

Research Article

Dynamic MRR Function Optimization Using Calculus of Variations

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A dynamic machining model to optimize the control of material removal rate (MRR) for a cutting tool undergoing the considerations of fixed tool life and maximum machining rate is established in this paper. This study not only applies material removal rate mathematically into the objective function, but also implements Calculus of Variations to comprehensively optimize the control of material removal rate. In addition, the optimal solution for the dynamic machining model to gain the maximum profit is provided, and the decision criteria for selecting the optimal solution of the dynamic machining model are then recommended. Moreover, the computerized analyses to simulate both the dynamic and traditional machining models for a numerical example are also promoted. This study definitely contributes an applicable approach to the dynamic function of material removal rate and provides the efficient tool to concretely optimize the profit of a cutting tool for operation planning and control in modern machining industry with profound insight.

1. Introduction

The cutting conditions of a cutting tool have been the most critical variables in the machining process. Cutting speed, feed rate, and depth of cut were considered as three factors of input cutting parameters. Liew and Ding [1] have expressed several methods to be used under stepwise constant variation in feed, speed, or depth of cut, but none is practically applicable when two or more cutting conditions are changed. Hence, the research of controlling cutting conditions with fixed material removal rate has been introduced [2, 3]. For most studies with this viewpoint, the material removal rate is fixed because of the expensive observation of control. However, through the computer-integrated interface to program the feed rate on modern computer numerical controlled (CNC) machines [2] with fixed cutting speed and depth of cut, the material removal rate is capable of being dynamically controlled.

In addition, Cámara et al. [3] described that tool life is also a critical parameter of the cutting process. Iqbal et al. [4] proposed a suitable implementation to predict tool life. Ferrer et al. [5] proposed a method of optimal control to ensure maximum tool life. Kim et al. [6] also provided a modified Taylor tool life equation to minimize tool cost. However, the maximum tool life or the minimum tool cost will not guarantee the maximum profit of a cutting tool. Besides, the various tool inspecting periods for a tool change from different machine tool operators will decrease the productivity and increase the cost during machining significantly. In order to well manage the productivity and the consumption of cutting tools, a fixed tool life is practically considered into the cutting process.

Moreover, the cost to machine each part is a function of the machining time, and the marginal cost of production is a linear function of production rate. Therefore, the marginal cost of operation for the machining process is also proposed to be a linear function of the material removal rate in this study. This explicates that more machining rate causes more operational cost such as machine maintenance, loading-unloading, and machine depreciation costs.

According to Zong et al. [7], the dependency of a reliability model on the cutting conditions is the aim to optimize the manufacturing system [8]. Although several time series modeling on the control of machining process are mentioned for decision making [9], none is capable to achieve the maximum profit. They are all emphasizing on the maximum tool usage or minimum tool cost. Practically, the profit and the productivity of machining operation are the mostly concerned problems confronting the manufacturing industry. Besides, the need of operating *CNC* machines efficiently to obtain the required payback is even greater in the case of rough machining, since a greater amount of material is removed thus increasing possible savings [10]. With the reasons above, there is an economic demand to control the material removal rate of rough machining operation on *CNC* machines. Thus, the essentiality of finding the optimal solution to achieve the maximum profit of a cutting tool with fixed tool life is absolutely arising.

2. Assumptions and Notations

Before formulating the problem, several assumptions and notations are to be constructed. They are described as follows.

2.1. Assumptions

- (1) The cutting process is a continuous rough turning operation with one type of tool.
- (2) Each tool performs the same fixed length of cutting time (tool life).
- (3) The upper limit of material removal rate is generated from the maximum cutting conditions suggested in the handbook, and the fixed tool life is derived from the Taylor's tool life equation [11] with these maximal conditions. Thus, no tool will break before this fixed tool life even with the upper limit of material removal rate.
- (4) The marginal cost of operation is a linear function of the material removal rate [12].
- (5) There is no chattering or scrapping of parts occurring during the machining process.
- (6) All chips from cutting and finished parts are held in the machine until a tool change.
- (7) All machined parts are paid at a given price immediately at the tool change.

2.2. Notations

One has the following:

a : average volume of material machined per unit part,

B : upper limit of material removal rate,

c : overall holding cost for unit chip per unit time in the machine, where $c = h_1 + (h_2/a)$,

$bM'(t)$: marginal operation cost at the material removal rate $M'(t)$; where b is a constant,

$bM^2(t)$: operational cost at time t ,

h_1 : chip holding cost for unit chip per unit time,

h_2 : part holding cost for unit finished part per unit time,

P : contribution per unit part machined,

$[0, T]$: controlling time interval of the tool life.

2.3. Decision Functions

One has the following

$M(t)$: cumulated volume of material machined during time interval $[0, t]$,

$M'(t)$: material removal rate at time t .

3. Model Formulation

When considering the machining profit of an individual cutting tool under fixed tool life, the mathematical model representing the machining profit under dynamic *MRR* control is formulated. In the mathematical model; P represents the revenue of each machined product, T is the selected fixed tool life of a cutting tool, $M(t)$ describes the cumulated amount of material machined during $[0, t]$, $M'(t)$ illustrates the *MRR* at time t , $M(T)$ denotes the cumulated volume of material machined before the fixed tool life T , and a indicates the average volume of material to be machined per unit part. Therefore, $P(M(T)/a)$ explains the contribution of an individual cutting tool under the machining operation with a fixed life T .

Since the marginal operation cost of production is a linear increasing function of production rate [12], the marginal operation cost of a machine is proposed in direct proportion to the *MRR* that is described as $bM'(t)$, where b is the operational constant that is obtained through the equipment management strategy of the *CNC* machine. It can often be evaluated by dividing the initial purchasing cost of the *CNC* machine from the value of multiplying the expected life of the machine with the suggested average *MRR* from the equipment manufacturer.

Whilst the marginal operation cost of production is a linear increasing function of production rate and the operational cost is directly proportional to the square of the production rate [12], the operational cost of a machine is also proposed in direct proportion

to the square of the material removal rate in this study. Thus, $\int_0^T bM'^2(t)dt$ signifies the operational cost during time interval $[0, T]$.

As all chips from cutting and finished products are usually held and stored at the machine until a tool replacement, $\int_0^T h_1M(t)dt$ denotes the chip holding cost during time interval $[0, T]$, where h_1 is the chip holding cost per unit chip per unit time. And, $\int_0^T h_2(M(t)/a)dt$ denotes the finished part holding cost during time interval $[0, T]$, where h_2 is the finished part holding cost per unit finished part per unit time. Therefore, the objective function and its constraints are listed as shown below:

$$\text{Max} \left\{ P \frac{M(T)}{a} - \int_0^T \left[bM'^2(t) + h_1M(t) + h_2 \frac{M(t)}{a} \right] dt \right\}. \quad (3.1)$$

Let $c = h_1 + (h_2/a)$ be the overall holding cost for unit chip per unit time. Thus, the dynamic machining model is then developed as shown below:

$$\begin{aligned} & \text{Max} \left\{ P \frac{M(T)}{a} - \int_0^T \left[bM'^2(t) + cM(t) \right] dt \right\}, \\ & \text{s.t.} \\ & M(0) = 0, \\ & M(T) \text{ is free,} \\ & 0 \leq M'(t) \leq B, \\ & \forall t \in [0, T]. \end{aligned} \quad (3.2)$$

4. Optimal Solution

Let $M^{*'}(t)$ and $M^*(t)$ be the optimal solution of the dynamic machining model, and assume that time interval $(0, \tilde{t})$ is the maximal subinterval of $[0, T]$ satisfying the Euler Equation [12, 13] $F_M(t, M^*(t), M^{*'}(t)) = dF_{M'}(t, M^*(t), M^{*'}(t))/dt$.

There are two probable situations to be discussed in this study.

4.1. Situation 1

$M^{*'}(t)$ will not touch B before T .

The optimal solution for *Situation 1* is shown as follows:

$$\begin{aligned} M^{*'}(t) &= \frac{c}{2b}t + \frac{1}{2b} \left(\frac{P}{a} - cT \right), \\ M^*(t) &= \frac{c}{4b}t^2 + \frac{1}{2b} \left(\frac{P}{a} - cT \right)t. \end{aligned} \quad (4.1)$$

The detail is developed in Appendix A.

From (4.1), it is found that the optimal material removal rate, $M^{*'}(t)$, is a strictly increasing linear function of t . In addition, by considering the constraint $M'(t) \geq 0$ into (4.1) at $t = 0$, it is obtained that

$$cT \leq \frac{P}{a}. \quad (4.2)$$

This denotes that the overall holding cost of unit chip for the period T is smaller than the contribution per unit material machined. Otherwise, the optimal material removal rate will touch zero.

Applying the constraint $M'(t) \leq B$ into (4.1) at $t = T$, we have

$$\frac{P}{a} \leq 2bB. \quad (4.3)$$

This represents that twice of the marginal cost at upper limit B must be larger than the contribution per unit material machined. Otherwise, the optimal material removal rate will touch the upper limit B .

Before solving the optimal solution for *Situation 2*, one Property is proposed and illustrated as follows.

Property 1. If the line $y = M^{*'}(t)$ touches the line $y = B$, these two lines should overlap to be $y = B$ from the touch point \tilde{t} to the end point T .

The proof of the Property is illustrated in Appendix B.

4.2. Situation 2

$M^{*'}(t)$ will touch B at time \tilde{t} : $\tilde{t} \in [0, T)$.

The optimal solution for *Situation 2* is shown as follows:

$$\begin{aligned} \tilde{t} &= T - \frac{1}{c} \left(\frac{P}{a} - 2bB \right), \\ M^{*'}(t) &= \begin{cases} \frac{c}{2b}t + \frac{1}{2b} \left(\frac{P}{a} - cT \right), & \text{if } t \in [0, \tilde{t}], \\ B, & \text{if } t \in (\tilde{t}, T], \end{cases} \end{aligned} \quad (4.4)$$

$$M^*(t) = \begin{cases} \frac{c}{4b}t^2 + \frac{1}{2b} \left(\frac{P}{a} - cT \right)t, & \text{if } t \in [0, \tilde{t}], \\ M(\tilde{t}) + B(t - \tilde{t}), & \text{if } t \in (\tilde{t}, T]. \end{cases} \quad (4.5)$$

The detail is developed in Appendix C.

4.3. Decision Criteria

From (4.4) and (4.5), the maximum value of $M^*(t)$ is found at $\tilde{t} = 0$ when $t = T$. Therefore, the following criteria are then made.

When $P/a \leq 2bB$, it means that $\tilde{t} \geq T$. This denies the assumption of *Situation 2*. It is that the optimal material removal rate will not reach the upper limit B within the tool life. The optimal solution is *Situation 1*.

When $P/a > 2bB$, it means that $\tilde{t} < T$. This is that the optimal material removal rate will reach the upper limit B within the tool life. Therefore, the optimal solution is *Situation 2*.

5. Computerized Analyses

For a specific turning operation, there are bounds for cutting conditions suggested in the machining handbook. Therefore, there must exist a maximum material removal rate U and a minimum material removal rate L derived from the maximum and the minimum cutting conditions, respectively. Each material removal rate between U and L can feasibly be selected as the upper limit B for the dynamic machining model. From the well-known Taylor's expression of the tool life [8], it is then modified as $B \times T^n = k$, when the cutting speed and depth of cut are selected fixed. Therefore, with this reconstructed equation, each fixed tool life T is then obtained with respect to every feasible upper limit B selected. Thus, for each feasible upper limit, there exist an optimal profit for the dynamic machining model. In addition, the traditional machining model with a fixed MRR between U and L is also potential to be optimal. To locate the maximum profit solution among all these possibilities, a computer program written in MATLAB to analyze the problem is then constructed. The concept of the flow chart is shown as follows: P, a, b, c, U, L, n, k should be given before the following algorithm. Initialize $B = L$.

Step 1. Compute T for both models, then compute the profit

$$O_b = P(BT/a) - (bB^2T + (cB/2)T^2) \quad (5.1)$$

and the production quantity $Q = BT/a$ for the traditional model.

Go to *Step 2*.

Step 2. Plot $C(B, O_b)$ and $F(B, Q)$, then go to *Step 3*.

Step 3. If $P/a > 2bB$, go to *Step 5*.

Otherwise, go to *Step 4*.

Step 4. Compute the profit

$$\bar{O}_b = (PM(T)/a) - \int_0^T [bM^2(t) + cM(t)] dt \quad (5.2)$$

and the production quantity $\bar{Q} = M(T)/a$ for *Situation 1* of the dynamic model.

Go to *Step 6*.

Step 5. Compute \tilde{t} , the profit

$$\bar{O}_b = (PM(T)/a) - \int_0^{\tilde{t}} [bM'^2(t) + cM(t)] dt - \int_{\tilde{t}}^T [bB^2 + c \left(M(\tilde{t}) + \int_{\tilde{t}}^t B dt \right)] dt \quad (5.3)$$

and the production quantity $\bar{Q} = M(T)/a$ for *Situation 2* of the dynamic model.

Go to Step 6.

Step 6. Plot $\bar{C}(B, \bar{O}_b)$ and $\bar{F}(B, \bar{Q})$, then go to Step 7.

Step 7. If $B \geq U$, stop the program.

Otherwise, set $B = B + 0.05$ as initialize, and return to Step 1.

6. Simulated Results

With a machining project from AirTAC Corporation in Taipei, Taiwan, the numerical example referenced to a rough turning operation of specific cylindrical parts with step diameters is studied. The machining operation is assigned to a CNC lathe with FANUC controller. All data collected are transformed and listed as follows:

$$\begin{aligned} P &= 13.00 \text{ dollars (US)}, \quad a = 1.00 \text{ in}^3, \quad b = 1.05 \frac{\text{dollars (US)} - \text{min}}{\text{in}^3}, \\ c &= 0.03 \frac{\text{dollars (US)}}{\text{in}^3}, \quad U = 12.00 \frac{\text{in}^3}{\text{min}}, \quad L = 10.00 \frac{\text{in}^3}{\text{min}}, \\ n &= 0.2, \quad k = 16.0. \end{aligned} \quad (6.1)$$

The simulated results of the MATLAB program are presented as the profit analysis and production quantity analysis in Figures 2 and 3, respectively. From Figure 2, it is noticed that the dynamic model is superior to the traditional model on the aspect of profit, when the selected upper limit B exceeds approximately 10.45 (in^3/min). In addition, the profit of dynamic model is much more steady within the whole range of allowable machining rate, since the profit of traditional model is greatly decreasing as the MRR increases. Therefore, it is deduced that the dynamic machining model is much more profitable than the traditional machining model for the higher range of machining rate, which is usually used in the real-world machining industry for higher productivity.

From Figure 3, it is also observed that the production quantity of a cutting tool under dynamic function is always less than what is under traditional control. However, as the machining rate increases, the difference of production quantity between two models is discovered to be decreasing. Thus, it is derived that the dynamic function will turn more competitive to the traditional control on the aspect of production quantity and productivity for the higher range of machining rate.

With the analyses of the dynamic model above, when there is a need for the cutting tool to acquire other productivity than the maximum profit solution, it is always feasible to slightly compromise on the profit for the required productivity. This will fractionally change the profit, but satisfactorily increases the productivity for the dynamic machining model.

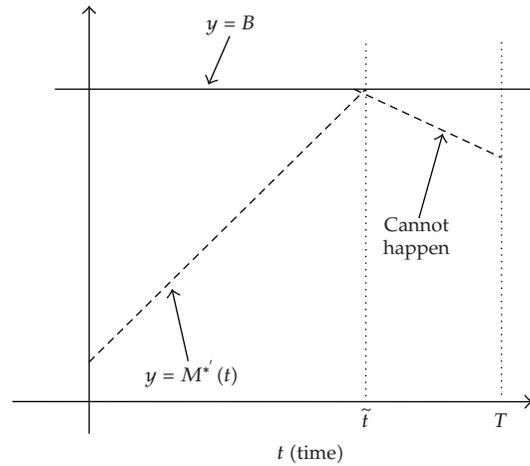


Figure 1: Possible condition of lines $y = B$ and $y = M^*(t)$.

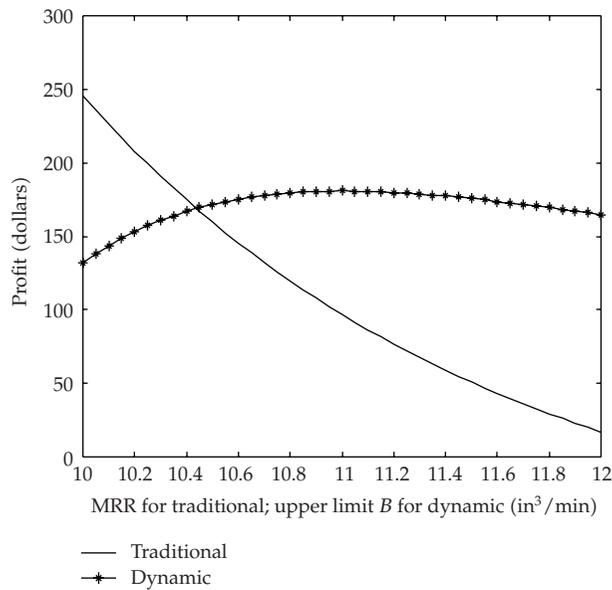


Figure 2: Profit analysis of a cutting tool with different models.

7. Conclusions

The tool life, operational cost, holding costs, contribution per unit part machined, average volume of material machined per unit part, and upper speed limit are considered simultaneously to dynamically optimize the control of material removal rate. It is an extremely hard-solving and complicated issue. However, through the dynamic machining model, the problem becomes concrete and solvable.

In addition, four characteristics of the dynamic machining model are illustrated as follows. First, the optimal material removal rate $M^*(t)$ is a strictly increasing linear function

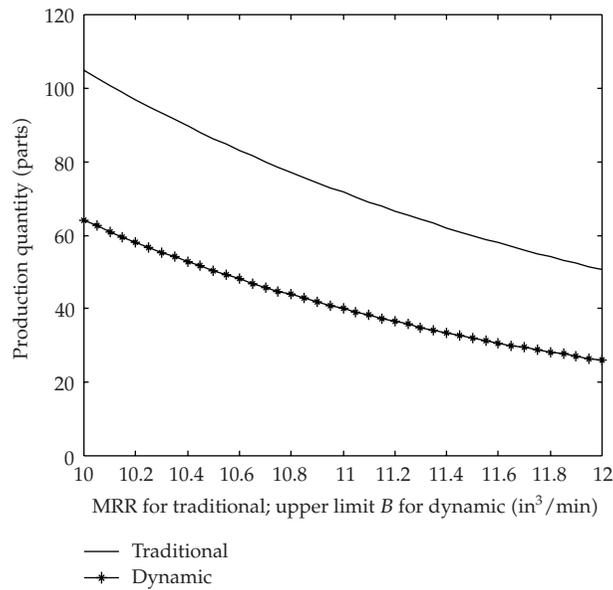


Figure 3: Production quantity analysis of a cutting tool with different models.

of t before reaching the upper limit B . Second, by PROPERTY, if the optimal material removal rate $M^*(t)$ touches the upper limit B , it will stay to be B for the rest of tool life T . Third, by (4.2), the overall holding cost of unit chip for the tool life period T must be smaller than the contribution per unit material machined. Otherwise, the optimal material removal rate will touch zero. Fourth, with (4.3), two times of the marginal cost at upper limit B must be larger than the contribution per unit material machined. Otherwise, the optimal material removal rate will touch the upper limit B .

Moreover, from the computerized analyses and numerical simulation with the MATLAB program, the four remarks are then provided. First, the dynamic model receives much more balanced profit than the traditional model within the whole range of allowable machining rate. Second, the dynamic machining model is much more profitable than the traditional machining model for the higher range of machining rate. Third, when the machining rate increases, the difference of production quantity between two models will decrease. Fourth, when there is a need to obtain higher productivity than the optimal solution, it is always feasible for the dynamic machining model to slightly compromise on the profit in achieving the productivity. With these remarks above, the applicability and flexibility of the dynamic machining model are significantly expended.

The material removal rate is an important control factor of machining operation, and the control of machining rate is also critical for production planning. This study not only delivers the idea of controlling the material removal rate to the machining technology, but also leads a cutting tool towards to achieve maximum profit. This study focuses on the modeling and optimization of the MRR in CNC turning. Future researches with the cutting experiments to prove the proposed model as well as the modeling of dynamic optimization on multitool machining process and machining project control are absolutely encouraged. In summary, this study surely generates a reliable and applicable concept of machining control to the techniques, and also provides a better and practical solution to this field.

Appendices

A. Appendix A

The optimal solution for *Situation 1*.

Suppose that the material removal rate $M^*(t)$ will not reach the upper limit B before tool life T .

From the Euler Equation [12, 13], $F_M(t, M^*(t), M^{*'}(t)) = dF_{M'}(t, M^*(t), M^{*'}(t))/dt$, it is derived that

$$c = \frac{d}{dt}(2bM^{*'}(t)). \quad (\text{A.1})$$

There is a constant k_1 satisfying

$$M^{*'}(t) = \frac{c}{2b}t + k_1, \quad \forall t \in [0, \tilde{t}]. \quad (\text{A.2})$$

With the transversality condition of salvage value for free end value $M(T)$ [12, 13], it is obtained that

$$M^{*'}(T) = \frac{P}{2ab} \quad (\text{A.3})$$

Integrating (A.2) with t , then

$$M^*(t) = \frac{c}{4b}t^2 + k_1t + k_2 \quad (\text{A.4})$$

Using the boundary condition $M(0) = 0$ and (A.4), we have

$$k_2 = 0 \quad (\text{A.5})$$

Comparing (A.3) and (A.2) at $t = T$, hence

$$k_1 = \frac{1}{2b} \left(\frac{P}{a} - cT \right) \quad (\text{A.6})$$

Substituting (A.5) and (A.6) into (A.2) and (A.4), the optimal solution for *Situation 1* is then derived.

B. Appendix B

The proof of Property.

Proof. From (4.1), the optimal material removal rate $M^{*'}(t)$ is a strictly increasing linear function of t . It holds for any subinterval of $[0, T]$ subject to $0 \leq M^{*'}(t) \leq B$. Therefore, the

straight line in the time interval $[\tilde{t}, T]$ (shown in Figure 1) cannot exist, because it contradicts the Euler Equation [12, 13] to be a decreasing linear function of t , the PROPERTY is then verified. \square

C. Appendix C

The optimal solution for *Situation 2*.

It is assumed that the material removal rate $M^*(t)$ will touch the upper limit B before tool life T . Before reaching the upper limit, the optimal solution for *Situation 1* is satisfied for this situation either.

The objective function of the dynamic machining model is then modified as

$$\begin{aligned} & \text{Max} \left\{ \frac{PM(T)}{a} - \int_0^{\tilde{t}} [bM'^2(t) + cM(t)] dt - \int_{\tilde{t}}^T \left[bB^2 + c \left(M(\tilde{t}) + \int_{\tilde{t}}^t B dt \right) \right] dt \right\} \\ & = - \text{Min} \left\{ \int_0^{\tilde{t}} [bM'^2(t) + cM(t)] dt + bB^2(T - \tilde{t}) + cM(\tilde{t})(T - \tilde{t}) + \frac{cB}{2}(T^2 - \tilde{t}^2) \right. \\ & \quad \left. - cB\tilde{t}(T - \tilde{t}) - \frac{P}{a}M(\tilde{t}) - \frac{P}{a}B(T - \tilde{t}) \right\}. \end{aligned} \quad (\text{C.1})$$

With the transversality condition of salvage value for free end point \tilde{t} [12, 13] and Property; the time \tilde{t} to reach the upper limit is derived.

Thus, with \tilde{t} and the Property; the optimal solution for *Situation 2* is then obtained.

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