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# A Three Species Ecosystem Consisting of a Prey, Predator and a Host Commensal to the Prey

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**Abstract**

***The present paper deals with an investigation on a three species ecosystem consisting of a prey ($N_1$); a predator ($N_2$) surviving on $N_1$ and a host ($N_3$) commensal with the prey $N_1$ only. The mathematical model equations constitute a set of three first order non-linear simultaneous equations in $N_1$, $N_2$ and $N_3$. The equation for $N_3$ is non-linear but de-coupled with $N_1$and $N_2$ .In all the eight equilibrium points of the model are identified and criteria for their stability are discussed. Trajectories of the perturbations over the equilibrium points are illustrated***.

**Keywords:** *Ecosystem, prey, predator, commensal, host, Equilibrium points, Normal steady state, stability, and threshold diagrams*.

## 1 Introduction

Mathematical modeling of ecosystems was initiated in 1925 by Lotka [9] and in 1931 by Volterra [15]. The general concepts of modeling have been presented in the treatises of Meyer [10], Kapur [3, 4]. The ecological symbiosis can be broadly classified as Prey-predation, competition, mutualism, commensalisms and so on. N.C. Srinivas [14] studied the competitive ecosystems of two species and three species with limited and unlimited resources. Lakshminarayan and Pattabhiramacharyulu [5, 6] Investigated Prey-predator Ecological models with a partial cover for the prey and alternate food for the predator. These authors are also analysed a prey-predator model with alternate food for the predator,harvesting of both the the species[7]. Lakshminarayan and apparao investigated a prey-predator model with a cover linearly varying with the prey population and an alternative food for the predator [8]. Recently, stability analysis of competitive species was carried out by Archana Reddy, Pattabhiramachryulu and Gandhi [1] and by Bhaskara Rama Sarma and Pattabhiramacharyulu [2], while the mutualism between to species was examined by Ravindra Reddy [12].Following this Phanikumar, Seshagirirao and Pattabhiramacharyulu studied the commensalism of two species with limited resources[11,13 ].

The present investigation is an analytical study of three species system: commensal – prey - predator and host system. In all eight equilibrium points are identified based on the model equations and these are spread over three distinct classes: (i) Fully washed out (ii) Semi/partially washed out and (iii) Co-existent states. Criteria for the asymptotic stability of the states have been derived. It is noticed that all the states are stable except in the following states:

i. Fully washed out state.
ii. The prey and predator are washed out but not the host.
iii. The predator and host are washed out but not the prey.
iv. The prey and the host are washed out but not the predator.
v. The predator is washed out and the prey and the host are not.

Some threshold results have also been established to highlight the regions of the asymptotic stability / instability.

## 2 Notation Adopted

$N_1$ (t) : The population of the Prey-Commensal Species.

$N_2$ (t) : The population of the predator striving of the prey $N_1$

$N_3$ (t) : The Population of the host to the prey $N_1$

$a_i$ : The natural growth rates of $N_i$, i = 1,2,3

$a_{ii}$ : The rate of decrease of $N_i$ due to insufficient resources of, $N_i$ i = 1,2,3

$a_{12}$ : The rate of decrease of the prey ($N_1$) due to inhibition by the predator ($N_2$)

$a_{13}$ : The rate of increase of the commensal ($N_1$) due to its successful promotion by host ($N_3$)

$a_{21}$ : The rate of increase of the predator ($N_2$) due to its successful attacks on the prey ($N_1$)

$K_i = a_i / a_{ii}$ : Carrying capacities of $N_i$ i = 1, 2, 3.

$C = a_{13}/a_{11}$ : Co-efficient of commensalism.

$P = a_{12}/a_{11}$ : Co-efficient of prey inhibition (suffering)

$Q = a_{21}/a_{22}$ : Co-efficient of predator consumption of the prey.

# 3 Basic Equations

The model equations for a three species multi-reactive ecosystem given by the following system of non-linear ordinary differential equations.

(i) Equation for the growth rate of Prey-commensal species ($N_1$):

$$\frac{dN_1}{dt} = a_{11}N_1 ( K_1 - N_1 - PN_2 + CN_3 ) \quad (1)$$

(ii) Equation for the growth rate of predator species ($N_2$):

$$\frac{dN_2}{dt} = a_{22} N_2 ( K_2 - N_1 + Q N_1 ) \quad (2)$$

(iii) Equation for the growth rate of host species ($N_3$):

$$\frac{dN_3}{dt} = a_{33} N_3 ( K_3 - N_3 ) \quad (3)$$

Further the variables $N_1$, $N_2$ and $N_3$ are non-negative and the model parameters $a_1$, $a_2$, $a_3$ $a_{11}$, $a_{22}$, $a_{33}$, $a_{12}$, $a_{13}$, $a_{21}$, $K_1$, $K_2$, $K_3$, C, P, and Q are all assumed to be non-negative constants.

# 4 Equilibrium States

The system under investigation has eight equilibrium states given by $\frac{dN}{dt} = 0$

A. Fully washed out state:

(i) $\overline{N_1} = 0, \overline{N_2} = 0; \overline{N_3} = 0$ (4)

B. States in which two of the three species are washed out and third is not.

(ii) $\overline{N_1} = 0;\ \overline{N_2} = 0;\ \overline{N_3} = K_3$ (5)

(iii) $\overline{N_1} = 0;\ \overline{N_2} = K_2;\ \overline{N_3} = 0$ (6)

(iv) $\overline{N_1} = K_1;\ \overline{N_2} = 0;\ \overline{N_3} = 0$ (7)

C. Only one of the three species is washed out while the other two are not

(v) $\overline{N_1} = 0;\ \overline{N_2} = K_2;\ \overline{N_3} = K_3$ (8)

(vi) $\overline{N_1} = K_1 + CK_3;\ \overline{N_2} = 0;\ \overline{N_3} = K_3$ (9)

(vii) $\overline{N_1} = \dfrac{K_1 - PK_2}{1+PQ};\ \overline{N_2} = \dfrac{QK_1 + K_2}{1+PQ}; \overline{N_3} = 0$ (10)

This state would exists only when with $K_1 > PK_2$

D. The co-existent state or normal steady state

(viii) $\overline{N_1} = \dfrac{K_1 - PK_2 + CK_3}{1+PQ};\ \overline{N_2} = \dfrac{QK_1 + K_2 + QCK_3}{1+PQ};\ \overline{N_3} = K_3$ (11)

This would exist only when $K_1 + CK_3 > PK_2$

## 5 Equilibrium States and Criteria for Their Stability

The basic equations (1), (2) and (3) are linearised to obtain the equations for the perturbed state,

$$\frac{du}{dt} = Au, \tag{12}$$

with

$$A = \begin{bmatrix} K_1a_{11} - 2a_{11}\overline{N_1} - Pa_{11}\overline{N_2} + Ca_{11}\overline{N_3} & -Pa_{11}\overline{N_1} & Ca_{11}\overline{N_1} \\ Qa_{22}\overline{N_2} & K_2a_{22} - 2a_{22}\overline{N_2} + Qa_{22}\overline{N_1} & 0 \\ 0 & 0 & K_3a_{33} - 2a_{33}\overline{N_3} \end{bmatrix} \tag{13}$$

where $N = (N_1, N_2, N_3) = \overline{N} + u$, with $u = (u_1, u_2, u_3)$ is a small perturbation over the equilibrium state $\overline{N} = (\overline{N_1}, \overline{N_2}, \overline{N_3})$, $u_{10}$, $u_{20}$ and $u_{30}$ are initial values of $u_1\ u_2$ & $u_3$ respectively

The equilibrium state $\overline{N}$ is stable, if all the eigen values of the characteristic matrix $A$ are negative or have negative real parts according as the roots are real or complex.

## 5.1 Fully Washed Out State:

In this case we have

$$A = \begin{bmatrix} K_1 a_{11} & 0 & 0 \\ 0 & K_2 a_{22} & 0 \\ 0 & 0 & K_3 a_{33} \end{bmatrix}$$

the characteristic roots of which are $K_1a_{11}$, $K_2a_{22}$, $K_3a_{33}$, and these are all positive. Hence the state is **unstable.** The equations (12) yield the solution curves:

The variations of the perturbations with respect to time is presented here

$$u_1 = u_{10}\, e^{K_1 a_{11} t};\quad u_2 = u_{20}\, e^{K_2 a_{22} t};\quad u_3 = u_{30} e^{K_3 a_{33} t} \qquad (14)$$

and the solution curves are illustrated in figures (1) to (5) and the conclusions are presented here.

## Trajectories of Perturbed Species:

The trajectories of (14) in the $u_1$- $u_2$ and $u_2$- $u_3$ planes are given by

$$\left(\frac{u_1}{u_{10}}\right)^{K_2 a_{22}} = \left(\frac{u_2}{u_{20}}\right)^{K_1 a_{11}} \quad and \quad \left(\frac{u_2}{u_{20}}\right)^{K_3 a_{33}} = \left(\frac{u_3}{u_{30}}\right)^{K_2 a_{22}}$$

$$\Rightarrow \quad \left(\frac{u_1}{u_{10}}\right)^{\frac{1}{a_1}} = \left(\frac{u_2}{u_{20}}\right)^{\frac{1}{a_2}} = \left(\frac{u_3}{u_{30}}\right)^{\frac{1}{a_3}}$$

## 5.2 Prey-Commensal And Predator Washed Out State:

In this Case we have

$$A = \begin{bmatrix} a_{11}(K_1 + CK_3) & 0 & 0 \\ 0 & K_2 a_{22} & 0 \\ 0 & 0 & -K_3 a_{33} \end{bmatrix}$$

and the characteristic roots are $a_{11}$ ($K_1$+$CK_3$), $K_2a_{22}$, - $K_3a_{33}$. As such the state is **unstable.**

The equations (12) yield the solution curves:

$$u_1 = u_{10}\, e^{a_{11}(K_1+CK_3)t}\,;\ u_2 = u_{20}\, e^{K_2 a_{22}\, t}\,;\ u_3 = u_{30}\, e^{-K_3 a_{33}\, t} \qquad (15)$$

and these are illustrated in figures (6) to (9) and conclusions are presented here.

## Trajectories of Perturbed species:

The trajectories of (15) in the $u_1 - u_2$ plane and $u_2 - u_3$ plane are given by

$$\left[\left(\frac{u_1}{u_{10}}\right)^{K_2 a_{22}} = \left(\frac{u_2}{u_{20}}\right)^{a_{11}(K_1+CK_3)}\right]^{K_3 a_{33}} \; and \left[\left(\frac{u_{30}}{u_3}\right)^{K_2 a_{22}} = \left(\frac{u_2}{u_{20}}\right)^{K_3 a_{33})}\right]^{a_{11}(K_1+CK_3)}$$

$$\left(\frac{u_1}{u_{10}}\right) 1/a_1 = \left(\frac{u_2}{u_{20}}\right) (1+CK_3/k_1) / a_2 = \left(\frac{u_3}{u_{30}}\right) (1+ck_3 / k_1) / a_3$$

## 5.3 Prey-Commensal and Host Washed Out State:

In this Case we have

$$A = \begin{bmatrix} a_{11}(K_1 - PK_2) & 0 & 0 \\ QK_2 a_{22} & -K_2 a_{22} & 0 \\ 0 & 0 & K_3 a_{33} \end{bmatrix}$$

and the characteristic roots of which are $-a_{11}(PK_2 - K_1)$, $-K_2a_{22}$, $K_3a_{33}$. Since one of these three roots is positive, hence the state is **unstable.** The equations (12) yield the solution curves.

$$\left.\begin{aligned} u_1 &= u_{10}\, e^{-a_{11}(PK_2 - K_1)t} \\ u_2 &= \frac{QK_2 a_{22} u_{10}}{K_1 a_{11} + K_2(a_{22} - Pa_{11})} e^{-a_{11}(PK_2 - K_1)t} + \left[u_{20} - \frac{QK_2 a_{22} u_{10}}{K_1 a_{11} + K_2(a_{22} - Pa_{11})}\right] e^{-K_2 a_{22} t} \\ u_3 &= u_{30}\, e^{K_3 a_{33} t} \end{aligned}\right\} \quad (16)$$

**Case – A When $PK_2 - K_1 > 0$ and $\frac{K_1}{K_2} = P - \frac{a_{22}}{a_{11}}$**

In this case (16) becomes $u_1 = u_{10}\ e^{-K_2 a_{22} t}$; $u_2 = u_{20}\, e^{-K_2 a_{22} t}$; $u_3 = u_{30}\, e^{K_3 a_{33} t}$

**Case – B When $PK_2 - K_1 = 0$**

In this case (16) becomes

$$u_1 = u_{10};\ u_2 = \frac{QK_2 a_{22} u_{10}}{K_1 a_{11} + K_2(a_{22} - Pa_{11})}\left[1 - e^{-K_2 a_{22} t}\right] + u_{20}\, e^{-K_2 a_{22} t},\ u_3 = u_{30}\, e^{K_3 a_{33} t}$$

and these are illustrated in figures (10) to (14) and the conclusions are presented here.

## Trajectories of Perturbed species:

**Case A:** When $PK_2 - K_1 > 0$, the trajectories of (16) in $u_1$-$u_2$ and $u_2 - u_3$ plane are

$$\left(\frac{u_1}{u_{10}}\right)^{K_3 a_{33}} \cdot \left(\frac{u_3}{u_{30}}\right)^{a_{11}(PK_2 - K_1)} = 1 \text{ and}$$

$$u_2 = \frac{QK_2 a_{22}}{K_2 a_{22} - a_{11}(PK_2 - K_1)} + \left[u_{20} - \frac{QK_2 a_{22} u_{10}}{K_2 a_{22} - a_{11}(PK_2 - K_1)}\right] \cdot \left(\frac{u_3}{u_{30}}\right)^{\frac{-K_2 a_{22}}{K_3 a_{33}}}$$

**Case B**: When $Pk_2 - K_1 = 0$, the trajectories of (16) are $\left(\frac{u_1}{u_{10}}\right)^{k_3 a_{33}} = 1$ and

$$u_2 = Q + [\, u_{20} - Qu_{10}] \left(\frac{u_3}{u_{30}}\right)^{\frac{-k_2 a_{22}}{k_3}}$$

## 5.4 Predator And Host Washed Out State:

In this case we have

$$A = \begin{bmatrix} -K_1 a_{11} & -Pa_{11}K_1 & Ca_{11}K_1 \\ 0 & a_{22}(K_2 + QK_1) & 0 \\ 0 & 0 & K_3 a_{33} \end{bmatrix}$$

and the characteristic roots of which are $-K_1a_{11}$, $a_{22}(K_2 + QK_1)$, $K_3a_{33}$. Since two of the three roots are positive, the state is **unstable**. The equations (12) yield the solution curves:

$$u_1 = u_{10} + u_{20} \frac{P}{a_{22}(K_2 + QK_1) + K_1 a_{11}} \left[1 - e^{a_{22}(K_2 + QK_1)t}\right] + u_{30} \cdot \frac{CK_1 a_{11}}{K_3 a_{33} + K_1 a_{11}} \left[e^{K_3 a_{33} t} - 1\right]$$

$$u_2 = u_{20}\, e^{a_{22}(K_2 + QK_1)t} \; ; \; u_3 = u_{30}\, e^{K_3 a_{33} t} \quad (17)$$

and these are illustrated in figure (15) to (18) and the conclusions are presented here .

## Trajectories of Perturbed species:

The trajectories of (17) are given by

$$\left(\frac{u_2}{u_{20}}\right)^{K_3 a_{33}} = \left(\frac{u_3}{u_{30}}\right)^{a_{22}(QK_1 + K_2)} \text{ and}$$

$$u_1 = u_{10} + u_{20}\ \frac{P}{a_{22}(K_2 + QK_1) + a_{11}K_1}\left(1 - \frac{u_2}{u_{20}}\right) + u_{30}\frac{CK_1a_{11}}{K_3a_{33} + K_1a_{11}}\left(\frac{u_3}{u_{30}} - 1\right)$$

## 5.5 Prey-Commensal Washed Out State:

In this case we have

$$A = \begin{bmatrix} a_{11}(K_1 - PK_2 + CK_3) & 0 & 0 \\ Qa_{22}K_2 & -K_2a_{22} & 0 \\ 0 & 0 & -K_3a_{33} \end{bmatrix}$$

and the characteristic roots of which are $a_{11}$ ($K_1$ - $PK_2$ + $CK_3$), - $K_2$ $a_{22}$, -$K_3a_{33}$

**Case A:** **When $K_1$ + $CK_3$ > $PK_2$**

One of the three roots is positive so that the state is **unstable**

**Case B:** **When $K_1$+$CK_3$ < $PK_2$**

All the three roots are negative hence the state is **stable:**

The equations (12) yield the solution curves in both cases

**For Case A**

$$\left.\begin{aligned} u_1 &= u_{10}\, e^{(K_1+CK_3-PK_2)\,t} \\ u_2 &= \frac{u_{20}\,Qa_{22}}{(K_1+CK_3-PK_2+K_2a_{22})}e^{(K_1+CK_3-PK_2)t} + \left[u_{20} - \frac{Qa_{22}u_{10}}{(K_1+CK_3-PK_2+K_2a_{22})}e^{-K_2a_{22}t}\right] \\ u_3 &= u_{30}.e^{-K_3a_{33}t.} \end{aligned}\right\} \quad (18)$$

**For Case B**

$$\left.\begin{aligned} u_1 &= u_{10}\, e^{-(K_1+cK_3-PK_2)\,t.} \\ u_2 &= \frac{u_{10}\,Qa_{22}}{K_2a_{22}-(K_1+CK_3-PK_2)}e^{-(K_1+CK_3-PK_2)t} + \left[u_{20} - \frac{Qa_{22}\,u_{10}}{K_2a_{22}-(K_1+CK_3PK_2)}e^{-K_2a_{22}t}\right] \\ u_3 &= u_{30}.\, e^{-K_3a_{33}t.} \end{aligned}\right\} \quad (19)$$

And these are illustrated in figures (19) to (22) and the conclusions are presented here.

When $\mathbf{u_{20}} = \frac{Qa_{22}u_{10}}{(K_1 + CK_3 - PK_2) + K_2a_{22}}$ then (18) becomes

$$u_1 = u_{10}\, e^{(K_1 + CK_3 - PK_2)\,t};\ u_2 = u_{20}\, e^{(K_1+CK_3-PK_2)\,t};\ u_3 = u_{30}\ e^{-K_3a_{33}\,t}$$

when $\mathbf{u_{20}} = \dfrac{Qa_{22}u_{10}}{K_2a_{22} - A}$ where A = $K_1 + CK_3 - PK_2 < 0$ then (19) becomes

$u_1 = u_{10}e^{-At}$ ; $u_2 = \dfrac{Qa_{22}u_{10}}{K_2a_{22} - A} e^{-At} = u_{20}e^{-At}$ ; $u_3 = u_{30}\, e^{-K_3 a_{33} t}$

## Trajectories of Perturbed species:

**Case A:** When A = $K_1 + CK_3 - PK_2 > 0$, the trajectories of (18) in $u_1$-$u_2$ and $u_2 - u_3$ plane are given by

$$\left(\frac{u_1}{u_{10}}\right) = \left(\frac{u_2}{u_{20}}\right) \text{ and } \left(\frac{u_2}{u_{20}}\right)^{\frac{1}{K_1+CK_3-PK_2}} \left(\frac{u_2}{u_{20}}\right)^{\frac{1}{K_3a_{33}}} = 1$$

**Case B:** When A = $K_1 + CK_3 - PK_2 < 0$, the trajectories of (19) in $u_1$-$u_2$ and $u_2 - u_3$ plane are given by

$$\left(\frac{u_1}{u_{10}}\right) = \left(\frac{u_2}{u_{20}}\right) \text{ and } \left(\frac{u_2}{u_{20}}\right)^{K_3a_{33}} = \left(\frac{u_3}{u_{30}}\right)^{A}$$

## 5.6 Predator Washed Out State:

In this case we have

$$A = \begin{bmatrix} -a_{11}(K_1+CK_3) & -Pa_{11}(K_1+CK_3) & Ca_{11}(K_1+CK_3) \\ 0 & a_{22}(K_2+Q(K_1+cK_3)) & 0 \\ 0 & 0 & -K_3a_{33} \end{bmatrix}$$

and the characteristic roots of which are $-a_{11}(K_1 + CK_3)$, $a_{22}(K_2 + Q(K_1 + CK_3))$, $-K_3a_{33}$ as one of these three roots is positive, the state is **unstable**.

The equations (12) yield The Solution Curves:

$$\left.\begin{aligned} u_1 &= a_{11}(K_1+CK_3)\left[\frac{C.u_{30}}{a_{11}(K_1+CK_3)-K_3a_{33}} e^{-K_3a_{33}t} - \frac{P.u_{20}}{a_{22}K_2+(K_1+CK_3)(Q+a_{11})} e^{(K_2a_{22}+(K_1+CK_3)Q)t}\right] \\ &\quad + u_{10} - a_{11}(K_1+cK_3)\left[\frac{Cu_{30}}{a_{11}(K_1+CK_3)-K_3a_{33}} - \frac{P.u_{20}}{a_{22}K_2+(K_1+cK_3)(q_1a_{11})}\right] \\ u_2 &= u_{20}\, e^{a_{22}(K_2 + Q(K_1 + CK_3))t} \\ u_3 &= u_{30}\, e^{-K_3a_{33}t} \end{aligned}\right\} \quad (20)$$

and these are illustrated in figures (23) to (25) and the conclusions are presented here.

## Trajectories of perturbed species:

The trajectories of (20) are given by

$$\left(\frac{u_2}{u_{20}}\right)^{K_3 a_{33}} \cdot \left(\frac{u_3}{u_{30}}\right)^{a_{22}(K_2+Q(K_1+CK_3))} = 1 \text{ and}$$

$$u_1 = u_{10} + a_{11}(K_1 + CK_3)\left[\frac{C.u_{30}}{a_{11}(K_1+CK_3) - K_3 a_{33}}\left(\frac{u_3}{u_{30}}\right) - \frac{P.u_{20}}{a_{22}K_2 + (K_1+CK_3)(Q+a_{11})}\left(1 - \frac{u_2}{u_{20}}\right)\right]$$

## 5.7 Host Washed Out State:

In this case we have

$$A = \begin{bmatrix} \frac{-a_{11}(K_1 - PK_2)}{1+PQ} & \frac{-a_{11}P(K_1 - PK_2)}{1+PQ} & \frac{a_{11}C(k_1 - PK_2)}{1+PQ} \\ \frac{a_{22}Q_2(K_2 + QK_3)}{1+PQ} & \frac{-a_{22}(K_2 + QK_1)}{1+PQ} & 0 \\ 0 & 0 & K_3 a_{33} \end{bmatrix}$$

The characteristic equation is :

$$\left[\left(\lambda + \frac{a_{11}(K_1 - PK_2)}{1+PQ}\right)\left(\lambda + \frac{a_{22}(K_2 - QK_1)}{1+PQ}\right) + a_{11}a_{22}\frac{PQ(K_1 - PK_2)(K_2 + QK_1)}{(1+PQ)^2}\right](\lambda - K_3 a_{33}) = 0$$

**Case A: When [ $a_{22}$ ($K_2$ + $QK_1$) + $a_{11}$ ($K_1$ – $PK_2$) $]^2$ = 4$a_{11}$ $a_{22}$ ($K_2$+$QK_1$) ($K_1$-$PK_2$) (1+PQ)**

$$\lambda_1 = \frac{-\left[a_{11}(K_1 - PK_2) + a_{22}(K_2 + QK_1)\right]}{2(1+PQ)},\ \lambda_2 = \frac{-\left[a_{11}(K_1 - PK_2) + a_{22}(K_2 + QK_1)\right]}{2(1+PQ)},\ \lambda_3 = k_3 a_{33}$$

as such the state is **unstable**.

**Case B:When [ $a_{22}$ ($K_2$ + $QK_1$) + $a_{11}$ ($K_1$ – $PK_2$) $]^2$ > 4$a_{11}$ $a_{22}$ ($K_2$+$QK_1$) ($K_1$-$PK_2$) (1+PQ)**

In this case the first two roots are real and unequal and third root is positive hence the state is **unstable.**

**Case C: When [a22 ($K_2$ + $QK_1$) + $a_{11}$ ($K_1$ – $PK_2$) $]^2$ < 4$a_{11}$ $a_{22}$ ($K_2$+$QK_1$) ($K_1$-$PK_2$) (1+PQ)**

In this case the first two roots are complex with negative real part but the third root is **positive**, hence the state is **unstable.**

The equations (12) yield the solution curves

$$u_1(t) = \frac{(\lambda_1+\beta_1)[u_{10}(\lambda_1-K_3a_{33})+\alpha_1u_{30}]-\alpha_1pu_{20}(\lambda_1-K_3a_{33})}{(\lambda_1-\lambda_2)(\lambda_1-K_3a_{33})}e^{\lambda_1t}$$

$$+\frac{(\lambda_2+\beta_1)[u_{10}(\lambda_2-K_3a_{33})+\alpha_1u_{30}]-\alpha_1Pu_{20}(\lambda_2-K_3a_{33})}{(\lambda_2-\lambda_1)(\lambda_2-K_3a_{33})}e^{\lambda_2t}+\frac{(K_3a_{33}+\beta_1)\alpha_1u_{30}}{(\lambda_1-K_3a_{33})(\lambda_2-K_3a_{33})}e^{K_3a_{33}t}$$

$$u_2(t)=\frac{(\lambda_1+\alpha_1)u_{10}(\lambda_1-K_3a_{33})-\beta_1Q[(\lambda_1-K_3a_{33})u_{10}+\alpha_1u_{30}}{(\lambda_1-\lambda_2)(\lambda_1-K_3a_{33})}e^{\lambda_1t}$$

$$+\frac{(\lambda_2+\alpha_1)u_{20}(\lambda_2-K_3a_{33})-\beta_1Q[(\lambda_2-K_3a_{33})u_{10}+\alpha_1u_{30}}{(\lambda_2-\lambda_1)(\lambda_2-K_3a_{33})}e^{\lambda_2t}-\frac{\beta_1q\alpha_1u_{30}}{(\lambda_2-K_3a_{33})(\lambda_2-K_3a_{33})}e^{K_3a_{33}t} \quad (21)$$

$u_3(t) = u_{30}\, e^{k_3a_{33}t}$

Where $\alpha_1=\frac{a_{11}(K_1-PK_2)}{1+PQ}, \beta_1=\frac{a_{22}(K_2+QK_1)}{1+PQ}$

## 5.8 Co-Existent State:

In this case were have

$$A = \begin{bmatrix} \frac{-a_{11}(K_1+CK_3-PK_2)}{1+PQ} & \frac{-a_{11}P(K_1+CK_3-PK_2)}{1+PQ} & a_{11}\frac{C(K_1+CK_3-PK_2)}{1+PQ} \\ \frac{a_{22}Q(QK_1+K_2+QCK_3}{1+PQ} & \frac{-a_{22}(QK_1+K_2+QCK_3)}{1+PQ} & 0 \\ o & 0 & -K_3a_{33} \end{bmatrix}$$

the characteristic equation is

$$\left[\left(\lambda+\frac{a_{11}(K_1+CK_3-PK_2}{1+PQ}\right)\left(\lambda+\frac{a_{22}(QK_1+K_2+QCK_3}{1+PQ}\right)+\frac{a_{11}a_{22}PQ(K_1+CK_3-PK_2)(QK_1+K_2+QCK_3)}{(1+PQ)^2}\right]$$

$(\lambda+K_3a_{33})=0$

and the characteristic roots are :

$\lambda = -k_3a_{33},\ \lambda=\frac{-(\alpha+\beta)\pm\sqrt{(\alpha+\beta)^2-4\alpha\beta(1+pq)}}{2}$ where

$$\alpha=\frac{a_{11}(K_1+CK_3-PK_2)}{1+PQ}, \beta=\frac{a_{22}(QK_1+K_2+QCK_3)}{1+PQ}$$

**Case A: When** $(\alpha+\beta)^2 > 4\alpha\beta(1+pq)$

The roots are real and negative, hence the state is **stable.**

**Cast B: When** $(\alpha+\beta)^2 = 4\alpha\beta(1+PQ)$

The roots are $\lambda_1 = -\dfrac{(\alpha+\beta)}{2}, \lambda_2 = -\dfrac{(\alpha+\beta)}{2}, \lambda_3 = -k_3 a_{33}$, hence the state is **stable.**

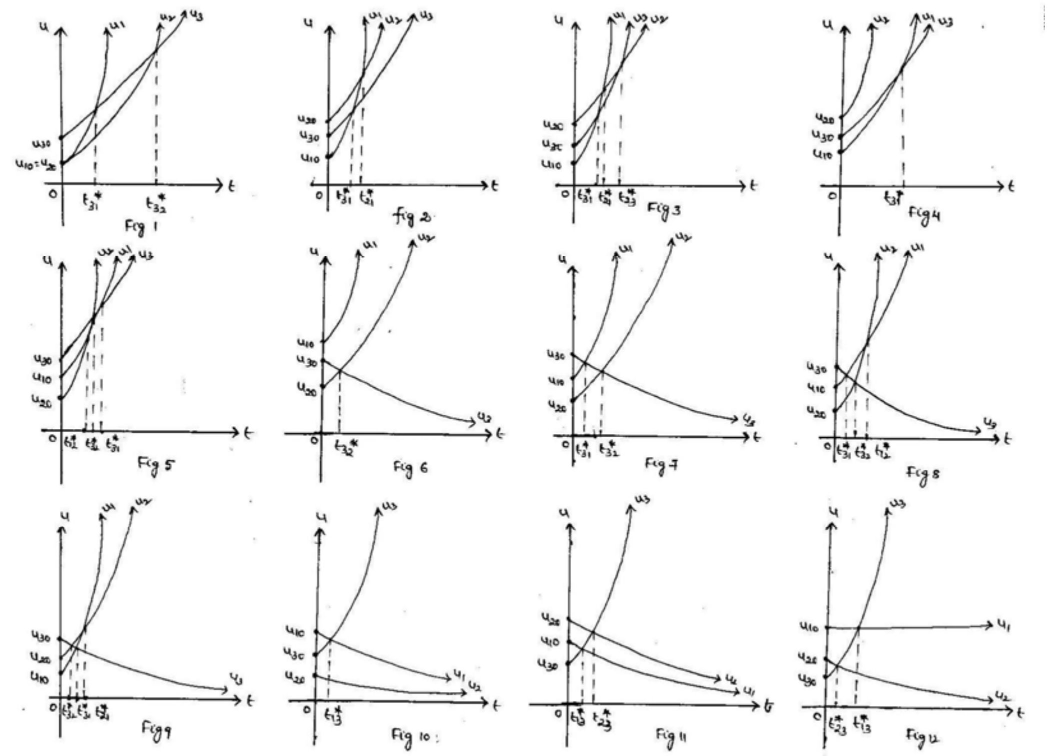
**Case C: When** $(\alpha+\beta)^2 < 4\alpha\beta(1+pq)$

The roots are complex with negative real part**,** hence the state is **stable.** The equations (12) yield the solution curves:

$$\left.\begin{aligned}
u_1 &= \frac{(\lambda_1+\beta)[u_{10}(\lambda_1+K_3a_{33})+C\alpha u_{30}]-\alpha\beta u_{20}(\lambda_1+K_3a_{33})}{(\lambda_1-\lambda_2)(\lambda_1+K_3a_{33})}e^{\lambda_1 t} \\
&+\frac{(\lambda_2+\beta)[u_{10}(\lambda_2+K_3a_{33})+C\alpha u_{30}]-\alpha\beta u_{20}(\lambda_2+K_3a_{33})}{(\lambda_2-\lambda_1)(\lambda_2+K_3a_{33})}e^{\lambda_2 t}+\frac{(\beta-K_3a_{33})C\alpha u_{30}}{(\lambda_1+K_3a_{33})(\lambda_2+K_3a_{33})}e^{-K_3a_{33}t} \\
u_2 &= \frac{(\lambda_1+K_3a_{33})(\lambda_1+\alpha)(\lambda_1+\beta)+(\lambda_1+K_3a_{33})\beta q u_{10}+\beta QC\alpha u_{30}}{(\lambda_1-\lambda_2)(\lambda_1+K_3a_{33})}e^{\lambda_1 t} \\
&+\frac{(\lambda_2+K_3a_{33})(\lambda_2+\alpha)(\lambda_2+\beta)+(\lambda_2+K_3a_{33})\beta Q u_{10}+\beta QC\alpha u_{30}}{(\lambda_2-\lambda_1)(\lambda_2+K_3a_{33})}e^{\lambda_2 t}+\frac{\beta QC\alpha u_{30}}{(\lambda_1+K_3a_{33})(\lambda_2+K_3a_{33})}e^{-K_3a_{33}t} \\
u_3 &= u_{30}\, e^{-K_3a_{33}t}
\end{aligned}\right\} \quad (22)$$

and these are illustrated in figures (26) to (28) and conclusions are presented.

## 6 Perturbation Graphs:



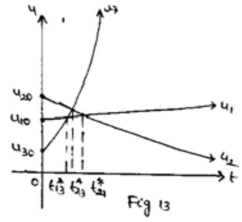


Fig 13

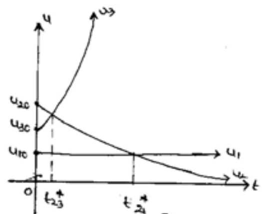


Fig 14

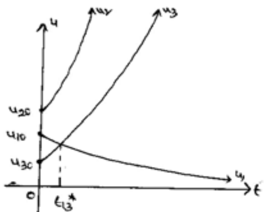


Fig 15

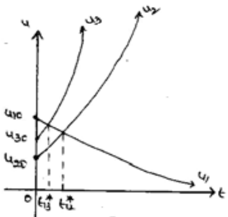


Fig 16

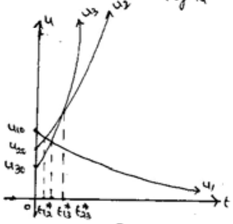


Fig 17

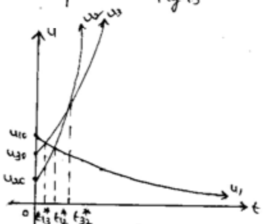


Fig 18

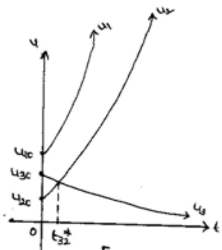


Fig 19

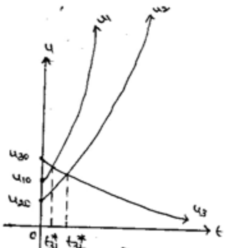


Fig 20

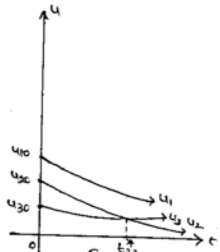


Fig 21

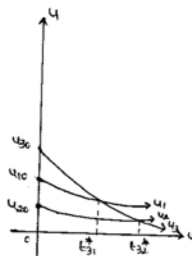
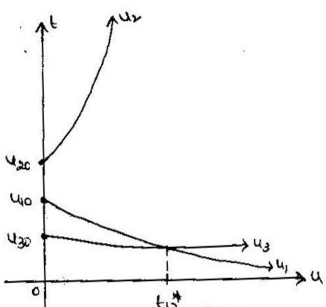


Fig 22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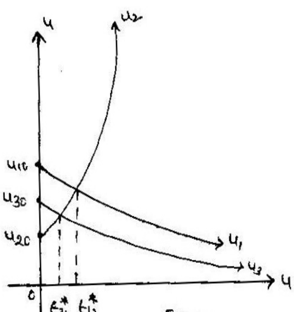
Fig 24

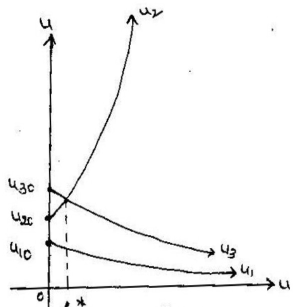
Fig 25

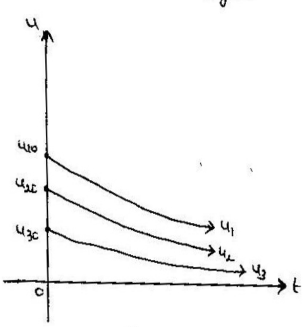
Fig 26

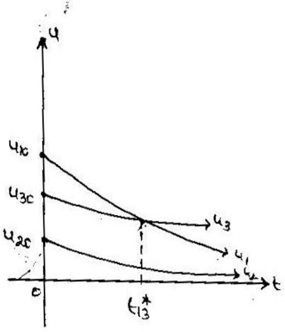
Fig 27

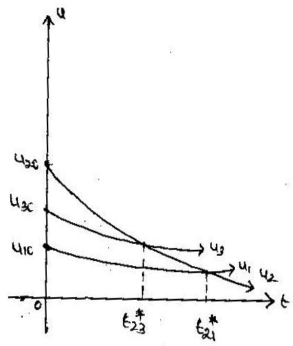
Fig 28

# 7 Conclusions of the Above Perturbation Graphs

## 7.1) Conclusions from Figure (1) to Figure (5)

**Case (i): $u_{10}=u_{20}<u_{30};\ a_1 > a_2 > a_3$**

Inspite of its initial lead ($u_3$), the host is out numbered by the commensal-prey at time $t_{31}^* = \frac{1}{K_3 a_{33} - K_1 a_{11}} \log\left(\frac{u_{10}}{u_{30}}\right)$ and by the predator at a time $t_{32}^* = \frac{1}{K_3 a_{33} - K_2 a_{22}} \log\left(\frac{u_{20}}{u_{30}}\right)$. This out numbering of commensal- prey over the host and predator as well is mainly due to its higher natural birth rate as shown in Fig.1.

**Case (ii): $u_{20}>u_{30}>u_{10};\ a_1> a_2> a_3$**

Even if the low initial strength the commensal prey ($u_{10}$) out numbers both the host at time $t_{31}^* = \frac{1}{K_3 a_{33} - K_2 a_{11}} \log\left(\frac{u_{10}}{u_{20}}\right)$ and $t_{21}^* = \frac{1}{K_2 a_{21} - K_1 a_{11}} \log\left(\frac{u_{10}}{u_{20}}\right)$.This is illustrated in Fig.2.

**Case (iii): $u_{20}>u_{30}>u_{10};\ a_1> a_3>a_2$**

With the low initial strength $u_{10}$ the commensal prey ($u_{10}$), it out numbers the host and predator at time $t_{31}^* = \frac{1}{K_3 a_{33} - K_1 a_{11}} \log\left(\frac{u_{10}}{u_{30}}\right)$and $t_{21}^* = \frac{1}{K_2 a_{22} - K_1 a_{11}} \log\left(\frac{u_{10}}{u_{20}}\right)$, and host out number the predator at $t_{23}^* = \frac{1}{K_2 a_{22} - K_3 a_{33}} \log\left(\frac{u_{30}}{u_{20}}\right)$ as shown in Fig.3.

**Case (iv): $u_{20}>u_{30}>u_{10};\ a_2> a_1>a_3$**

The predator always out numbers both the host and prey, because of its exceeding natural growth rate. The commensal-prey exceeds the host from time $t_{31}^* = \frac{1}{K_3 a_{33} - K_1 a_{11}} \log\left(\frac{u_{10}}{u_{30}}\right)$ on wards as shown in Fig.4.

**Case (v): $u_{30}>u_{10}>u_{20};\ a_2> a_1>a_3$**

Inspite of the low initial strength of predator numbers the commensal prey and host at time $t_{12}^* = \frac{1}{K_2 a_{22} - K_1 a_{11}} \log\left(\frac{u_{20}}{u_{10}}\right)$ and $t_{32}^* = \frac{1}{K_3 a_{33} - K_2 a_{22}} \log\left(\frac{u_{20}}{u_{30}}\right)$ and commensal prey out number the host at time $t_{31}{}^* = \frac{1}{K_3 a_{33} - K_1 a_{11}} \log\left(\frac{u_{10}}{u_{30}}\right)$. This could be attributed because of the lowest natural growth rate of the host and the highest growth rate of the predator. This is illustrated in Fig.5.

## 7.2 Conclusions from Figure (6) to Figure (9)

**Case (i): $u_{10}>u_{30}>u_{20}$**

The prey dominates the predator because of higher natural growth rate and initial take off, the predator out numbers the host at time $t_{32}{}^* = \frac{1}{K_3 a_{33} - K_2 a_{22}} \log\left(\frac{u_{30}}{u_{20}}\right)$ as shown in Fig.6.

**Case (ii): $u_{30}>u_{10}>u_{20}$**

The prey and predator both dominates the host at times $t_{31}{}^* = \frac{1}{a_{11}(K_1 + K_3) + K_3 a_{33}} \log\left(\frac{u_{30}}{u_{10}}\right)$ and $t_{32}{}^* = \frac{1}{K_3 a_{33} - K_2 a_{22}} \log\left(\frac{u_{30}}{u_{20}}\right)$ as shown in Fig.7.

**Case (iii): $u_{30}>u_{10}>u_{20}$**The predator out numbers both host and prey at time $t_{32}{}^* = \frac{1}{K_3 a_{33} - K_2 a_{22}} \log\left(\frac{u_{30}}{u_{20}}\right)$ and $t_{12}{}^* = \frac{1}{K_2 a_{22} - K_1 + a_{11}} \log\left(\frac{u_{10}}{u_{20}}\right)$.Further the prey out numbers the host at time $t_{31}{}^* = \frac{1}{a_{11}(K_1 + K_3) + K_3 a_{33}} \log\left(\frac{u_{30}}{u_{10}}\right)$.This is illustrated in Fig.8.

**Case (iv): $u_{30} > u_{20} > u_{10}$**

Inspite of the low initial strength of the prey, it out number the host and predator at time $t_{31}{}^* = \frac{1}{a_{11}(K_1 + K_3) + K_3 a_{33}} \log\left(\frac{u_{30}}{u_{10}}\right)$and $t_{21}{}^* = \frac{1}{K_1 a_{11} - K_2 a_{22}} \log\left(\frac{u_{20}}{u_{10}}\right)$ and predator out number the host at time $t_{32}{}^* = \frac{1}{K_3 a_{33} - K_2 a_{22}} \log\left(\frac{u_{30}}{u_{20}}\right)$.This happens due to a higher natural growth rate of prey compared to that of the predator. This is illustrated in Fig.9.

## 7.3 Conclusions from Figure (10) to Figure (14)

**Case (i):** $\boldsymbol{u_{10}>u_{30}>u_{20}}$The commensal-prey out number the host at time $t_{13}^{*}=\frac{1}{a_{22}u_2-a_{33}K_3}\log\left(\frac{u_{10}}{u_{30}}\right)$ alter that the host dominates, and the prey-commensal declines further as shown in Fig.10.

**Case (ii):** $\boldsymbol{u_{20}>u_{10}>u_{30}}$

Even with the low initial strength of the host it out numbers the prey-commensal and later predator at time $t_{13}^{*}$ and $t_{23}^{*}$ respectively. Where $t_{13}^{*}=\frac{1}{a_{22}K_2+a_{33}K_3}\log\left(\frac{u_{10}}{u_{30}}\right)$, $t_{23}^{*}=\frac{1}{a_{22}K_2+a_{33}K_3}\log\left(\frac{u_{20}}{u_{30}}\right)$ as sshown in Fig.11.

**Case (iii):** $\boldsymbol{u_{10}>u_{20}>u_{30}}$

Inspite of the low initial strength of the host it out number the predator and later commensal-prey at time $t_{23}^{*}$ (obtained by solving u1 & u2 in case B) and $t_{13}^{*}=\frac{1}{K_3a_{33}}\log\left(\frac{u_{10}}{u_{30}}\right)$ respectively. This is illustrated in Fig.12.

**Case (iv):** $\boldsymbol{u_{20}>u_{10}>u_{30}}$

Even with the low initial strength of the host it out number the prey-commensal and alter the predator at time $t_{13}^{*}=\frac{1}{K_3a_{33}}\log\left(\frac{u_{10}}{u_{30}}\right)$ and $t_{23}^{*}$(obtained by solving u1 & u2 in case B) respectively as shown in Fig.13.

**Case (v):** $\boldsymbol{u_{20}>u_{30}>u_{10}}$

The host exceeds the predator from time $t_{23}^{*}$(obtained by solving u1 & u2 in case B) and the predator declines further and asymptotic to the equilibrium point after $t_{21}^{*}$ (obtained by solving u1 & u2 in case B). This is illustrated in Fig.14.

## 7.4 Conclusions from Figure (15) to Figure (18)

**Case (i): $u_{20} > u_{10} > u_{30}$**

The predator always and number both the host and prey-commensal because of exceeding natural growth rate. The host exceeds the commensal -prey from time $t_{13}^*$ (obtained by solving $u_1$& $u_3$) as shown in Fig.15.

**Case (ii): $u_{10} > u_{30} > u_{20}$**

The host and predator both dominates the prey-commensal at time $t_{13}^*$ and $t_{12}^*$ (obtained by solving $u_1$& $u_3$, $u_1$& $u_2$).This illustrated in Fig.16.

**Case (iii): $u_{10} > u_{20} > u_{30}$**

The host out number the commensal-prey and predator at time $t_{13}^*$ and $t_{23}^* = \log\left(\frac{u_{20}}{u_{30}}\right)\frac{1}{K_3 a_{33} - a_{22}(K_2 + QK_1)}$ Further the commensal – prey declines asymptotic to the equilibrium point as shown in Fig.17.

**Case (iv): $u_{10} > u_{30} > u_{20}$**

The predator out number the commensal-prey and host at time $t_{12}^*$ (obtained by solving $u_1$& $u_2$) and $t_{32}^* = \log\left(\frac{u_{30}}{u_{20}}\right)\frac{1}{K_3 a_{33} - a_{22}(K_2 + QK_1)}$ Further the commensal – prey declines asymptotic to the equilibrium point as shown in Fig.18.

## 7.5 Conclusions from Figure (19) to Figure (22)

**Case (i): $u_{10} > u_{30} > u_{20}$**

The prey-commensal always out number both the host and predator, because of its exceeding natural growth rate. The predator exceeds the host from time $t_{32}^* = \log\left(\frac{u_{30}}{u_{20}}\right)\frac{1}{K_1 + CK_3 - PK_2 + K_3 + a_{33}}$ ,as shown in Fig.19.

**Case (ii): $u_{30} > u_{10} > u_{20}$**

Inspite of the low initial strength of the predator, it out number the host and prey at time $t_{32}^* = \frac{1}{K_1 + CK_3 - PK_2 + K_3 a_{33}}\log\left(\frac{u_{30}}{u_{20}}\right)$ and commensal-prey out number the host at time $t_{31}^{*} = \frac{1}{K_1 + CK_3 - PK_2 + K_3 a_{33}} \log\left(\frac{u_{30}}{u_{10}}\right)$ This happens due to higher natural growth rate of prey-commensal compared to that of predator. This illustrated in Fig.20.

**Case (iii):** $u_{10} > u_{20} > u_{30}$

The prey-commensal always out number both the host and predator because of its exceeding natural growth rate. The host exceeds the predator from time $t_{23}^{*} = \frac{1}{A - K_3 a_{33}} \log\left(\frac{u_{20}}{u_{30}}\right)$ onwards where $A = K_1 + CK_3 - PK_2 < 0$.How ever all the three converge asymptotically to the equilibrium point. This illustrated in Fig.21.

**Case (iv):** $u_{30} > u_{10} > u_{20}$

The prey-commensal and predator both dominates the host at time $t_{31}^{*} = \frac{1}{K_3 a_{33} - a_3} \log\left(\frac{u_{30}}{u_{10}}\right)$ and $t_{32}^{*} = \frac{1}{K_3 a_{33} - a_3} \log\left(\frac{u_{30}}{u_{20}}\right)$ onwards, where $A = K_1 + CK_3 - PK_2 < 0$. However all the three converge asymptotically to the equilibrium point. This illustrated in Fig.22.

## 7.6 Conclusions from Figure (23) to Figure (25)

**Case (i):** $u_{20} > u_{10} > u_{30}$

The prey-commensal out number the host at time $t_{13}^{*}$ (obtained by solving $u_1$& $u_3$), after that the host dominates and both declines further as shown in Fig.23.

**Case (ii):** $u_{10} > u_{30} > u_{20}$

Even with the low initial strength of the predator it out numbers the host and later prey-commensal at time $t_{32}^{*}$ and $t_{12}^{*}$ (obtained by solving $u_1$& $u_2$) respectively. Where $t_{32}^{*} = \frac{1}{a_{22}\left(K_2 + Q\left(K_1 + CK_3\right)\right) + K_3 a_{33}} \log\left(\frac{u_{30}}{u_{20}}\right)$ as shown in Fig.24.

**Case (iii):** $u_{30} > u_{20} > u_{10}$

The host out number the predator at time $t_{23}^{*} = \frac{1}{\left(a_{22}\left(K_2 + Q\left(K_1 + CK_3\right)\right) + K_3 a_{33}\right)} \log\left(\frac{u_{20}}{u_{30}}\right)$ after that the predator dominates, and the host declines further as shown in Fig.25.

## 7.7 Conclusions from Figure (26) to Figure (28)

**Case (i):** $u_{10} > u_{20} > u_{30}$

The prey-commensal dominates both predator and host in natural growth as well as in its initial population strength. In this case the prey-commensal continues out numbering both predator and host as shown in Fig.26. However all the three converge asymptotically to the equilibrium point.

**Case (ii):** $u_{10} > u_{30} > u_{20}$

The host exceeds the prey-commensal from time $t_{13}^{*}$ onwards .How ever all the three converge asymptotically to the equilibrium point. This illustrated in Fig.27.

**Case (iii):** $u_{20} > u_{30} > u_{10}$

The prey-commensal and host both dominates the predator at time $t_{23}^{*}$ and $t_{21}^{*}$ (obtained by solving $u_2, u_3$ and $u_2, u_1$ respectively) onwards. How ever all the three converge asymptotically to the equilibrium point. This illustrated in Fig.28

## 8. Threshold Diagrams

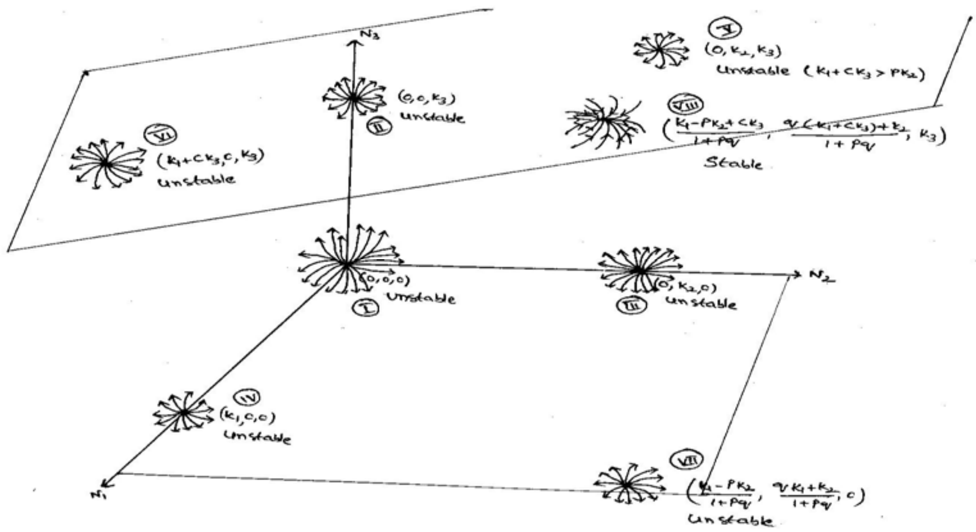


**Fig-29**

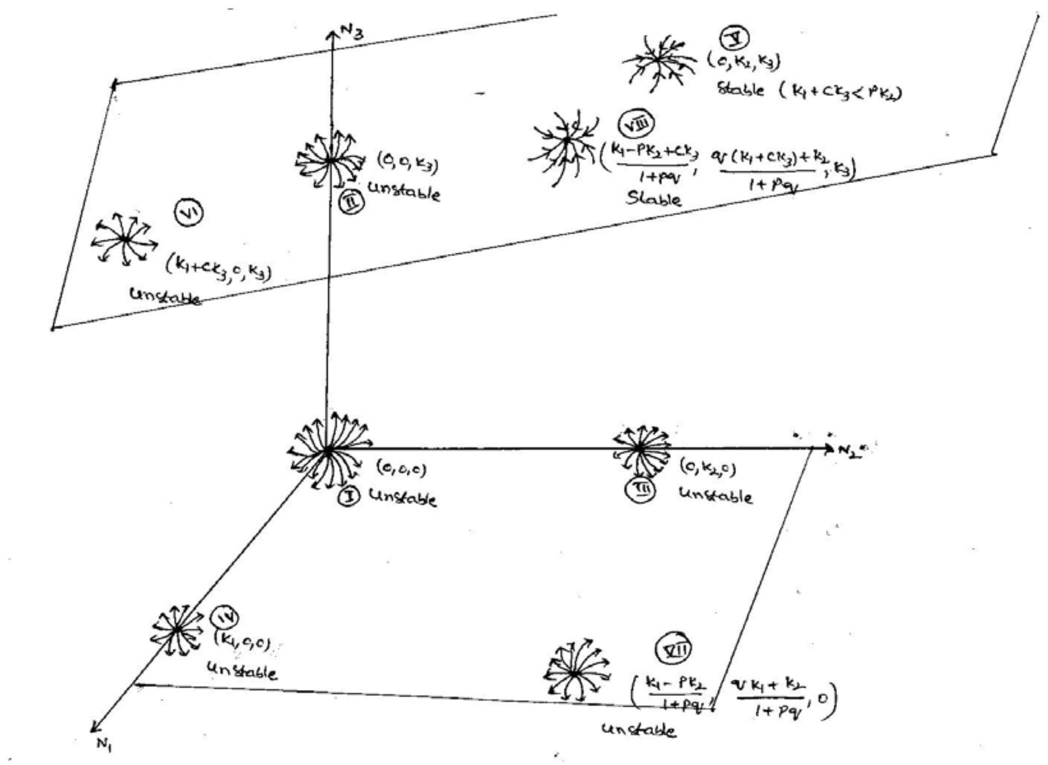
N3
N2
N1
(0,0,0)
I Unstable
(0,K2,0)
III Unstable
(0,0,K3)
II Unstable
(K1,0,0)
IV Unstable
(K1+cK3,0,K3)
VI Unstable
(0,K2,K3)
V Stable (K1+cK3<pK2)
VIII
Stable
VII
Unstable

**Fig-30**

# 6 Open Problems

An ecosystem consisting of a prey ,a predator with unlimited resources and a host commensal to the prey with resources (i) unlimited and (ii) limited and also with harvesting of one or other species.

An ecosystem consisting of a predator that survives on an immigrating prey that is also a commensal to a host. It is also relevant to examine the delayed effect in the above mentioned problems.

A three species ecosystem consisting of a prey with limited resources, a harvested predator with unlimited resources and a harvested host commensal to the prey, the effect of delay(s) of the interaction in the above cases.

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