Fixed Point Theorems of Compatible Mappings of Type(R) in Metric Spaces

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Abstract

In this paper we proved some common fixed point theorems for compatible mappings of Type(R) introduced by Y. Rohen and M. R. Singh [7] in metric spaces.

Keywords: common fixed point, compatible mappings, compatible mappings of type(P), compatible mappings of type(R).

1 Introduction and Preliminaries


Following are the various types of compatible mappings.

Definition [1]: Let \( S \) and \( T \) be mappings from a complete metric space \( X \) into itself. The mappings \( S \) and \( T \) are said to be compatible if
\[
\lim_{n \to \infty} d(STx_n, TSx_n) = 0
\]
whenever \( \{x_n\} \) is a sequence in \( X \) such that
\[
\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \quad \text{for some} \quad t \in X.
\]
Definition [5]: Let $S$ and $T$ be mappings from a complete metric space $X$ into itself. The mappings $S$ and $T$ are said to be compatible of type (P) if
$$\lim_{n \to \infty} d(SSx_n, TTx_n) = 0$$
whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t$ for some $t \in X$.

Definition [7]: Let $S$ and $T$ be mappings from a complete metric space $X$ into itself. The mappings $S$ and $T$ are said to be compatible of type (R) if
$$\lim_{n \to \infty} d(STx_n, TSx_n) = 0$$
and
$$\lim_{n \to \infty} d(SSx_n, TTx_n) = 0$$
whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t$ for some $t \in X$.

2 Main Result

We prove the following propositions.

Proposition 2.1. Let $S$, $T : (X, d) \to (X, d)$ be mappings. If $S$ and $T$ are compatible mappings of type (R) and $Sz = Tz$ for some $z \in X$, then $SSz = STz = TSz = TTz$.

Proof: Let $\{x_n\}$ be a sequence in $X$ defined by $x_n = z$, $n = 1, 2, \ldots$, and $Sz = Tz$ for some $z \in X$. Then we have $Sx_n, Tx_n \to Sz$ as $n \to \infty$. Since $S$ and $T$ are compatible mappings of type (R), we have
$$d(SSz, TTz) = \lim_{n \to \infty} d(SSx_n, TTx_n) = 0.$$ Therefore, $SSz = TTz$. But $Sz = Tz$ implies $SSz = STz = TSz = TTz$. This completes the proof.

Proposition 2.2. Let $S$, $T : (X, d) \to (X, d)$ be mappings. Let $S$ and $T$ are compatible mappings of type (R) and let $Sx_n, Tx_n \to z$ as $n \to \infty$ for some $z \in X$. Then we have the followings:

(i) $\lim_{n \to \infty} TTx_n = Sz$ if $S$ is continuous at $z$.
(ii) $\lim_{n \to \infty} SSx_n = Tz$ if $T$ is continuous at $z$.
(iii) $\lim_{n \to \infty} STx_n = Tz$ if $T$ is continuous.
(iv) $\lim_{n \to \infty} TSx_n = Sz$ if $S$ is continuous.
(v) $STz = TSz$ and $Sz = Tz$ if $S$ and $T$ are continuous at $z$.

Proof. (i) Suppose that $S$ is continuous at $z$. Since
\[
\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = z
\]
for some \(z \in X\), we have \(SSx_n \to Sz\) as \(n \to \infty\). Again, since \(S\) and \(T\) are compatible of type (R), we have \(\lim_{n \to \infty} d(TTx_n, SSx_n) = 0\) and so, since we have
\[
d(TTx_n, Sz) \leq d(TTx_n, SSx_n) + d(SSx_n, Sz),
\]
it follows that \(TTx_n \to Sz\) as \(n \to \infty\).

Proof of (ii), (iii) and (iv) can be done in the similar process as in (i).

(v) Suppose that \(S\) and \(T\) are continuous at \(z\). Since \(Tx_n \to z\) as \(n \to \infty\) and \(S\) is continuous at \(z\), by (i), \(TTx_n \to Sz\) as \(n \to \infty\). On the other hand, since \(Tx_n \to z\) as \(n \to \infty\) and \(T\) is also continuous at \(z\), \(TTx_n \to Tz\). Thus, we have \(Sz = Tz\) by the uniqueness of the limit and so, by proposition 2.1, \(TSz = STz\).

This completes the proof.

**Theorem 2.3.** Let \((X, d)\) be a complete metric space and \(A, B, S\) and \(T\) be mappings from \(X\) into itself. Suppose that \(S\) and \(T\) are continuous mappings satisfying the following conditions:

- \(A(X) \subset T(X)\) and \(B(X) \subset S(X)\),
- The pairs \(\{A, S\}\) and \(\{B, T\}\) are compatible of type (R),
- \(d(Ax, By) \leq \Phi(\max\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \frac{1}{2} [d(Sx, By) + d(Ty, Ax)]\})\) for all \(x, y \in X\), where \(\Phi: [0, \infty) \to [0, \infty)\) is a non-decreasing and upper semi continuous function and \(\Phi(t) < t\) for all \(t > 0\).

Then \(A, B, S\) and \(T\) have a unique common fixed point in \(X\).

**Proof.** Since \(A(X) \subset T(X)\) and \(B(X) \subset S(X)\), we can choose a sequence \(\{x_n\}\) in \(X\) such that \(Sx_{2n} = Bx_{2n-1}\) and \(Tx_{2n-1} = Ax_{2n-2}\) for \(n = 1, 2, 3, \ldots\). Suppose that
\[
y_{2n-1} = Tx_{2n-1} = Ax_{2n-2} \quad \text{and} \quad y_{2n} = Sx_{2n} = Bx_{2n-1}\]
for \(n = 1, 2, 3, \ldots\). By using the technique of Chang [8], we can prove that \(\{y_n\}\) is a Cauchy sequence in \(X\) and so, since \(X\) is complete, it converges to a point \(z\) in \(X\). On the other hand, the subsequences \(\{Ax_{2n-2}\}, \{Bx_{2n-1}\}, \{Sx_{2n}\}\) and \(\{Tx_{2n-1}\}\) of \(\{y_n\}\) also converge to the point \(z\).

Since \(\{A, S\}\) and \(\{B, T\}\) are compatible of type (R), it follows from the continuity of \(S\) and \(T\), (4) and Proposition 2.2 that
\[
Ty_{2n} \to Tz, \quad By_{2n} = BBx_{2n-1} \to Tz, \quad
Sy_{2n-1} \to Sz, \quad Ay_{2n-1} = AAx_{2n-2} \to Sz
\]
as \(n \to \infty\). By (3) and (4), we have
d (Ay_{2n-1}, By_{2n}) \leq \Phi \left( \max \{d(Sy_{2n-1}, Ty_{2n}), d(Sy_{2n-1}, Ay_{2n-1}), d(Ty_{2n}, By_{2n}), \frac{1}{2} [d(Sy_{2n-1}, By_{2n}) + d(Ty_{2n}, Ay_{2n-1})]\} \right).

By the upper semicontinuity of \Phi (t), (4) and (5), if Sz \neq Tz, then we have

d(Sz, Tz) \leq \Phi \left( \max \{d(Sz, Tz), 0, 0, d(Sz, Tz)\} \right)
= \Phi \left( d(Sz, Tz) \right) < d(Sz, Tz),

which is contradiction. Thus it follows that Sz = Tz.

Similarly, from (3), (4), (5) and the upper semicontinuity of \Phi, we can obtain Sz = Bz and Tz = Az. Hence we have

Az = Bz = Sz = Tz. \quad (6)

From (3) and (4), we have also

d(Ax_{2n}, Bz) \leq \Phi \left( \max \{d(Sx_{2n}, Tz), d(Sx_{2n}, Ax_{2n}), d(Tz, Bz), \frac{1}{2} [d(Sx_{2n}, Bz) + d(Tz, Ax_{2n})]\} \right).

This implies that, if Bz \neq z, then

d(z, Bz) \leq \Phi \left( d(z, Bz) \right) < d(z, Bz),

which is a contradiction. Therefore, we have z = Az = Bz = Sz = Tz. The uniqueness of the fixed point z is obvious from (2). This completes the proof.

From Theorem (2.3), we have the following:

**Theorem 2.4.** Let \((X, d)\) be a complete metric space and \(A, B\) be mappings from \(X\) into itself satisfying the following condition

\[ d(Ax, By) \leq \Phi \left( \max \{d(x, y), d(x, Ax), d(y, By), \frac{1}{2} [d(x, By) + d(y, Ax)]\} \right). \quad (7) \]

for all \(x, y\) in \(X\), where \(\Phi (t)\) is the same as in Theorem (2.3) then \(A\) and \(B\) have a unique common fixed point in \(X\).

**Proof.** Define a sequence \(\{x_n\}\) in \(X\) by

\[ x_{2n-1} = Ax_{2n-2} \text{ and } x_{2n} = Bx_{2n-2} \quad (8) \]

for \(n = 1, 2, 3, \ldots\). Then it is easy to show that \(\{x_n\}\) is a Cauchy sequence in \(X\). Since \(X\) is complete, letting \(x_n \to z \in X\) as \(n \to \infty\), we know that \(\{x_{2n-1}\}\) and \(\{x_{2n}\}\) converge to \(z\), too. By (7) and (8), we have

\[ d(Az, x_{2n}) \leq d(Az, Bx_{2n-2}) \leq \Phi \left( \max \{d(z, x_{2n-2}), d(z, Az), d(x_{2n-2}, x_{2n}), \frac{1}{2} [d(z, x_{2n}) + d(x_{2n-2}, Az)]\} \right). \]
By the upper semicontinuity of $\Phi(t)$, if $Az \neq z$, then we have

$$d(Az, z) \leq \Phi(d(z, Az)) < d(z, Az),$$

which is contradiction and so $z = Az$. Similarly, we have $z = Bz$. This completes the proof.

The following result is an immediate consequence of Theorem 2.3.

**Theorem 2.5.** Let $(X, d)$ be a complete metric space and $S$, $T$ and $A_n$ be mappings from $X$ into itself, $n = 1, 2, \ldots$. Suppose further that $S$ and $T$ are continuous and, for every $n \in \mathbb{N}$, the pairs $\{A_{2n-1}, S\}$ and $\{A_{2n}, T\}$ are compatible of type (R), $A_{2n-1}(X) \subset T(X)$ and $A_{2n}(X) \subset S(X)$ and, for any $n \in \mathbb{N}$, the set of positive integers, the following condition is satisfied:

$$d(A_n x, A_{n+1} y) \leq \Phi \left( \max \{d(Sx, Ty), d(Sx, A_n x), d(Ty, A_{n+1} y), \frac{1}{2} [d(Sx, A_{n+1} y) + d(Ty, A_n x)] \} \right). \quad (9)$$

for all $x, y \in X$, where $\Phi(t)$ is the same as in Theorem 2.3. Then $S$, $T$ and $\{A_n\}$, $n \in \mathbb{N}$, have a unique common fixed point in $X$.

**Open Problem**

It remains open to check the result using Biased mappings.

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**References**


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