Journal of Applied Mathematics and Stochastic Analysis 5, Number 4, Winter 1992, 375-380

NOTE ON THE INEQUALITIES OF J. KAZDAN[1]

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ABSTRACT

In this note, we prove the Kazdan's inequalities without using what is called the Heisenberg uncertainty principle. Instead we prove it using Garofalo-Lin inequality among other things.

Key words: Heisenberg uncertainty principle, unique continuation theorem, Garofalo-Lin inequality, Schwarz inequality, Poincare inequality.

AMS (MOS) subject classifications: Primary: 35, Secondary: 49.

1. INTRODUCTION

In [4], J. Kazdan has shown strong unique continuation theorem (Theorem 1.8 of [4]) whose proof is mainly based on his main lemma (Lemma 2.4 of [4]). Several analytic as well as geometric inequalities were used to prove the main lemma. Among them are the following inequalities:

There exist constants C_1, C_2, C_3, C_4, C_5 and r_0 such that for all $r \in (0, r_0)$

$$|I_j(r)| \leq C_j f(r)(H(r) + D(r)) \; (j = 1,2) \tag{1}_j$$

$$\frac{1}{r^{n-2}} \int_{\partial B_r} |\nabla u|^2 dS \leq rB(r) + C_3 H(r) + D(r) \tag{2}$$

$$|I_3(r) \leq C_4 f(r)(H(r) + D(r) + \sqrt{rH(r)B(r)}) \tag{3}$$

$$|I_4(r)| \leq C_5 f(r)(H(r) + D(r) + rB(r)). \tag{4}$$

[1]Received: February, 1992. Revised: July, 1992.

Here $f(r), I_1(r), I_2, I_3(r), H(r), D(r), B(r)$ are defined as follows: let f be a smooth increasing function with $f(0)=0$ satisfying $\int_0^{r_0} \frac{f(r)}{r} dr < \infty$ and let u satisfy for $n \geq 3$ the differential inequality with a and b constants:

$$| \Delta u(x) | \leq \frac{af(r)}{r^2} | u(x) | + \frac{bf(r)}{r} | \nabla u(x) | \tag{5}$$

$$I_1(r) = \frac{1}{r^{n-2}} \int_{B_r} u \Delta u dV \tag{6}$$

$$I_2(r) = \frac{2}{r^{n-2}} \int_{B_r} \rho u_\rho \Delta u dV \tag{7}$$

$$I_3(r) = \frac{1}{r^{n-3}} \int_{\partial B_r} u \Delta u dS \tag{8}$$

$$H(r) = \frac{1}{r^{n-1}} \int_{\partial B_r} | u |^2 dS \tag{9}$$

$$D(r) = \frac{1}{r^{n-2}} \int_{B_r} | \nabla u |^2 dV \tag{10}$$

$$B(r) = \frac{2}{r^{n-2}} \int_{\partial B_r} u_\rho^2 dS. \tag{11}$$

In his proof of inequalities $(1)_j$-(2), Kazdan relies on what is called the Heisenberg uncertainty principle (see [2], [3], & [4]):

$$\int_{B_r} \frac{w^2}{\rho^2} dV \leq \frac{c}{r} \int_{\partial B_r} w^2 dS + \tilde{C} \int_{B_r} | \nabla w |^2 dV, \ n \geq 3 \tag{12}$$

$$\int_{B_r} \frac{2 | w | \, | \nabla w |}{\rho} dV \leq \frac{C'}{r} \int_{\partial B_r} w^2 dS + \tilde{C}' \int_{B_r} | \nabla w |^2 dV, \ n \geq 3 \tag{13}$$

where C and $\tilde{C}$ are dimensional constants. Inequality (13) is an easy consequence of (12). Indeed a straightforward computation shows that $C' = \frac{C\lambda}{r}$, $\tilde{C}' = \lambda(\frac{1}{\lambda^2} + \tilde{C})$ for any $\lambda > 0$.

Since there is nothing to comment on the proofs of inequalities (3) and (4), we prove inequalities $(1)_j$ and (2) without using the Heisenberg uncertainty principle (12)-(13). Instead we use the following lemma which is the Garofalo-Lin inequality (see 4.11 of [2]) applied to the operator L where

$$Lu = -\Delta u + b(x).\nabla u + V(x)u = 0. \tag{14}$$

Here $b(x)$ and V are majorized by with constants a and b:

$$|b(x)| \le \frac{bf(r)}{r}, \quad |V(x)| \le \frac{af(r)}{r^2}. \tag{15}$$

Lemma: *Let $u \in w^{1,2}_{loc}$ satisfy equation (14). Then there exists a small constant $r_0 \in (0,1)$ depending on n, b, V and u such that for all $r \in (0, r_0)$*

$$r \int_{\partial B_r} u^2 dS \ge \int_{B_r} u^2 dV. \tag{16}$$

Proof: First observe that

$$\int_{B_r} u(b(x).u)(r^2 - |x|^2)dV$$

$$\le \int_{B_r} |u|\,|b(x)|\,|\nabla u|(r^2 - |x|^2)dV$$

$$\le \|b\|_{L^\infty}\Big(\int_{B_r} u^2(r^2 - |x|^2)dV\Big)^{1/2}\Big(\int_{B_r} |\nabla u|^2(r^2 - |x|^2)dV\Big)^{1/2}$$

(Schwarz inequality)

$$\le \|b\|_{L^\infty} r_0^2\Big(\int_{B_r} u^2 dV\Big)^{1/2}\Big(\int_{B_r} |\nabla u|^2 dV\Big)^{1/2}$$

$$\le C\|b\|_{L^\infty} r_0^2\Big(\int_{B_r} |\nabla u|^2 dV\Big) \qquad \text{(Poincare inequality)}$$

where C is a dimensional constant. Consequently we obtain

$$\int_{B_r} u(b(x).\nabla u)(r^2 - |x|^2 dV \ge -C\|b\|_{L^\infty} r_0^2 \int_{B_r} |\nabla u|^2 dV. \tag{17}$$

Choose r_0 so small that

$$r_0^2 \le 1/(C\|b\|_{L^\infty} \int_{B_r} |\nabla u|^2 dV \int_{B_r} u^2 dV). \tag{18}$$

Inequalities (17)-(18) then reveal that

$$\int_{B_r} u(b(x).\nabla u)(r^2-|x|^2)dV \geq -\int_{B_r} u^2 dV. \tag{19}$$

Secondly we have

$$\int_{B_r} Vu^2(r^2-|x|^2)dV \geq -\|V\|_{L^\infty} r_0^2 \int_{B_r} u^2 dV. \tag{20}$$

Choose r_0 such that

$$r_0^2 \leq (n-2)/\|V\|_{L^\infty}. \tag{21}$$

Inequalities (20)-(21) then show that

$$\int_{B_r} Vu^2(r^2-|x|^2)dV \geq -(n-2)\int_{B_r} u^2 dV. \tag{22}$$

Finally integration by parts and equation (14) give us the following identity:

$$\int_{B_r} (|\nabla u|^2 + ub(x).\nabla u + Vu^2)(r^2-|x|^2)dV = r\int_{\partial B_r} u^2 dS - n\int_{B_r} u^2 dV. \tag{23}$$

Equation (23) combined with inequalities (19) and (22) shows that

$$\begin{aligned} r\int_{\partial B_r} u^2 dS &\geq \int_{B_r} |\nabla u|^2(r^2-|x|^2)dV - \underset{B_r}{\nabla} u^2 dV - (n-2)\int_{B_r} u^2 dV \\ &\quad + n\int_{B_r} u^2 dV \\ &\geq -\int_{B_r} u^2 dV - (n-2)\int_{B_r} u^2 dV + n\int_{B_r} u^2 dV \\ &= \int_{B_r} u^2 dV \end{aligned}$$

for all $r \in (0, r_0)$ where r_0 is chosen to be the minimum of the right hand sides of inequalities (18) and (21). This completes the proof.

We give the proof of $(I)_1$only as the proofs of $(I)_2$ and (2) are essentially the same.

Proof of $(I)_1$: $$|I_1(r)| \leq \frac{1}{r^{n-2}}\int_{B_r} |u|\,|\Delta u|\, dV$$

$$\leq \frac{1}{r^{n-2}}\int_{B_r} |u|\left(\frac{af(r)}{r^2}|u| + \frac{bf(r)}{r}|\nabla u|\right)dV \qquad \text{(by (5))}$$

$$\leq \frac{af(r)}{r^{n-1}} \int_{\partial B_r} u^2 dS + \frac{bf(r)}{r^{n-1}} \left(\int_{B_r} u^2 dV \right)^{1/2} \left(\int_{B_r} |\nabla u|^2 dV \right)^{1/2} \qquad \text{(Lemma and Schwarz)}$$

$$\leq \frac{af(r)}{r^{n-1}} \int_{\partial B_r} u^2 dS + \frac{bf(r)}{r^{n-1}} \left(\frac{1}{2r} \int_{B_r} u^2 dV + \frac{r}{2} \int_{B_r} |\nabla u|^2 dV \right)$$

$$\leq af(r)H(r) + \frac{b}{2} f(r)H(r) + \frac{b}{2} f(r)D(r) \qquad \text{(Lemma, (9) \& (10))}$$

$$\leq C_1 f(r)(H(r) + D(r))$$

where $C_1 = a + b/2$. This complete the proof of $(I)_1$.

A simple computation shows inequalities $(I)_2$, (2), (3) and (4) are satisfied with $C_2 = a + 2b$, $C_3 = (n-2) + (n+2)\gamma + C_2 f(r)$, $C_4 = a + (3b/2)\sqrt{C_3}$, $C_5 = (3/2)C_4$, where γ satisfies $0(r) < (n+2)\gamma$.

REFERENCES

[1] M.S. Baouendi and E.C. Zachmanoglou, "Unique continuation of partial differential equations and inequalities from manifolds of any dimension", *Duke Math. J.* **45**, (1978), pp. 1-13.

[2] N. Garofalo and F.H. Lin, "Monotonicity properties of variational integrals A_p weights and unique continuations", *Indiana Univ. Math. J.* **35**, (1986), pp. 245-268.

[3] ____________________, "Unique continuation for elliptic operators: A geometric-variational approach", *Comm. Pure & Applied Math* **XL**, (1987), pp. 347-366.

[4] J.L. Kazdan, "Unique continuation in geometry", *Comm. Pure & Applied Math.* **XLI**, (1988), pp. 667-681.

[5] D. Gilbarg and N.S. Trudinger, "*Elliptic Partial Differential Equations of Second Order*", Springer-Verlag, 1983.