p(0) = 1 \neq \frac{x-1}{x+1}$$

for all $|x| = 1$. Since (13) implies that

$$\frac{e^{i\lambda}(zf'(z) - \alpha g(z)) - i(1-\alpha)g(z)\sin\lambda}{(1-\alpha)\cos\lambda} \neq \frac{x-1}{x+1} g(z),$$

we obtain that

$$(x+1)\left\{e^{i\lambda}(zf'(z)-\alpha g(z))-i(1-\alpha)g(z)\sin\lambda\right\}\neq(x-1)(1-\alpha)g(z)\cos\lambda$$

or

$$(x+1)e^{i\lambda}zf'(z)-\alpha e^{i\lambda}g(z)-x\alpha e^{i\lambda}g(z)-i(1-\alpha)g(z)\sin\lambda-ix(1-\alpha)g(z)\sin\lambda \qquad (14)$$
$$\neq x(1-\alpha)g(z)\cos\lambda-(1-\alpha)g(z)\cos\lambda.$$

The relation (14) is equivalent to

$$(x+1)e^{i\lambda}zf'(z)-\alpha e^{i\lambda}g(z)-x\alpha e^{i\lambda}g(z)-x(1-\alpha)e^{i\lambda}g(z)+(1-\alpha)e^{-i\lambda}g(z)\neq 0$$

that is,

$$(1+x)e^{i\lambda}zf'(z)+(e^{-i\lambda}-xe^{i\lambda}-2\alpha\cos\lambda)g(z)\neq 0.$$

Note that the above relation can be weitten with

$$(x+1)e^{i\lambda}\left(z+\sum_{n=2}^{\infty}na_nz^n\right)+(e^{-i\lambda}-xe^{i\lambda}-2\alpha\cos\lambda)\left(z+\sum_{n=2}^{\infty}b_nz^n\right)\neq 0$$

or

$$2(1-\alpha)\cos\lambda\ z\left\{1+\sum_{n=2}^{\infty}\frac{n(x+1)a_n+(e^{-2i\lambda}-x-2\alpha e^{-i\lambda}\cos\lambda)b_n}{2(1-\alpha)e^{-i\lambda}\cos\lambda}z^{n-1}\right\}\neq 0. \qquad (15)$$

Dividing the both sides of (15) by $2(1-\alpha)\cos\lambda\ z$ $(z\neq 0)$ and noting

$$e^{-2i\lambda}=-1+2e^{-i\lambda}\cos\lambda, \qquad (16)$$

we know that

$$1+\sum_{n=2}^{\infty}\frac{na_n+(2(1-\alpha)e^{-i\lambda}-1)b_n+x(na_n-b_n)}{2(1-\alpha)e^{-i\lambda}\cos\lambda}z^{n-1}\neq 0.$$

This completes the proof of the lemma. □

Applying Lemma 2.7, we obtain

Theorem 2.8 *If $f(z)\in\mathcal{A}$ satisfies the following condition*

$$\sum_{n=2}^{\infty}\left[\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}\left(ja_j+((1-\alpha)e^{-2i\lambda}-\alpha)b_n\right)(-1)^{k-j}\binom{\beta}{k-j}\right\}\binom{\gamma}{n-k}\right|\right.$$
$$\left.+\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}(ja_j-b_n)(-1)^{k-j}\binom{\beta}{k-j}\right\}\binom{\gamma}{n-k}\right|\right]\leqq 2(1-\alpha)\cos\lambda$$

for some α $(0 \leqq \alpha < 1)$, λ $\left(-\frac{\pi}{2} < \lambda < \frac{\pi}{2}\right)$, $\beta \in \mathbb{R}$, $\gamma \in \mathbb{R}$ *and* $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*$, *then* $f(z) \in \mathcal{CC}_\lambda(\alpha; g(z))$.

Proof. Applying the same method of the proof in Theorem 2.1, we know that $f(z)$ belongs to $\mathcal{CC}_\lambda(\alpha; g(z))$ if $f(z) \in \mathcal{A}$ satisfies

$$\sum_{n=2}^{\infty} \left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} B_j (-1)^{k-j} c_{k-j} \right\} d_{n-k} \right| \leqq 1$$

where $c_n = \binom{\beta}{n}$, $d_n = \binom{\gamma}{n}$ and B_j is the coefficient defined by Lemma 2.7.

Now, we consider that

$$\frac{1}{|2(1-\alpha)e^{-i\lambda}\cos\lambda|} \sum_{n=2}^{\infty} \left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} \Big(ja_j + (2(1-\alpha)e^{-i\lambda}\cos\lambda - 1)b_j + x(ja_j - b_j) \Big) (-1)^{k-j} c_{k-j} \right\} d_{n-k} \right|$$

$$\leqq \frac{1}{2(1-\alpha)\cos\lambda} \sum_{n=2}^{\infty} \left[\left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} \Big(ja_j + (2(1-\alpha)e^{-i\lambda}\cos\lambda - 1)b_j \Big) (-1)^{k-j} c_{k-j} \right\} d_{n-k} \right| \right.$$
$$\left. + |x| \left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} (ja_j - b_j)(-1)^{k-j} c_{k-j} \right\} d_{n-k} \right| \right]$$

$$\leqq 1.$$

This implies that if $f(z) \in \mathcal{A}$ satisfies

$$\sum_{n=2}^{\infty} \left[\left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} \Big(ja_j + (2(1-\alpha)e^{-i\lambda}\cos\lambda - 1)b_j \Big) (-1)^{k-j} \binom{\beta}{k-j} \right\} \binom{\gamma}{n-k} \right| \right.$$
$$\left. + \left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} (ja_j - b_j)(-1)^{k-j} \binom{\beta}{k-j} \right\} \binom{\gamma}{n-k} \right| \right] \leqq 2(1-\alpha)\cos\lambda,$$

then $f(z) \in \mathcal{CC}_\lambda(\alpha; g(z))$. This completes the proof of Theorem 2.8. □

Considering $g(z) = f(z)$ in Theorem 2.8 and noting (16), we have the following corollary.

Corollary 2.9 ([1], Theorem 3) *If* $f(z) \in \mathcal{A}$ *satisfies the following inequality*

$$\sum_{n=2}^{\infty}\left[\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}(j-\alpha+(1-\alpha)e^{-2i\lambda})(-1)^{k-j}\left(\begin{array}{c}\beta\\k-j\end{array}\right)a_j\right\}\left(\begin{array}{c}\gamma\\n-k\end{array}\right)\right|\right.$$
$$\left.+\left|\sum_{k=1}^{\infty}\left\{\sum_{j=1}^{k}(j-1)(-1)^{k-j}\left(\begin{array}{c}\beta\\k-j\end{array}\right)a_j\right\}\left(\begin{array}{c}\gamma\\n-k\end{array}\right)\right|\right] \leqq 2(1-\alpha)\cos\lambda$$

for some α $(0 \leqq \alpha < 1)$, λ $\left(-\frac{\pi}{2} < \lambda < \frac{\pi}{2}\right)$, $\beta \in \mathbb{R}$ *and* $\gamma \in \mathbb{R}$, *then* $f(z) \in \mathcal{SP}(\lambda, \alpha)$.

Furthermore, setting $\lambda = 0$ in Theorem 2.8, we obtain the following condition for $\mathcal{CC}_0(\alpha; g(z))$.

Corollary 2.10 *If* $f(z) \in \mathcal{A}$ *satisfies the following condition*

$$\sum_{n=2}^{\infty}\left[\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}(ja_j+(1-2\alpha)b_j)(-1)^{k-j}\left(\begin{array}{c}\beta\\k-j\end{array}\right)\right\}\left(\begin{array}{c}\gamma\\n-k\end{array}\right)\right|\right.$$
$$\left.+\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}(ja_j-b_j)(-1)^{k-j}\left(\begin{array}{c}\beta\\k-j\end{array}\right)\right\}\left(\begin{array}{c}\gamma\\n-k\end{array}\right)\right|\right] \leqq 2(1-\alpha)$$

for some α $(0 \leqq \alpha < 1)$, $\beta \in \mathbb{R}$, $\gamma \in \mathbb{R}$ *and* $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*$, *then* $f(z) \in \mathcal{CC}_0(\alpha; g(z))$.

3 Coefficient conditions for functions in the class $\mathcal{STS}(\mu_1, \mu_2)$

In this section, we consider the subclass $\mathcal{STS}(\mu_1, \mu_2)$ of $\mathcal{A}$ due to Takahashi and Nunokawa [4] as follows:

$$\mathcal{STS}(\mu_1,\mu_2) = \left\{f(z) \in \mathcal{A} \ : \ \frac{\pi\mu_1}{2} < \arg\frac{zf'(z)}{f(z)} < \frac{\pi\mu_2}{2} \quad (-1 \leqq \mu_1 < 0 < \mu_2 \leqq 1)\right\}.$$

Now, taking $\mu_1 = -\mu$ and $\mu_2 = \mu$ for some μ $(0 < \mu \leqq 1)$, we have the class $\mathcal{STS}(\mu)$ of strongly starlike functions of order μ in $\mathbb{U}$ defined by

$$\mathcal{STS}(\mu) = \left\{f(z) \in \mathcal{A} \ : \ \left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi\mu}{2} \quad (0 < \mu \leqq 1)\right\}.$$

Similary, we also define the subclasses $\mathcal{STC}(\mu_1, \mu_2)$ and $\mathcal{STC}(\mu)$ of $\mathcal{A}$ by

$$\mathcal{STC}(\mu_1,\mu_2) = \left\{f(z) \in \mathcal{A} \ : \ \frac{\pi\mu_1}{2} < \arg\left(1+\frac{zf''(z)}{f'(z)}\right) < \frac{\pi\mu_2}{2} \quad (-1 \leqq \mu_1 < 0 < \mu_2 \leqq 1)\right\}$$

and

$$\mathcal{STC}(\mu) = \left\{ f(z) \in \mathcal{A} \ : \ \left| \arg \left(1 + \frac{zf''(z)}{f'(z)} \right) \right| < \frac{\pi\mu}{2} \quad (0 < \mu \leqq 1) \right\},$$

respectively.
Now, we derive

Theorem 3.1 *If $f(z) \in \mathcal{A}$ saitsfies the both conditions of*

$$\sum_{n=2}^{\infty} \left[\left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} (j - e^{i\pi\mu_1})(-1)^{k-j} \binom{\beta}{k-j} a_j \right\} \binom{\gamma}{n-k} \right| + \left| \sum_{k=1}^{\infty} \left\{ \sum_{j=1}^{k} (j-1)(-1)^{k-j} \binom{\beta}{k-j} a_j \right\} \binom{\gamma}{n-k} \right| \right] \leqq -2 \sin \frac{\pi\mu_1}{2} \tag{17}$$

and

$$\sum_{n=2}^{\infty} \left[\left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} (j - e^{i\pi\mu_2})(-1)^{k-j} \binom{\beta}{k-j} a_j \right\} \binom{\gamma}{n-k} \right| + \left| \sum_{k=1}^{\infty} \left\{ \sum_{j=1}^{k} (j-1)(-1)^{k-j} \binom{\beta}{k-j} a_j \right\} \binom{\gamma}{n-k} \right| \right] \leqq 2 \sin \frac{\pi\mu_2}{2} \tag{3.2}$$

for some μ_1 $(-1 \leqq \mu_1 < 0)$, μ_2 $(0 < \mu_2 \leqq 1)$, $\beta \in \mathbb{R}$ and $\gamma \in \mathbb{R}$, then $f(z) \in \mathcal{STS}(\mu_1, \mu_2)$.

Proof. Setting $\lambda = -\dfrac{1+\mu_1}{2}\pi$ or $\lambda = \dfrac{1-\mu_2}{2}\pi$ and taking $\alpha = 0$ in Corollary 2.9, we obtain the inequality (17) or (18).
Thus, it follows that

$$-\frac{\pi}{2} < \arg \left(e^{-\frac{1+\mu_1}{2}\pi} \frac{zf'(z)}{f(z)} \right) < \frac{\pi}{2} \quad \textit{and} \quad -\frac{\pi}{2} < \arg \left(e^{\frac{1-\mu_2}{2}\pi} \frac{zf'(z)}{f(z)} \right) < \frac{\pi}{2}$$

that is, that

$$\frac{\pi\mu_1}{2} < \arg \frac{zf'(z)}{f(z)} < \frac{\pi(2+\mu_1)}{2} \quad \textit{and} \quad -\frac{\pi(2-\mu_2)}{2} < \arg \frac{zf'(z)}{f(z)} < \frac{\pi\mu_2}{2}.$$

Therefore, we have

$$\frac{\pi\mu_1}{2} < \arg \frac{zf'(z)}{f(z)} < \frac{\pi\mu_2}{2}.$$

This completes the proof of Theorem3.1. □

Letting $\mu_1 = -\mu$ and $\mu_2 = \mu$ for some μ $(0 < \mu \leqq 1)$ in Theorem 3.1, we know the following corollary.

Corollary 3.2 *If $f(z) \in \mathcal{A}$ satisfies the both inequalities of*

$$\sum_{n=2}^{\infty}\left[\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}(j-e^{-i\pi\mu})(-1)^{k-j}\binom{\beta}{k-j}a_j\right\}\binom{\gamma}{n-k}\right|\right.$$
$$\left.+\left|\sum_{k=1}^{\infty}\left\{\sum_{j=1}^{k}(j-1)(-1)^{k-j}\binom{\beta}{k-j}a_j\right\}\binom{\gamma}{n-k}\right|\right] \leqq 2\sin\frac{\pi\mu}{2}$$

and

$$\sum_{n=2}^{\infty}\left[\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}(j-e^{i\pi\mu})(-1)^{k-j}\binom{\beta}{k-j}a_j\right\}\binom{\gamma}{n-k}\right|\right.$$
$$\left.+\left|\sum_{k=1}^{\infty}\left\{\sum_{j=1}^{k}(j-1)(-1)^{k-j}\binom{\beta}{k-j}a_j\right\}\binom{\gamma}{n-k}\right|\right] \leqq 2\sin\frac{\pi\mu}{2}$$

for some μ $(0 < \mu \leqq 1)$, $\beta \in \mathbb{R}$ and $\gamma \in \mathbb{R}$, then $f(z) \in \mathcal{STS}(\mu)$.

In particular, putting $\mu_1 = -1$ and $\mu_2 = 1$ in Theorem 3.1, we see the following result.

Corollary 3.3 ([1], Corollary 2) *If $f(z) \in \mathcal{A}$ satisfies the following condition*

$$\sum_{n=2}^{\infty}\left[\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}(j+1)(-1)^{k-j}\binom{\beta}{k-j}a_j\right\}\binom{\gamma}{n-k}\right|\right.$$
$$\left.+\left|\sum_{k=1}^{\infty}\left\{\sum_{j=1}^{k}(j-1)(-1)^{k-j}\binom{\beta}{k-j}a_j\right\}\binom{\gamma}{n-k}\right|\right] \leqq 2$$

for some $\beta \in \mathbb{R}$ and $\gamma \in \mathbb{R}$, then $f(z) \in \mathcal{S}^*$.

Finally, noting that

$$f(z) \in \mathcal{STC}(\mu_1,\mu_2) \iff zf'(z) \in \mathcal{STS}(\mu_1,\mu_2),$$

we have the following results.

Theorem 3.4 *If $f(z) \in \mathcal{A}$ satisfies the both conditions of*

$$\sum_{n=2}^{\infty}\left[\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k} j(j-e^{i\pi\mu_1})(-1)^{k-j}\begin{pmatrix}\beta\\k-j\end{pmatrix}a_j\right\}\begin{pmatrix}\gamma\\n-k\end{pmatrix}\right|\right.$$
$$\left.+\left|\sum_{k=1}^{\infty}\left\{\sum_{j=1}^{k} j(j-1)(-1)^{k-j}\begin{pmatrix}\beta\\k-j\end{pmatrix}a_j\right\}\begin{pmatrix}\gamma\\n-k\end{pmatrix}\right|\right] \leqq -2\sin\frac{\pi\mu_1}{2}$$

and

$$\sum_{n=2}^{\infty}\left[\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k} j(j-e^{i\pi\mu_2})(-1)^{k-j}\begin{pmatrix}\beta\\k-j\end{pmatrix}a_j\right\}\begin{pmatrix}\gamma\\n-k\end{pmatrix}\right|\right.$$
$$\left.+\left|\sum_{k=1}^{\infty}\left\{\sum_{j=1}^{k} j(j-1)(-1)^{k-j}\begin{pmatrix}\beta\\k-j\end{pmatrix}a_j\right\}\begin{pmatrix}\gamma\\n-k\end{pmatrix}\right|\right] \leqq 2\sin\frac{\pi\mu_2}{2}$$

for some μ_1 $(-1 \leqq \mu_1 < 0)$, μ_2 $(0 < \mu_2 \leqq 1)$, $\beta \in \mathbb{R}$ *and* $\gamma \in \mathbb{R}$, *then* $f(z) \in \mathcal{STC}(\mu_1, \mu_2)$.

Corollary 3.5 *If* $f(z) \in \mathcal{A}$ *satisfies the both relations of*

$$\sum_{n=2}^{\infty}\left[\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k} j(j-e^{-i\pi\mu})(-1)^{k-j}\begin{pmatrix}\beta\\k-j\end{pmatrix}a_j\right\}\begin{pmatrix}\gamma\\n-k\end{pmatrix}\right|\right.$$
$$\left.+\left|\sum_{k=1}^{\infty}\left\{\sum_{j=1}^{k} j(j-1)(-1)^{k-j}\begin{pmatrix}\beta\\k-j\end{pmatrix}a_j\right\}\begin{pmatrix}\gamma\\n-k\end{pmatrix}\right|\right] \leqq 2\sin\frac{\pi\mu}{2}$$

and

$$\sum_{n=2}^{\infty}\left[\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k} j(j-e^{i\pi\mu})(-1)^{k-j}\begin{pmatrix}\beta\\k-j\end{pmatrix}a_j\right\}\begin{pmatrix}\gamma\\n-k\end{pmatrix}\right|\right.$$
$$\left.+\left|\sum_{k=1}^{\infty}\left\{\sum_{j=1}^{k} j(j-1)(-1)^{k-j}\begin{pmatrix}\beta\\k-j\end{pmatrix}a_j\right\}\begin{pmatrix}\gamma\\n-k\end{pmatrix}\right|\right] \leqq 2\sin\frac{\pi\mu}{2}$$

for some μ $(0 < \mu \leqq 1)$, $\beta \in \mathbb{R}$ *and* $\gamma \in \mathbb{R}$, *then* $f(z) \in \mathcal{STC}(\mu)$.

Corollary 3.6 ([1], Corollary 4) *If* $f(z) \in \mathcal{A}$ *satisfies the following condition*

$$\sum_{n=2}^{\infty}\left[\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k} j(j+1)(-1)^{k-j}\begin{pmatrix}\beta\\k-j\end{pmatrix}a_j\right\}\begin{pmatrix}\gamma\\n-k\end{pmatrix}\right|\right.$$
$$\left.+\left|\sum_{k=1}^{\infty}\left\{\sum_{j=1}^{k} j(j-1)(-1)^{k-j}\begin{pmatrix}\beta\\k-j\end{pmatrix}a_j\right\}\begin{pmatrix}\gamma\\n-k\end{pmatrix}\right|\right] \leqq 2$$

for some $\beta \in \mathbb{R}$ *and* $\gamma \in \mathbb{R}$, *then* $f(z) \in \mathcal{K}$.

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