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Powers of p-Hyponormal Operators

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Applying Furuta's and Hansen's inequalities, it is shown that if T is a p-hyponormal operator, then T^n is (p/n)-hyponormal. Applications are obtained.

Keywords: p-Hyponormal operator; Furuta's inequality; Hansen's inequality

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1 INTRODUCTION

Let H be a complex Hilbert space and $L(H)$ be the algebra of bounded linear operators on H. An operator $T \in L(H)$ is said to be p-hyponormal, $p > 0$, if $(T^*T)^p \geq (TT^*)^p$. A p-hyponormal operator is said to be hyponormal if $p = 1$; semi-hyponormal if $p = 1/2$. The well known Löwner–Heinz inequality implies that every p-hyponormal operator is q-hyponormal for any $0 < q \leq p$. Hyponormal operators have been studied by many authors, such as Halmos [7], Stampfli [10,11] and Xia [13]. Semi-hyponormality was introduced by Xia [12]. See [13] for properties of semi-hyponormal operators. For p-hyponormal operators, see [1,2].

Throughout this paper we assume $0 < p \leq 1$ and use a capital letter to denote an operator in $L(H)$. In [7, Problem 164], Halmos gave an example of a hyponormal operator A whose square A^2 is not

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hyponormal. Here we use Furuta's [5] and Hansen's [6] inequalities to show that if T is p-hyponormal, then T^2 is $(p/2)$-hyponormal. In fact, we will show that for any positive integer n, the operator T^n is (p/n)-hyponormal. Applications of our result are also obtained.

2 THE RESULT

LEMMA 1 (Furuta's inequality [5]) *If $A \geq B \geq 0$, then the inequalities*

$$(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$$

and

$$A^{(p+2r)/q} \geq (A^r B^p A^r)^{1/q}$$

hold for $p, r \geq 0$, $q \geq 1$ with $(1+2r)q \geq p+2r$.

LEMMA 2 (Hansen's inequality [6]) *If $A \geq 0$ and $\|B\| \leq 1$, then*

$$(B^* A B)^p \geq B^* A^p B$$

for $0 \leq p \leq 1$.

THEOREM 1 *Let T be a p-hyponormal operator. The inequalities*

$$(T^{n^*} T^n)^{p/n} \geq (T^* T)^p \geq (T T^*)^p \geq (T^n T^{n^*})^{p/n}$$

hold for all positive integer n.

Proof Let $T = U|T|$ be the polar decomposition of T. For each positive integer n, let $A_n = (T^{n^*} T^n)^{p/n}$ and $B_n = (T^n T^{n^*})^{p/n}$. We will use induction to establish the inequalities

$$A_n \geq A_1 \geq B_1 \geq B_n. \tag{1}$$

The inequalities (1) clearly hold for $n = 1$. Assume (1) hold for $n = k$. The induction hypothesis and the assumption that T is p-hyponormal imply

$$U^* A_k U \geq U^* A_1 U \geq A_1.$$

Let $C_k = (U^* A_k^{k/p} U)^{p/k}$. Hansen's inequality implies $C_k \geq U^* A_k U \geq A_1$. Thus

$$\begin{aligned} A_{k+1} &= (T^{*^{k+1}} T^{k+1})^{(p/k+1)} \\ &= (T^*(T^{*^k} T^k)T)^{(p/k+1)} \\ &= (|T|U^* A_k^{k/p} U|T|)^{(p/k+1)} \\ &= (A_1^{1/2p} C_k^{k/p} A_1^{1/2p})^{(p/k+1)} \\ &\geq A_1 \end{aligned}$$

by Furuta's inequality. On the other hand, the induction hypothesis implies

$$B_k \leq B_1 \leq A_1.$$

Thus

$$\begin{aligned} B_{k+1} &= (T^{k+1} T^{*^{k+1}})^{(p/k+1)} \\ &= (T(T^k T^{*^k})T^*)^{(p/k+1)} \\ &= (U|T|B_k^{k/p}|T|U^*)^{(p/k+1)} \\ &= U(|T|B_k^{k/p}|T|)^{(p/k+1)} U^* \\ &= U(A_1^{1/2p} B_k^{k/p} A_1^{1/2p})^{(p/k+1)} U^* \\ &\leq U A_1 U^* \\ &= B_1, \end{aligned}$$

where the inequality follows from Furuta's inequality. Therefore,

$$A_{k+1} \geq A_1 \geq B_1 \geq B_{k+1}$$

and hence, by induction, inequalities (1) hold for $n \geq 1$. The proof is complete.

COROLLARY 1 *If the operator T is p-hyponormal, then T^n is (p/n)-hyponormal.*

Concrete examples of non-hyponormal p-hyponormal operators are hard to come by. In [12], Xia gave an example of a singular integral operator which is semi-hyponormal but not hyponormal. Corollary 1 allows us to give another example of a semi-hyponormal operator which is not hyponormal. Let A be the operator in Halmos' [7, Problem 164]. Thus, A is hyponormal but A^2 is not hyponormal. By Corollary 1, A^2 is semi-hyponormal. Moreover, A^{2n} is $(1/2n)$-hyponormal.

3 APPLICATIONS

In [10, Theorem 5], Stampfli proved that if T is hyponormal and T^n is normal for some positive integer n, then T is normal. Stampfli's result had been extended by Ando [3] to the case where T is paranormal. Although not as broad as Ando's extension, Theorem 1 can easily be used to extend Stampfli's result to p-hyponormal operators as follows.

COROLLARY 2 *Let the operator T be p-hyponormal. If T^n is normal, then T is normal.*

Proof By Theorem 1 and the assumption that T^n is normal,

$$(T^{n^*}T^n)^{p/n} = (T^*T)^p = (TT^*)^p = (T^nT^{n^*})^{p/n}.$$

Whence $T^*T = TT^*$. The proof is complete.

In [9, Theorem 7], Putnam proved that if T is hyponormal, and $r \geq 0$ is such that $r^2 \in \sigma(T^*T)$, then there is a $z \in \sigma(T)$ such that $|z| = r$. Recently, Chō and Itoh [4, Theorem 4] generalized Putnam's result to the case where the operator T is p-hyponormal. Theorem 1 can be utilized to give a generalization of the result of Chō and Itoh as follows.

THEOREM 2. *Let T be a p-hyponormal operator and n be a positive integer. If $r \geq 0$ is such that $r^2 \in \sigma(T^{n^*}T^n)$, then there is a $z \in \sigma(T)$ such that $|z|^n = r$.*

Proof Theorem 1 implies T^n is (p/n)-hyponormal. Therefore, by [4, Theorem 4], there is a $w \in \sigma(T^n)$ such that $|w| = r$. Since $\sigma(T^n) = \{z^n : z \in \sigma(T)\}$, there is a $z \in \sigma(T)$ such that $z^n = w$. Clearly $|z|^n = r$ and the proof is complete.

As an extension of the well-known Putnam's area inequality for hyponormal operators [8], Xia [13, Theorem XI.5.1] proved the following Theorem 3 for the case in which T is p-hyponormal with $p \geq 1/2$ and $n = 1$. In [4, Theorem 5], Chō and Itoh extended Xia's result to p-hyponormal operators with $0 < p \leq 1/2$.

THEOREM 3 *Let T be p-hyponormal. If $\sigma(T) \subseteq \{re^{i\theta}: 0 \leq \theta < 2\pi/m\}$ for some positive integer m, then*

$$\|(T^{n^*}T^n)^{p/n} - (T^nT^{n^*})^{p/n}\| \leq \frac{np}{\pi} \iint\limits_{\sigma(T)} \rho^{2p-1}\, \mathrm{d}\rho\, \mathrm{d}\theta$$

for positive integers $n \leq m$.

Proof By Theorem 1, T^n is (p/n)-hyponormal. It follows from [4, Theorem 5] that

$$\|(T^{n^*}T^n)^{p/n} - (T^nT^{n^*})^{p/n}\| \leq \frac{p}{n\pi} \iint\limits_{\sigma(T^n)} r^{2(p/n)-1}\, \mathrm{d}r\, \mathrm{d}\phi.$$

Since $\sigma(T^n) = \{\rho^n e^{in\theta}: \rho e^{i\theta} \in \sigma(T)\}$, the result follows by the substitutions $r = \rho^n$ and $\phi = n\theta$.

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