A FINITE CAPACITY BULK SERVICE QUEUE WITH SINGLE VACATION AND MARKOVIAN ARRIVAL PROCESS

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Received 29 March 2004 and in revised form 11 June 2004

Vacation time queues with Markovian arrival process (MAP) are mainly useful in modeling and performance analysis of telecommunication networks based on asynchronous transfer mode (ATM) environment. This paper analyzes a single-server finite capacity queue wherein service is performed in batches of maximum size "b" with a minimum threshold "a" and arrivals are governed by MAP. The server takes a single vacation when he finds less than "a" customers after service completion. The distributions of buffer contents at various epochs (service completion, vacation termination, departure, arbitrary and pre-arrival) have been obtained. Finally, some performance measures such as loss probability and average queue length are discussed. Numerical results are also presented in some cases.

1. Introduction

Queueing systems with single and multiple vacation(s) have wide applications in many areas including computer communications and manufacturing systems. An excellent survey on this topic can be found in [5]. In this connection, see also the books by Takagi [20, 21, 22] and the references therein. Most of the studies on such queues have been carried out in the past by assuming Poisson input and considering infinite capacity. However, in recent years, there has been a growing interest to analyze queues with Markovian arrival process (MAP) as input process which is a very good representation of the bursty and correlated traffic arising in telecommunication networks based on asynchronous transfer mode (ATM), and is a rich class of point processes containing many familiar arrival processes such as Poisson process, PH-renewal process, Markov-modulated Poisson process (MMPP), and so forth. For more details on these point processes and their importance in stochastic modeling, see Neuts [16, 17]. The $MAP/G/1$ queue with vacation(s) has been analyzed by several authors; see, for example, Lucantoni et al. [12], Kasahara et al. [9], and Lee et al. [11]. Few authors, such as Matendo [14, 15], Ferrandiz [6], Schellhaas [19], and so forth, have studied queueing systems with vacation(s) assuming input as a batch Markovian arrival process (BMAP).

Journal of Applied Mathematics and Stochastic Analysis 2004:4 (2004) 337–357
2000 Mathematics Subject Classification: 60K25, 90B22
URL: http://dx.doi.org/10.1155/S1048953304403025

Finite-buffer queues are more realistic in real-life situations and the need to analyze such queues has been stressed upon from time to time by practitioners of queueing theory. For example, in telecommunication networks, messages/packets are stored in the system if a server is busy. In such situations, one of the main concerns of the system designer is the estimation of blocking probability which, in general, is kept small to avoid loss of packets. Blondia [1] analyzed the $MAP/G/1/N$ queue with multiple vacations and obtained the queue length distributions at departure epochs and arbitrary and pre-arrival epochs. He discussed two types of vacation models: (i) exhaustive service discipline and (ii) limited service discipline. Further, he obtained the Laplace-Stieltjes transform (LST) of virtual and actual waiting time distributions. A more general $MAP/G/1/N$ queue with single (multiple) vacation(s) along with setup and close-down times has been discussed by Niu and Takahashi [18] using supplementary variable technique whereby they obtained queue length distributions at arbitrary epochs and the LST of virtual and actual waiting time distributions.

Most queueing models assume that customers are served singly. But this assumption is far from the truth when we consider those numerous real-world situations in which customers are served in batches. In such queues, customers are served by a single server (multiple servers) in batches of maximum size "b" with a minimum threshold value "a." Such type of service rule is referred to as general bulk service (GBS) rule. The bulk service queues have potential applications in many areas, for example, in loading and unloading of cargoes at a seaport, in traffic signal systems, and in computer networks, jobs are processed in batches with a limit on the number of jobs taken at a time for processing. However, there are many instances where, after the completion of the service of a batch, if the server finds less than "a" customers in the queue, he leaves for a vacation. This time may be utilized by the server to carry out some additional work. On return from a vacation, if he finds "a" or more customers waiting, he takes them for service. Otherwise, he may remain idle (dormant) and continue to do so until the queue length reaches "a." In queueing literature, such types of queues are known as bulk service queues with single vacation. Bulk service queues are, generally speaking, hard to analyze. Often the finite capacities in the bulk service queues increase the complexities of the solution and it becomes more complex if vacation(s) is taken into consideration. The $MAP/G/1$ bulk service finite capacity queue has been discussed by Chakravarthy [2] and Gupta and Vijaya Laxmi [8]. It may be mentioned here that analysis of the $M/G^{(a,b)}/1/N$ queue with single vacation has been recently carried out by Gupta and Sikdar [7].

In this paper, we consider the $MAP/G^{(a,b)}/1/N$ queue with single vacation. The analytic analysis of this queue is carried out and the distributions of the number of customers in the queue at service completion, vacation termination, and departure epochs have been obtained using the imbedded Markov chain technique. The supplementary variable (with remaining service time of a batch in service and remaining vacation time of the server as supplementary variables) method is used to develop the relations between the queue length distributions when the server is busy or on vacation at arbitrary and service completion/vacation termination epochs. These relations can also be obtained using other methods, such as renewal theory; see, for example, [1]. The advantage of using the supplementary variable method over other methods is that one can obtain several other results as a by-product by using simple algebraic manipulation of transform equations. One such result is the mean length of the idle period which has been discussed in the appendix. Moreover, one can derive relations among the queue length distributions at various epochs in a simple, elegant, and straightforward manner without involving any integration. For more advantages of the supplementary variable technique, see, [18, page 2] and [4, page 87].

The rest of the paper is organized as follows. Section 2 gives a brief review of MAP. Afterwards, in Sections 3 and 4, we discuss the model, develop the steady-state matrix differential equations, and obtain the queue distributions at various epochs. Some useful performance measures and computational procedures are presented in Sections 5 and 6, respectively. We end this paper by presenting some numerical results.

2. Markovian arrival process

The MAP is a generalization of the Poisson process where the arrivals are governed by an underlying m-state Markov chain. Let c_{ij}, $i \neq j$, $1 \leq i, j \leq m$, be the state transition rate from state i to state j in the underlying Markov chain without an arrival, and let d_{ij}, $1 \leq i, j \leq m$, be the state transition rate from state i to state j in the underlying Markov chain with an arrival. The matrix $\mathbf{C} = [c_{ij}]$ has nonnegative off-diagonal and negative diagonal elements, and the matrix $\mathbf{D} = [d_{ij}]$ has nonnegative elements. Let $A(t)$ denote the number of customers arriving in $(0,t]$ and let $J(t)$ be the state of the underlying Markov chain at time t with state space $\{i : 1 \leq i \leq m\}$. Then $\{A(t), J(t)\}$ is a two-dimensional Markov process with state space $\{(n,i) : n \geq 0,\ 1 \leq i \leq m\}$. The infinitesimal generator of the above Markov process is given by

$$\mathbf{Q} = \begin{pmatrix} \mathbf{C} & \mathbf{D} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{C} & \mathbf{D} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{C} & \mathbf{D} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2.1}$$

Then $\{A(t), J(t)\}$ is called the MAP. Since $\mathbf{Q}$ is the infinitesimal generator of the MAP, we have

$$(\mathbf{C}+\mathbf{D})\mathbf{e} = \mathbf{0}, \tag{2.2}$$

where $\mathbf{e}$ is an $m \times 1$ vector with all its elements equal to 1. Since $(\mathbf{C}+\mathbf{D})$ is the infinitesimal generator of the underlying Markov chain $\{J(t)\}$, there exists a stationary probability vector $\overline{\boldsymbol{\pi}}$ such that

$$\overline{\boldsymbol{\pi}}(\mathbf{C}+\mathbf{D}) = \mathbf{0}, \quad \overline{\boldsymbol{\pi}}\mathbf{e} = \mathbf{1}. \tag{2.3}$$

The fundamental arrival rate of the above Markov process is given by $\lambda^* = \overline{\boldsymbol{\pi}}\mathbf{D}\mathbf{e}$.

Further, we define $\{\mathbf{P}(n,t),\ n \geq 0,\ t \geq 0\}$ as an $m \times m$ matrix whose (i,j)th element is the conditional probability defined as

$$P_{i,j}(n,t) = \Pr\{A(t) = n,\ J(t) = j \mid A(0) = 0,\ J(0) = i\}. \tag{2.4}$$

These matrices satisfy the following system of difference-differential equations:

$$\frac{d}{dt}\mathbf{P}(0,t) = \mathbf{P}(0,t)\mathbf{C}, \qquad \frac{d}{dt}\mathbf{P}(n,t) = \mathbf{P}(n,t)\mathbf{C}+\mathbf{P}(n-1,t)\mathbf{D}, \quad n \geq 1, \tag{2.5}$$

with $\mathbf{P}(0,0) = \mathbf{I}$. The matrix-generating function $\mathbf{P}^*(z,t)$, defined by

$$\mathbf{P}^*(z,t) = \sum_{n=0}^{\infty} \mathbf{P}(n,t)z^n, \quad |z| \leq 1, \tag{2.6}$$

satisfies

$$\frac{d}{dt}\mathbf{P}^*(z,t) = \mathbf{P}^*(z,t)(\mathbf{C}+z\mathbf{D}), \qquad \mathbf{P}^*(z,0) = \mathbf{I}. \tag{2.7}$$

Solving the above matrix-differential equation, we get

$$\mathbf{P}^*(z,t) = e^{(\mathbf{C}+z\mathbf{D})t}, \quad |z| \leq 1,\ t \geq 0. \tag{2.8}$$

In the following section, we discuss the model and develop associated equations.

3. Modeling, analysis, and basic results

We consider a single-server queue, where the input process is the MAP characterized by the $m \times m$ matrices $\mathbf{C}$ and $\mathbf{D}$. The customers are served by a single server in batches of maximum size "b" with a minimum threshold value "a." Let $S(x)$, $\{s(x)\}$, $[S^*(\theta)]$ be the distribution function (DF), probability density function (pdf), the LST of the service time S of a typical batch. When the server finishes serving a batch and finds less than "a" customers in the queue, the server leaves for a vacation of random length V. On return from a vacation, if he finds "a" or more customers waiting, he takes maximum "b" customers for service. Otherwise, he remains idle (dormant) until the queue length reaches "a." The system has finite buffer (queue) capacity of size N ($\geq b$), that is, the maximum number of customers allowed in the system at any time is $(N+b)$. Let $V(x)$, $\{v(x)\}$, $[V^*(\theta)]$ be the DF, pdf, the LST of a typical vacation time V. The mean service [vacation] time is $\theta_s = -S^{*(1)}(0)$ $[\theta_v = -V^{*(1)}(0)]$, where $f^{*(j)}(d)$ is the jth ($j \geq 1$) derivative of $f^*(\phi)$ at $\phi = d$. The traffic intensity is given by $\rho = \lambda^*\theta_s/b$. Further, let ρ' be the probability that the server is busy. It may be noted that in the case of finite buffer queues, ρ and ρ' are different. For the sake of convenience, we denote the model by $MAP/G^{(a,b)}/1/N/SV$, where SV stands for "single vacation." The state of the system at time t is described by random variables, namely,

(i) $\xi(t) = 2$, 1, or 0 if the server is busy , on vacation, or in dormancy,
(ii) $N_q(t)$ = number of customers present in the queue not counting those in service,
(iii) $J(t)$ = state of the underlying Markov chain of the MAP,

(iv) $S(t)$ = remaining service time of the batch in service,
(v) $V(t)$ = remaining vacation time of the server who is on vacation.

We define, for $1 \le i \le m$,

$$\begin{aligned} \pi_i(n,x;t)\Delta x &= \Pr\{N_q(t)=n,\ J(t)=i,\ x<S(t)\le x+\Delta x,\ \xi(t)=2\}, \quad 0\le n\le N,\ x\ge 0,\\ \omega_i(n,x;t)\Delta x &= \Pr\{N_q(t)=n,\ J(t)=i,\ x<V(t)\le x+\Delta x,\ \xi(t)=1\}, \quad 0\le n\le N,\ x\ge 0,\\ \nu_i(n;t) &= \Pr\{N_q(t)=n,\ J(t)=i,\ \xi(t)=0\}, \quad 0\le n\le a-1. \end{aligned} \tag{3.1}$$

In the limiting case, that is, when $t\to\infty$, the above probabilities will be denoted by $\pi_i(n,x)$, $\omega_i(n,x)$, and $\nu_i(n)$, respectively. We further define the row vectors

$$\begin{aligned} \boldsymbol{\pi}(n,x) &= [\pi_1(n,x),\pi_2(n,x),\dots,\pi_m(n,x)], \quad 0\le n\le N,\\ \boldsymbol{\omega}(n,x) &= [\omega_1(n,x),\omega_2(n,x),\dots,\omega_m(n,x)], \quad 0\le n\le N,\\ \boldsymbol{\nu}(n) &= [\nu_1(n),\nu_2(n),\dots,\nu_m(n)], \quad 0\le n\le a-1. \end{aligned} \tag{3.2}$$

Relating the states of the system at two consecutive time epochs t and $t+\Delta t$ and using probabilistic arguments, we obtain a set of partial differential equations for $1\le i\le m$. Assuming that steady state exists and using matrices and vectors, those equations can be written as

$$\mathbf{0} = \boldsymbol{\nu}(0)\mathbf{C}+\boldsymbol{\omega}(0,0), \tag{3.3}$$

$$\mathbf{0} = \boldsymbol{\nu}(n)\mathbf{C}+\boldsymbol{\nu}(n-1)\mathbf{D}+\boldsymbol{\omega}(n,0), \quad 1\le n\le a-1, \tag{3.4}$$

$$-\frac{d}{dx}\boldsymbol{\pi}(0,x) = \boldsymbol{\pi}(0,x)\mathbf{C}+s(x)\sum_{n=a}^{b}\big(\boldsymbol{\pi}(n,0)+\boldsymbol{\omega}(n,0)\big)+s(x)\boldsymbol{\nu}(a-1)\mathbf{D}, \tag{3.5}$$

$$-\frac{d}{dx}\boldsymbol{\pi}(n,x) = \boldsymbol{\pi}(n,x)\mathbf{C}+\boldsymbol{\pi}(n-1,x)\mathbf{D}+s(x)\big(\boldsymbol{\pi}(n+b,0)+\boldsymbol{\omega}(n+b,0)\big), \quad 1\le n\le N-b, \tag{3.6}$$

$$-\frac{d}{dx}\boldsymbol{\pi}(n,x) = \boldsymbol{\pi}(n,x)\mathbf{C}+\boldsymbol{\pi}(n-1,x)\mathbf{D}, \quad N-b+1\le n\le N-1, \tag{3.7}$$

$$-\frac{d}{dx}\boldsymbol{\pi}(N,x) = \boldsymbol{\pi}(N-1,x)\mathbf{D}+\boldsymbol{\pi}(N,x)(\mathbf{C}+\mathbf{D}), \tag{3.8}$$

$$-\frac{d}{dx}\boldsymbol{\omega}(0,x) = \boldsymbol{\omega}(0,x)\mathbf{C}+v(x)\boldsymbol{\pi}(0,0), \tag{3.9}$$

$$-\frac{d}{dx}\boldsymbol{\omega}(n,x) = \boldsymbol{\omega}(n,x)\mathbf{C}+\boldsymbol{\omega}(n-1,x)\mathbf{D}+v(x)\boldsymbol{\pi}(n,0), \quad 1\le n\le a-1, \tag{3.10}$$

$$-\frac{d}{dx}\boldsymbol{\omega}(n,x) = \boldsymbol{\omega}(n,x)\mathbf{C}+\boldsymbol{\omega}(n-1,x)\mathbf{D}, \quad a\le n\le N-1, \tag{3.11}$$

$$-\frac{d}{dx}\boldsymbol{\omega}(N,x) = \boldsymbol{\omega}(N-1,x)\mathbf{D}+\boldsymbol{\omega}(N,x)(\mathbf{C}+\mathbf{D}). \tag{3.12}$$

We define the Laplace transforms of $\boldsymbol{\pi}(n,x)$ and $\boldsymbol{\omega}(n,x)$ as

$$\boldsymbol{\pi}^*(n,s)=\int_0^\infty e^{-sx}\boldsymbol{\pi}(n,x)dx,\quad \boldsymbol{\omega}^*(n,s)=\int_0^\infty e^{-sx}\boldsymbol{\omega}(n,x)dx,\quad 0\le n\le N,\ \operatorname{Re}s\ge 0, \tag{3.13}$$

so that

$$\boldsymbol{\pi}(n)\equiv\boldsymbol{\pi}^*(n,0)=\int_0^\infty \boldsymbol{\pi}(n,x)dx,\quad \boldsymbol{\omega}(n)\equiv\boldsymbol{\omega}^*(n,0)=\int_0^\infty \boldsymbol{\omega}(n,x)dx,\quad 0\le n\le N, \tag{3.14}$$

where $\boldsymbol{\pi}(n)$ ($\boldsymbol{\omega}(n)$), $0\le n\le N$, is the $1\times m$ vector whose ith component is $\pi_i(n)$ ($\omega_i(n)$) and it denotes the joint probability that there are n customers in the queue and the state of the arrival process is i ($1\le i\le m$) when the server is busy (on vacation) at arbitrary time.

Multiplying equations (3.5)–(3.12) by e^{-sx} and integrating with respect to x over 0 to ∞, we have

$$-s\boldsymbol{\pi}^*(0,s)+\boldsymbol{\pi}(0,0)=\boldsymbol{\pi}^*(0,s)\mathbf{C}+S^*(s)\sum_{n=a}^{b}\big(\boldsymbol{\pi}(n,0)+\boldsymbol{\omega}(n,0)\big)+S^*(s)\boldsymbol{\nu}(a-1)\mathbf{D}, \tag{3.15}$$

$$\begin{aligned}-s\boldsymbol{\pi}^*(n,s)+\boldsymbol{\pi}(n,0)&=\boldsymbol{\pi}^*(n,s)\mathbf{C}+\boldsymbol{\pi}^*(n-1,s)\mathbf{D}\\&\quad+S^*(s)\big(\boldsymbol{\pi}(n+b,0)+\boldsymbol{\omega}(n+b,0)\big),\quad 1\le n\le N-b,\end{aligned} \tag{3.16}$$

$$-s\boldsymbol{\pi}^*(n,s)+\boldsymbol{\pi}(n,0)=\boldsymbol{\pi}^*(n,s)\mathbf{C}+\boldsymbol{\pi}^*(n-1,s)\mathbf{D},\quad N-b+1\le n\le N-1, \tag{3.17}$$

$$-s\boldsymbol{\pi}^*(N,s)+\boldsymbol{\pi}(N,0)=\boldsymbol{\pi}^*(N-1,s)\mathbf{D}+\boldsymbol{\pi}^*(N,s)(\mathbf{C}+\mathbf{D}), \tag{3.18}$$

$$-s\boldsymbol{\omega}^*(0,s)+\boldsymbol{\omega}(0,0)=\boldsymbol{\omega}^*(0,s)\mathbf{C}+V^*(s)\boldsymbol{\pi}(0,0), \tag{3.19}$$

$$\begin{aligned}-s\boldsymbol{\omega}^*(n,s)+\boldsymbol{\omega}(n,0)&=\boldsymbol{\omega}^*(n,s)\mathbf{C}+\boldsymbol{\omega}^*(n-1,s)\mathbf{D}\\&\quad+V^*(s)\boldsymbol{\pi}(n,0),\quad 1\le n\le a-1,\end{aligned} \tag{3.20}$$

$$-s\boldsymbol{\omega}^*(n,s)+\boldsymbol{\omega}(n,0)=\boldsymbol{\omega}^*(n,s)\mathbf{C}+\boldsymbol{\omega}^*(n-1,s)\mathbf{D},\quad a\le n\le N-1, \tag{3.21}$$

$$-s\boldsymbol{\omega}^*(N,s)+\boldsymbol{\omega}(N,0)=\boldsymbol{\omega}^*(N-1,s)\mathbf{D}+\boldsymbol{\omega}^*(N,s)(\mathbf{C}+\mathbf{D}). \tag{3.22}$$

Now, using equations (3.3)-(3.4) and (3.15)–(3.22), we will first derive certain results in the form of lemmas and theorems.

Lemma 3.1.

$$-\boldsymbol{\nu}(n)\mathbf{C}\mathbf{e}=\sum_{j=0}^{n}\boldsymbol{\omega}(j,0)\mathbf{e},\quad 0\le n\le a-1. \tag{3.23}$$

It may be noted here that the left-hand side represents the number of escapes from the dormancy state n per unit time, while the right-hand side represents the number of entrances into the dormancy state n per unit time.

Proof. Setting $n = 1$ in (3.4), postmultiplying it by $\mathbf{e}$, and using $(\mathbf{C}+\mathbf{D})\mathbf{e} = \mathbf{0}$ and $\boldsymbol{\nu}(0)\mathbf{C}\mathbf{e} + \boldsymbol{\omega}(0,0)\mathbf{e} = 0$ (from (3.3)), we get $\boldsymbol{\nu}(1)\mathbf{C}\mathbf{e} + \sum_{j=0}^{1}\boldsymbol{\omega}(j,0)\mathbf{e} = 0$. Recursively, for $n = 2,3,\ldots, a-1$, from (3.4) after simplification, we get the result of Lemma 3.1. □

Lemma 3.2.

$$\sum_{n=0}^{a-1}\boldsymbol{\pi}(n,0)\mathbf{e} = \sum_{n=0}^{N}\boldsymbol{\omega}(n,0)\mathbf{e}. \tag{3.24}$$

The left-hand side represents the entering rate to vacation state, while the right-hand side represents the departure rate from the vacation state.

Proof. Setting $s = 0$ in (3.15), (3.16), (3.17), and (3.18) and using (3.14), we get

$$\boldsymbol{\pi}(0,0) = \boldsymbol{\pi}(0)\mathbf{C} + \sum_{n=a}^{b}\left(\boldsymbol{\pi}(n,0)+\boldsymbol{\omega}(n,0)\right) + \boldsymbol{\nu}(a-1)\mathbf{D}, \tag{3.25}$$

$$\boldsymbol{\pi}(n,0) = \boldsymbol{\pi}(n)\mathbf{C} + \boldsymbol{\pi}(n-1)\mathbf{D} + \boldsymbol{\pi}(n+b,0) + \boldsymbol{\omega}(n+b,0), \quad 1 \le n \le N-b, \tag{3.26}$$

$$\boldsymbol{\pi}(n,0) = \boldsymbol{\pi}(n)\mathbf{C} + \boldsymbol{\pi}(n-1)\mathbf{D}, \quad N-b+1 \le n \le N-1, \tag{3.27}$$

$$\boldsymbol{\pi}(N,0) = \boldsymbol{\pi}(N-1)\mathbf{D} + \boldsymbol{\pi}(N)(\mathbf{C}+\mathbf{D}). \tag{3.28}$$

Postmultiplying (3.25), (3.26), (3.27), and (3.28) by $\mathbf{e}$, adding over all possible values of n, and using Lemma 3.1 and $(\mathbf{C}+\mathbf{D})\mathbf{e} = \mathbf{0}$, after simplification, we obtain the result of Lemma 3.2. □

Theorem 3.3.

$$\theta_s\sum_{n=0}^{N}\boldsymbol{\pi}(n,0)\mathbf{e} + \theta_v\sum_{n=0}^{N}\boldsymbol{\omega}(n,0)\mathbf{e} + \sum_{n=0}^{a-1}\boldsymbol{\nu}(n)\mathbf{e} = 1. \tag{3.29}$$

It may be noted here that the first term of the left-hand side represents the probability that the server is busy. The sum of the second and third terms is the probability that the server is idle. That is, $\theta_s\sum_{n=0}^{N}\boldsymbol{\pi}(n,0)\mathbf{e} = \sum_{n=0}^{N}\boldsymbol{\pi}(n)\mathbf{e} = \rho'$ *and* $\theta_v\sum_{n=0}^{N}\boldsymbol{\omega}(n,0)\mathbf{e} + \sum_{n=0}^{a-1}\boldsymbol{\nu}(n)\mathbf{e} = \sum_{n=0}^{N}\boldsymbol{\omega}(n)\mathbf{e} + \sum_{n=0}^{a-1}\boldsymbol{\nu}(n)\mathbf{e} = 1-\rho'$ *(for proof, see the appendix). The expression of* ρ' *is given in Lemma 3.4.*

Proof. Postmultiplying (3.15)–(3.22) by $\mathbf{e}$, adding over all possible values of n, and using Lemma 3.1 and $(\mathbf{C}+\mathbf{D})\mathbf{e} = \mathbf{0}$, we obtain

$$\begin{aligned}\sum_{n=0}^{N}&\left(\boldsymbol{\pi}^*(n,s)+\boldsymbol{\omega}^*(n,s)\right)\mathbf{e}\\ &= \frac{1-S^*(s)}{s}\sum_{n=0}^{N}\left(\boldsymbol{\pi}(n,0)+\boldsymbol{\omega}(n,0)\right)\mathbf{e} + \frac{S^*(s)-V^*(s)}{s}\sum_{n=0}^{a-1}\boldsymbol{\pi}(n,0)\mathbf{e}.\end{aligned} \tag{3.30}$$

Taking the limit as $s \to 0$ in (3.30) and using the normalization condition $\sum_{n=0}^{N}(\boldsymbol{\pi}(n) + \boldsymbol{\omega}(n))\mathbf{e} + \sum_{n=0}^{a-1}\boldsymbol{\nu}(n)\mathbf{e} = 1$ and Lemma 3.2, after simplification, we get the desired result. □

Lemma 3.4. *The probability that the server is busy is given by*

$$\rho' = \frac{\theta_s}{\theta_s + \theta_v K_1 + K_2}, \tag{3.31}$$

where $K_1 = \sum_{n=0}^{a-1}\mathbf{p}^+(n)\mathbf{e}$, $K_2 = \sum_{n=0}^{a-1}\sum_{j=0}^{n}\sum_{k=0}^{j}\mathbf{p}^+(j-k)\mathbf{V}(k)\overline{\mathbf{D}}^{n-j}(-\mathbf{C})^{-1}\mathbf{e}$, *and* $\mathbf{p}^+(n)$, $0 \le n \le N$, *is the* $1 \times m$ *vector whose ith component is* $p_i^+(n)$ *and it denotes the joint probability that there are n customers in the queue and the state of the arrival process is i* $(1 \le i \le m)$ *immediately after departure of a batch. Note that the* (i,j)*th element of* $\overline{\mathbf{D}} = (-\mathbf{C})^{-1}\mathbf{D}$ *is the conditional probability that, given that it was in state i at the arrival time of the last customer, the MAP is in state j at the time of a customer's arrival. Therefore, the factor* $\overline{\mathbf{D}}^n$ *represents the state transition matrix during the interarrival time of n customers; see Niu and Takahashi* [18, page 8].

Proof. Let Θ_b (resp., Θ_i/Θ_d) be the random variable denoting the length of a busy (resp., idle/dormant) period and let θ_b (resp., θ_i/θ_d) be the mean length of a busy (resp., idle/dormant) period; then we have

$$\rho' = \frac{\theta_b}{\theta_b + \theta_i}. \tag{3.32}$$

Following the argument given in [3, page 334], one can easily show that $\theta_b = \theta_s/K_1$. Further, $\theta_i = \theta_v + K_2/K_1$ (for proof, see the appendix); substituting these in (3.32), after simplification, we get the result of Lemma 3.4. □

One may note here that in single vacation, the idle period may consist of vacation (V) and dormant (Θ_d) periods, that is, $\Theta_i = V + \Theta_d$ and $\theta_i = \theta_v + \theta_d$.

4. Queue length distributions at various epochs

4.1. Queue length distributions at service completion, vacation termination, and departure epochs. Let $\boldsymbol{\pi}^+(n)$ (resp., $\boldsymbol{\omega}^+(n)$) $(0 \le n \le N)$ denote the row vector whose ith element represents the probability that there are n customers in the queue and the state of the arrival process is i $(1 \le i \le m)$ at the service completion (resp., vacation termination) epoch. As $\sum_{n=0}^{N}(\boldsymbol{\pi}^+(n) + \boldsymbol{\omega}^+(n))\mathbf{e} = 1$, it can be easily seen that $\boldsymbol{\pi}^+(n)$ $(\boldsymbol{\omega}^+(n))$ and $\boldsymbol{\pi}(n,0)$ $(\boldsymbol{\omega}(n,0))$ are connected by the relations

$$\boldsymbol{\pi}^+(n) = \frac{1}{\sigma}\boldsymbol{\pi}(n,0), \quad \boldsymbol{\omega}^+(n) = \frac{1}{\sigma}\boldsymbol{\omega}(n,0), \quad 0 \le n \le N, \tag{4.1}$$

where $\sigma = \sum_{n=0}^{N}(\boldsymbol{\pi}(n,0) + \boldsymbol{\omega}(n,0))\mathbf{e}$. The expression of σ in terms of ρ' is given in Lemma 4.2. In the following lemma, we will first obtain the value of $\sum_{n=0}^{N}\boldsymbol{\omega}(n,0)\mathbf{e}$ as it is needed to get σ.

LEMMA 4.1.

$$\sum_{n=0}^{N} \boldsymbol{\omega}(n,0)\mathbf{e} = \frac{(1-\rho')K_1}{\theta_v K_1 + K_2}. \tag{4.2}$$

Proof.

$$\begin{aligned} P\,(\text{server is in dormancy}) &= P\,(\text{server is idle})P\,(\text{server is in dormancy/server is idle}) \\ &= (1-\rho')\left[\frac{\theta_d}{\theta_i}\right] \\ &= (1-\rho')\frac{K_2}{\theta_v K_1 + K_2}. \end{aligned} \tag{4.3}$$

Again, we have

$$P\,(\text{server is in dormancy}) = \sum_{n=0}^{a-1} \boldsymbol{\nu}(n)\mathbf{e}. \tag{4.4}$$

Comparing (4.3) and (4.4) and using $\theta_v \sum_{n=0}^{N} \boldsymbol{\omega}(n,0)\mathbf{e} + \sum_{n=0}^{a-1} \boldsymbol{\nu}(n)\mathbf{e} = 1-\rho'$ (Theorem 3.3), after simplification, we get the result of Lemma 4.1. □

LEMMA 4.2.

$$\sigma = \frac{\rho'(\theta_v K_1 + K_2) + \theta_s(1-\rho')K_1}{\theta_s(\theta_v K_1 + K_2)}. \tag{4.5}$$

Proof. As $\sum_{n=0}^{N} \boldsymbol{\pi}(n,0)\mathbf{e} = \rho'/\theta_s$ (from Theorem 3.3) and $\sum_{n=0}^{N} \boldsymbol{\omega}(n,0)\mathbf{e}$ is known from Lemma 4.1, using them in $\sigma = \sum_{n=0}^{N}(\boldsymbol{\pi}(n,0)+\boldsymbol{\omega}(n,0))\mathbf{e}$, after simplification, we obtain the result. □

It can be seen from (4.1) that to get $\boldsymbol{\pi}^+(n)$ and $\boldsymbol{\omega}^+(n)$, we need to find out $\boldsymbol{\pi}(n,0)$ and $\boldsymbol{\omega}(n,0)$. As $\boldsymbol{\pi}(n,0)$ and $\boldsymbol{\omega}(n,0)$ are cumbersome to evaluate directly from (3.15)–(3.22), we obtain them using imbedded Markov chain technique. However, we will make use of (4.1) to derive relations between the distributions of the number of customers in the queue at service completion (vacation termination) and arbitrary epochs; see Section 4.2.

To obtain $\boldsymbol{\pi}^+(n)$ and $\boldsymbol{\omega}^+(n)$ using imbedded Markov chain technique, we first set up some necessary notation and establish some preliminary results.

LEMMA 4.3.

$$\int_0^{\infty} \mathbf{P}(n,t)\mathbf{D}\,dt = \overline{\mathbf{D}}^{n+1}, \quad n \ge 0. \tag{4.6}$$

Proof. The proof follows from (2.8). □

Next, consider $\mathbf{A}(n,x)$, $\mathbf{V}(n,x)$, and $\mathbf{B}(n,k,x)$, $x \geq 0$, as the $m \times m$ matrices of mass functions defined by

$$
\begin{aligned}
\mathbf{A}(n,x) &= \int_0^x \mathbf{P}(n,t)\,dS(t), \quad 0 \leq n \leq N-1, \\
\mathbf{A}'(n,x) &= \sum_{k=n}^{\infty} \mathbf{A}(k,x), \quad b \leq n \leq N, \\
\mathbf{V}(n,x) &= \int_0^x \mathbf{P}(n,t)\,dV(t), \quad 0 \leq n \leq N-1, \\
\mathbf{V}'(n,x) &= \sum_{k=n}^{\infty} \mathbf{V}(k,x), \quad N-a+1 \leq n \leq N, \\
\mathbf{B}(n,k,x) &= \int_0^x \mathbf{P}(a-1-n,x-u)\mathbf{D}\mathbf{A}(k,u)\,du, \quad 0 \leq n \leq a-1,\ 0 \leq k \leq N-1, \\
\mathbf{B}'(n,N,x) &= \sum_{k=N}^{\infty} \mathbf{B}(n,k,x), \quad 0 \leq n \leq a-1,
\end{aligned}
\tag{4.7}
$$

whose (i,j)th elements are given by

(1) $a_{ij}(n,x) = P\{$given a departure at time 0, which left at least "a" customers in the queue and the arrival process in phase i, the next departure occurs no later than time x with the arrival process in phase j, and during that service, there were n $(0 \leq n \leq N-1)$ arrivals$\}$,

(2) $v_{ij}(n,x) = P\{$given a departure at time 0, which left n $(0 \leq n \leq a-1)$ customers in the queue, vacation begins and also the arrival process in phase i; the end of the vacation occurs no later than time x with the arrival process in phase j, and during that vacation, there were n $(0 \leq n \leq N-1)$ arrivals$\}$,

(3) $b_{ij}(n,k,x) = P\{$given that the dormant period begins at time 0, there were n $(0 \leq n \leq a-1)$ customers in the queue and the arrival process in phase i; during the dormant period, say $x-u$ $(0 \leq u \leq x$, where x is the total time of dormant and service periods), there are $a-1-n$ $(0 \leq n \leq a-1)$ arrivals and service occurs no later than time u $(0 \leq u \leq x)$ with the arrival process in phase j, and during that service period, there were k $(0 \leq k \leq N-1)$ arrivals$\}$.

Further, we define

$$
\begin{aligned}
\mathbf{A}(n) &= \mathbf{A}(n,\infty), \quad 0 \leq n \leq N-1, \\
\mathbf{A}'(n) &= \mathbf{A}'(n,\infty) = \sum_{k=n}^{\infty} \mathbf{A}(k), \quad b \leq n \leq N,
\end{aligned}
\tag{4.8}
$$

$$
\begin{aligned}
\mathbf{V}(n) &= \mathbf{V}(n,\infty), \quad 0 \leq n \leq N-1, \\
\mathbf{V}'(n) &= \mathbf{V}'(n,\infty) = \sum_{k=n}^{\infty} \mathbf{V}(k), \quad N-a+1 \leq n \leq N,
\end{aligned}
\tag{4.9}
$$

$$
\mathbf{B}(n,k) = \mathbf{B}(n,k,\infty) = \overline{\mathbf{D}}^{a-n}\mathbf{A}(k), \quad 0 \leq n \leq a-1,\ 0 \leq k \leq N-1, \tag{4.10}
$$

$$
\mathbf{B}'(n,N) = \mathbf{B}'(n,N,\infty) = \overline{\mathbf{D}}^{a-n}\mathbf{A}'(N), \quad 0 \leq n \leq a-1. \tag{4.11}
$$

The results in (4.10) and (4.11) can be easily obtained using the expression of $\mathbf{B}(n,k,x)$, $\mathbf{B}'(n,N,x)$ and Lemma 4.3.

Now, observing two consecutive service completion and vacation termination epochs, using probabilistic arguments, and further using matrices and vectors notations, we get the following equations:

$$\boldsymbol{\pi}^+(0) = \sum_{k=0}^{a-1} \boldsymbol{\omega}^+(k)\mathbf{B}(k,0) + \sum_{k=a}^{b} (\boldsymbol{\pi}^+(k) + \boldsymbol{\omega}^+(k))\mathbf{A}(0), \tag{4.12}$$

$$\begin{aligned} \boldsymbol{\pi}^+(n) &= \sum_{k=0}^{a-1} \boldsymbol{\omega}^+(k)\mathbf{B}(k,n) + \sum_{k=a}^{b} (\boldsymbol{\pi}^+(k) + \boldsymbol{\omega}^+(k))\mathbf{A}(n) \\ &\quad + \sum_{k=1}^{n} (\boldsymbol{\pi}^+(b+k) + \boldsymbol{\omega}^+(b+k))\mathbf{A}(n-k), \quad 1 \le n \le N-b, \end{aligned} \tag{4.13}$$

$$\begin{aligned} \boldsymbol{\pi}^+(n) &= \sum_{k=0}^{a-1} \boldsymbol{\omega}^+(k)\mathbf{B}(k,n) + \sum_{k=a}^{b} (\boldsymbol{\pi}^+(k) + \boldsymbol{\omega}^+(k))\mathbf{A}(n) \\ &\quad + \sum_{k=0}^{N-(b+1)} (\boldsymbol{\pi}^+(N-k) + \boldsymbol{\omega}^+(N-k))\mathbf{A}(n-(N-b)+k), \quad N-b+1 \le n \le N-1, \end{aligned} \tag{4.14}$$

$$\begin{aligned} \boldsymbol{\pi}^+(N) &= \sum_{k=0}^{a-1} \boldsymbol{\omega}^+(k)\mathbf{B}'(k,N) + \sum_{k=a}^{b} (\boldsymbol{\pi}^+(k) + \boldsymbol{\omega}^+(k))\mathbf{A}'(N) \\ &\quad + \sum_{k=0}^{N-(b+1)} (\boldsymbol{\pi}^+(N-k) + \boldsymbol{\omega}^+(N-k))\mathbf{A}'(b+k), \end{aligned} \tag{4.15}$$

and

$$\boldsymbol{\omega}^+(n) = \sum_{k=0}^{n} \boldsymbol{\pi}^+(n-k)\mathbf{V}(k), \quad 0 \le n \le a-1, \tag{4.16}$$

$$\boldsymbol{\omega}^+(n) = \sum_{k=1}^{a} \boldsymbol{\pi}^+(a-k)\mathbf{V}(n-a+k), \quad a \le n \le N-1, \tag{4.17}$$

$$\boldsymbol{\omega}^+(N) = \sum_{k=0}^{a-1} \boldsymbol{\pi}^+(k)\mathbf{V}'(N-k), \tag{4.18}$$

where $\sum_{n=0}^{N}(\boldsymbol{\pi}^+(n) + \boldsymbol{\omega}^+(n))\mathbf{e} = 1$ is the normalization condition. Our aim is to find $\boldsymbol{\pi}^+(n)$ and $\boldsymbol{\omega}^+(n)$ $(0 \le n \le N)$, so that we can get $\boldsymbol{\pi}(n)$ and $\boldsymbol{\omega}(n)$ $(0 \le n \le N)$, using the relations to be developed in Section 4.2.

As evaluation of $\boldsymbol{\pi}^+(n)$ $(0 \le n \le N)$ is dependent on $\boldsymbol{\omega}^+(n)$ $(0 \le n \le N)$, therefore we first need to evaluate $\boldsymbol{\omega}^+(n)$ $(0 \le n \le N)$ using (4.16), (4.17), and (4.18). It is further seen that to get $\boldsymbol{\omega}^+(n)$ $(0 \le n \le N)$ from (4.16), (4.17), and (4.18), we need to find

$\boldsymbol{\pi}^+(n)$ $(0 \le n \le a-1)$. Finally, it may be noted here that getting $\boldsymbol{\pi}^+(n)$ $(0 \le n \le a-1)$ from (4.12)-(4.13) is difficult, if not impossible, as it involves $\sum_{n=a}^{b} \boldsymbol{\pi}^+(n)$ and other terms. This problem is resolved in the following lemma.

Lemma 4.4. *Let* $\mathbf{p}^+(n)$ $(0 \le n \le N)$ *be the* $1 \times m$ *vector whose ith component is* $p_i^+(n)$ *and which denotes the joint probability that there are n customers in the queue and the state of the arrival process is in phase i* $(1 \le i \le m)$ *immediately after departure of a batch. The relation between* $\mathbf{p}^+(n)$ *and* $\boldsymbol{\pi}^+(n)$ $(0 \le n \le N)$ *is given by*

$$\mathbf{p}^+(n) = \frac{\sigma\theta_s}{\rho'}\boldsymbol{\pi}^+(n), \quad 0 \le n \le N. \tag{4.19}$$

Proof. As $\mathbf{p}^+(n)$ and $\boldsymbol{\pi}^+(n)$ differ by a constant term and $\sum_{n=0}^{N} \mathbf{p}^+(n)\mathbf{e} = 1$, we get

$$\mathbf{p}^+(n) = \frac{\boldsymbol{\pi}^+(n)}{\sum_{n=0}^{N} \boldsymbol{\pi}^+(n)\mathbf{e}}, \quad 0 \le n \le N. \tag{4.20}$$

As $\sum_{n=0}^{N} \boldsymbol{\pi}(n,0)\mathbf{e} = \rho'/\theta_s$ (Theorem 3.3), dividing both sides by σ, we get

$$\sum_{n=0}^{N} \boldsymbol{\pi}^+(n)\mathbf{e} = \frac{\rho'}{\sigma\theta_s}. \tag{4.21}$$

Using (4.21) in (4.20), we obtain the result of Lemma 4.4. □

Now we will make use of (4.20) and $\boldsymbol{\pi}^+(n) = \mathbf{p}^+(n)\sum_{n=0}^{N} \boldsymbol{\pi}^+(n)\mathbf{e}$ to get $\boldsymbol{\pi}^+(n)$ $(0 \le n \le N)$. From (4.21), it is seen that $\sum_{n=0}^{N} \boldsymbol{\pi}^+(n)\mathbf{e}$ can be obtained if ρ' is known. Further, to get ρ' from (3.31), we need to know $\mathbf{p}^+(n)$ $(0 \le n \le a-1)$. It is further seen that even after getting $\sum_{n=0}^{N} \boldsymbol{\pi}^+(n)\mathbf{e}$, $\boldsymbol{\pi}^+(n)$ $(0 \le n \le N)$ can be obtained if $\mathbf{p}^+(n)$ $(0 \le n \le N)$ is known. The unknown quantities $\mathbf{p}^+(n)$ $(0 \le n \le N)$ can be obtained using the imbedded Markov chain approach which is discussed below.

Now, let $\mathbf{R}(n,k,x)$ be the matrix whose element $r_{i,j}(n,k,x)$ is the conditional probability that, given a departure at time 0, leaves the system with n customers $(0 \le n \le a-1)$ and the arrival process in phase i; the next departure occurs no later than time x with the arrival process in phase j and there were k $(0 \le k \le N-1)$ arrivals during the service time of that departure. We denote

$$\begin{aligned} \mathbf{R}(n,k) &= \mathbf{R}(n,k,\infty), \quad 0 \le n \le a-1,\ 0 \le k \le N-1, \\ \mathbf{R}'(n,N) &= \sum_{r=N}^{\infty} \mathbf{R}(n,r), \quad 0 \le n \le a-1. \end{aligned} \tag{4.22}$$

Then, considering all the possible cases, it can be easily seen that

$$\mathbf{R}(n,k) = \begin{cases} \sum_{i=0}^{a-1-n} \mathbf{V}(i)\mathbf{B}(n+i,k) + \sum_{i=a-n}^{b-n} \mathbf{V}(i)\mathbf{A}(k), & 0 \le n \le a-1,\ k=0, \\ \sum_{i=0}^{a-1-n} \mathbf{V}(i)\mathbf{B}(n+i,k) + \sum_{i=a-n}^{b-n} \mathbf{V}(i)\mathbf{A}(k) & \\ + \sum_{l=b-n+1}^{N-1} \mathbf{V}(l)\mathbf{A}(k-l+b-n) & \\ +\mathbf{V}'(N)\mathbf{A}(k-N+b-n), & 0 \le n \le a-1,\ 1 \le k \le N-1, \\ & k-l+b-n \ge 0,\ k-N+b-n \ge 0, \end{cases}$$

$$\begin{aligned} \mathbf{R}'(n,N) = {} & \sum_{i=0}^{a-1-n} \mathbf{V}(i)\mathbf{B}'(n+i,N) + \sum_{i=a-n}^{b-n} \mathbf{V}(i)\mathbf{A}'(N) + \sum_{l=b-n+1}^{N-1} \mathbf{V}(l)\mathbf{A}'(N-l+b-n) \\ & + \mathbf{V}'(N)\mathbf{A}'(b-n), \quad 0 \le n \le a-1,\ N-l+b-n \ge 0,\ b-n \ge 0. \end{aligned} \tag{4.23}$$

Let N_n^+ and J_n^+ denote, respectively, the number of customers in the queue and the phase of the arrival process immediately after the nth departure. Then $\{(N_n^+, J_n^+) : n \ge 0\}$ is a bivariate imbedded Markov chain with the state space $\{0,1,2,\ldots,N\} \times \{1,2,\ldots,m\}$. Now, observing the system immediately after departures, the transition probability matrix (TPM) is given by

$$\mathcal{P} = \begin{bmatrix} \mathbf{R}(0,0) & \mathbf{R}(0,1) & \cdots & \mathbf{R}(0,N-b) & \cdots & \mathbf{R}(0,N-1) & \mathbf{R}'(0,N) \\ \mathbf{R}(1,0) & \mathbf{R}(1,1) & \cdots & \mathbf{R}(1,N-b) & \cdots & \mathbf{R}(1,N-1) & \mathbf{R}'(1,N) \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ \mathbf{R}(a-1,0) & \mathbf{R}(a-1,1) & \cdots & \mathbf{R}(a-1,N-b) & \cdots & \mathbf{R}(a-1,N-1) & \mathbf{R}'(a-1,N) \\ \mathbf{A}(0) & \mathbf{A}(1) & \cdots & \mathbf{A}(N-b) & \cdots & \mathbf{A}(N-1) & \mathbf{A}'(N) \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ \mathbf{A}(0) & \mathbf{A}(1) & \cdots & \mathbf{A}(N-b) & \cdots & \mathbf{A}(N-1) & \mathbf{A}'(N) \\ \mathbf{0} & \mathbf{A}(0) & \cdots & \mathbf{A}(N-b-1) & \cdots & \mathbf{A}(N-2) & \mathbf{A}'(N-1) \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}(0) & \cdots & \mathbf{A}(b-1) & \mathbf{A}'(b) \end{bmatrix}. \tag{4.24}$$

The unknown quantities $\mathbf{p}^+(n)$ $(0 \le n \le N)$ can be obtained by solving the system of equations $\mathbf{p}^+ = \mathbf{p}^+\mathcal{P}$ with $\mathbf{p}^+\mathbf{e} = 1$, where $\mathbf{p}^+ = [\mathbf{p}^+(0), \mathbf{p}^+(1), \ldots, \mathbf{p}^+(N)]$ of order $(N+1)m$. Here, one may note that we have solved this system of equations using the Grassmann-Taksar-Heyman (GTH) algorithm given in [10]. Once $\mathbf{p}^+(n)$ $(0 \le n \le N)$ are known, $\boldsymbol{\pi}^+(n)$ and $\boldsymbol{\omega}^+(n)$ $(0 \le n \le N)$ can be obtained using (4.19) and (4.16)–(4.18), respectively.

4.2. Queue length distributions at arbitrary epoch. To obtain queue length distributions at arbitrary epoch, we will develop relations between distributions of the number of customers in the queue at service completion (vacation termination) and arbitrary epochs.

LEMMA 4.5. *The relation between* $\boldsymbol{\nu}(n)$ $(0 \le n \le a-1)$ *and* $\boldsymbol{\omega}^+(n)$ $(0 \le n \le a-1)$ *is given by*

$$\boldsymbol{\nu}(n) = \sigma \sum_{j=0}^{n} \boldsymbol{\omega}^+(j)\overline{\mathbf{D}}^{n-j}(-\mathbf{C})^{-1}, \quad 0 \le n \le a-1. \tag{4.25}$$

Proof. Multiplying (3.3) by $1/\sigma$ and using (4.1), after simplification, we get $\boldsymbol{\nu}(0) = \sigma\boldsymbol{\omega}^+(0)(-\mathbf{C})^{-1}$.

Setting $n = 1$ in (3.4), multiplying it by $1/\sigma$, and using (4.1) and $\boldsymbol{\nu}(0) = \sigma\boldsymbol{\omega}^+(0)(-\mathbf{C})^{-1}$, we get $\boldsymbol{\nu}(1) = \sigma\sum_{j=0}^{1} \boldsymbol{\omega}^+(j)\overline{\mathbf{D}}^{1-j}(-\mathbf{C})^{-1}$.

Proceeding in this way in general, we get the result of Lemma 4.5. □

LEMMA 4.6. *The relation between* $\boldsymbol{\omega}(n)$ $(0 \le n \le N-1)$, $\boldsymbol{\omega}^+(n)$ $(0 \le n \le N-1)$, *and* $\boldsymbol{\pi}^+(n)$ $(0 \le n \le a-1)$ *is given by*

$$\boldsymbol{\omega}(0) = \sigma\left[\boldsymbol{\pi}^+(0) - \boldsymbol{\omega}^+(0)\right](-\mathbf{C})^{-1}, \tag{4.26}$$

$$\boldsymbol{\omega}(n) = \left[\boldsymbol{\omega}(n-1)\mathbf{D} + \sigma\left(\boldsymbol{\pi}^+(n) - \boldsymbol{\omega}^+(n)\right)\right](-\mathbf{C})^{-1}, \quad 1 \le n \le a-1, \tag{4.27}$$

$$\boldsymbol{\omega}(n) = \left[\boldsymbol{\omega}(n-1)\mathbf{D} - \sigma\boldsymbol{\omega}^+(n)\right](-\mathbf{C})^{-1}, \quad a \le n \le N-1. \tag{4.28}$$

Proof. Setting $s = 0$ in (3.19), (3.20), and (3.21), we get

$$\boldsymbol{\omega}(0,0) = \boldsymbol{\omega}(0)\mathbf{C} + \boldsymbol{\pi}(0,0), \tag{4.29}$$

$$\boldsymbol{\omega}(n,0) = \boldsymbol{\omega}(n)\mathbf{C} + \boldsymbol{\omega}(n-1)\mathbf{D} + \boldsymbol{\pi}(n,0), \quad 1 \le n \le a-1, \tag{4.30}$$

$$\boldsymbol{\omega}(n,0) = \boldsymbol{\omega}(n)\mathbf{C} + \boldsymbol{\omega}(n-1)\mathbf{D}, \quad a \le n \le N-1. \tag{4.31}$$

Multiplying (4.29), (4.30), and (4.31) by $1/\sigma$ and using (4.1), after simplification, we get (4.26), (4.27), and (4.28), respectively. □

LEMMA 4.7. *The relation between* $\boldsymbol{\pi}(n)$ $(0 \le n \le N-1)$, $\boldsymbol{\pi}^+(n)$ $(0 \le n \le N)$, *and* $\boldsymbol{\omega}^+(n)$ $(a \le n \le N)$ *is given by*

$$\boldsymbol{\pi}(0) = \left[\boldsymbol{\nu}(a-1)\mathbf{D} + \sigma\left(\sum_{n=a}^{b}\left(\boldsymbol{\pi}^+(n) + \boldsymbol{\omega}^+(n) - \boldsymbol{\pi}^+(0)\right)\right)\right](-\mathbf{C})^{-1}, \tag{4.32}$$

$$\boldsymbol{\pi}(n) = \left[\boldsymbol{\pi}(n-1)\mathbf{D} + \sigma\left(\boldsymbol{\pi}^+(n+b) + \boldsymbol{\omega}^+(n+b) - \boldsymbol{\pi}^+(n)\right)\right](-\mathbf{C})^{-1}, \quad 1 \le n \le N-b, \tag{4.33}$$

$$\boldsymbol{\pi}(n) = \left[\boldsymbol{\pi}(n-1)\mathbf{D} - \sigma\boldsymbol{\pi}^+(n)\right](-\mathbf{C})^{-1}, \quad N-b+1 \le n \le N-1. \tag{4.34}$$

Proof. Multiplying (3.25), (3.26), and (3.27) by $1/\sigma$ and using (4.1), after simplification, we get (4.32), (4.33), and (4.34), respectively. □

LEMMA 4.8.

$$\boldsymbol{\omega}(N)\mathbf{e} = (1-\rho') - \left[\sum_{n=0}^{N-2}\boldsymbol{\omega}(n)\mathbf{D} + \sigma\left(\sum_{n=0}^{a-1}\boldsymbol{\pi}^{+}(n) - \sum_{n=0}^{N-1}\boldsymbol{\omega}^{+}(n) + \sum_{n=0}^{a-1}\sum_{j=0}^{n}\boldsymbol{\omega}^{+}(j)\overline{D}^{n-j}\right)\right](-\mathbf{C})^{-1}\mathbf{e}, \tag{4.35}$$

$$\boldsymbol{\pi}(N)\mathbf{e} = \rho' - \sum_{n=0}^{N-1}\boldsymbol{\pi}(n)\mathbf{e}. \tag{4.36}$$

Proof. Postmultiplying (4.25) and (4.26), (4.27), (4.28) by $\mathbf{e}$ and using these in $\sum_{n=0}^{N}\boldsymbol{\omega}(n)\mathbf{e} + \sum_{n=0}^{a-1}\boldsymbol{\nu}(n)\mathbf{e} = 1-\rho'$, we get (4.35). Further, using $\sum_{n=0}^{N}\boldsymbol{\pi}(n)\mathbf{e} = \rho'$, we get (4.36). □

LEMMA 4.9. *Let* $\mathbf{p}(n)$ $(0 \le n \le N)$ *be the* $1\times m$ *vector whose* j*th component* $p_j(n)$ *denotes the probability that there are* n *customers in the queue at an arbitrary epoch and, at that time, the arrival process is in state* j*. Then*

$$\mathbf{p}(n) = \boldsymbol{\nu}(n) + \boldsymbol{\omega}(n) + \boldsymbol{\pi}(n), \quad 0 \le n \le a-1, \tag{4.37}$$

$$\mathbf{p}(n) = \boldsymbol{\omega}(n) + \boldsymbol{\pi}(n), \quad a \le n \le N-1, \tag{4.38}$$

$$\mathbf{p}(N) = \overline{\boldsymbol{\pi}} - \sum_{n=0}^{a-1}\boldsymbol{\nu}(n) - \sum_{n=0}^{N-1}\left(\boldsymbol{\omega}(n) + \boldsymbol{\pi}(n)\right). \tag{4.39}$$

It may be noted here that, due to the singularity of $(\mathbf{C}+\mathbf{D})$, we cannot obtain $\boldsymbol{\pi}(N)$ and $\boldsymbol{\omega}(N)$ (from (3.18) and (3.22), after setting $s=0$) whereas $\mathbf{p}(N)$ can be obtained using the normalization condition. But if we find the row sums $\boldsymbol{\pi}(N)\mathbf{e}$ and $\boldsymbol{\omega}(N)\mathbf{e}$, then these are enough for calculations of key performance measures; see Section 5. This has also been pointed out by Niu and Takahashi [18, page 21].

4.3. Queue length distributions at pre-arrival epoch. Let $\mathbf{p}^{-}(n)$ $(0 \le n \le N)$ be the $1\times m$ vector whose jth component is given by $p_j^{-}(n)$ and which gives the steady-state probability that an arrival finds n customers in the queue and the arrival process is in state j. Then vector $\mathbf{p}^{-}(n)$ is given by

$$\mathbf{p}^{-}(n) = \frac{\mathbf{p}(n)\mathbf{D}}{\lambda^{*}}, \quad 0 \le n \le N. \tag{4.40}$$

As the distributions of the number of customers in the queue at arbitrary epoch are known from (4.37), (4.38), and (4.39), one can easily evaluate the pre-arrival epoch probabilities using (4.40).

5. Performance measures

As the steady-state probabilities at service completion, vacation termination, and departure and arbitrary epochs are known, various performance measures such as the average number of customers in the queue at any arbitrary epoch ($Lq = \sum_{i=0}^{N} i\mathbf{p}(i)\mathbf{e}$), the average number of customers in the queue when the server is busy ($Lq_2 = \sum_{i=0}^{N} i\boldsymbol{\pi}(i)\mathbf{e}$), the average number of customers in the queue when the server is on vacation ($Lq_1 = \sum_{i=0}^{N} i\boldsymbol{\omega}(i)\mathbf{e}$), and the average number of customers in the queue when the server is in dormancy ($Lq_0 = \sum_{i=0}^{a-1} i\boldsymbol{\nu}(i)\mathbf{e}$) can be easily obtained. The loss probability of a customer is given by $P_{\text{loss}} = \mathbf{p}^{-}(N)\mathbf{e} = \mathbf{p}(N)\mathbf{De}/\lambda^{*}$.

6. Computational procedures

In this section, we briefly discuss the necessary steps required for the computation of the matrices $\mathbf{A}(n)$, $\mathbf{V}(n)$, $\mathbf{B}(n,k)$, and $\mathbf{R}(n,k)$. The evaluation of $\mathbf{A}(n)$ ($\mathbf{V}(n)$), in general, for arbitrary service (vacation) time distribution requires numerical integration and can be carried out along the lines proposed by Neuts [16, pages 67–70] or by Lucantoni and Ramaswami [13]. However, when the service time distribution is of phase type (PH-distribution), these matrices can be evaluated without any numerical integration [16]. It may be noted here that PH-distribution is a rich class of distributions, and service (vacation) time distributions arising in real-world queueing problems can be easily approximated using it. The following theorem gives a procedure for the computation of the matrices $\mathbf{A}(n)$ and $\mathbf{V}(n)$.

Theorem 6.1. *Let $S(x)$ follow a PH-distribution with irreducible representation $(\boldsymbol{\beta},\mathbf{S})$, where $\boldsymbol{\beta}$ and $\mathbf{S}$ are of dimension γ_1, and the matrices $\mathbf{A}(n)$ are given by*

$$\begin{aligned} \mathbf{A}(n) &= \mathbf{M}(n)(\mathbf{I}\otimes\mathbf{S}^0), \quad 0\le n\le N-1,\\ \mathbf{A}'(n) &= \mathbf{M}'(n)(\mathbf{I}\otimes\mathbf{S}^0), \quad b\le n\le N, \end{aligned} \tag{6.1}$$

where

$$\begin{aligned} \mathbf{M}(0) &= -(\mathbf{I}\otimes\boldsymbol{\beta})[\mathbf{C}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{S}]^{-1},\\ \mathbf{M}(n) &= -\mathbf{M}(n-1)(\mathbf{D}\otimes\mathbf{I})[\mathbf{C}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{S}]^{-1}, \quad 1\le n\le N-1,\\ \mathbf{M}'(n) &= -\mathbf{M}(n-1)(\mathbf{D}\otimes\mathbf{I})\big[(\mathbf{C}+\mathbf{D})\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{S}\big]^{-1}, \quad b\le n\le N, \end{aligned} \tag{6.2}$$

and the symbol $\otimes$ denotes the Kronecker product of two matrices.

Similarly, let $V(x)$ follow a PH-distribution with irreducible representation $(\boldsymbol{\alpha},\mathbf{T})$, where $\boldsymbol{\alpha}$ and $\mathbf{T}$ are of dimension γ_2, and the matrices $\mathbf{V}(n)$ are given by

$$\mathbf{V}(n) = \mathbf{E}(n)(\mathbf{I}\otimes\mathbf{T}^0), \quad 0\le n\le N-1, \tag{6.3}$$

$$\mathbf{V}'(n) = \mathbf{E}'(n)(\mathbf{I}\otimes\mathbf{T}^0), \quad N-a+1\le n\le N, \tag{6.4}$$

where

$$\begin{aligned} \mathbf{E}(0) &= -(\mathbf{I}\otimes\boldsymbol{\alpha})[\mathbf{C}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{T}]^{-1}, \\ \mathbf{E}(n) &= -\mathbf{E}(n-1)(\mathbf{D}\otimes\mathbf{I})[\mathbf{C}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{T}]^{-1}, \quad 1\le n\le N-1, \\ \mathbf{E}'(n) &= -\mathbf{E}(n-1)(\mathbf{D}\otimes\mathbf{I})\big[(\mathbf{C}+\mathbf{D})\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{T}\big]^{-1}, \quad N-a+1\le n\le N. \end{aligned} \tag{6.5}$$

For proof, see [2, 8, 16]. One may note here that for the sake of convenience, we use the notation $L(n)$ instead of the usual notation L_n for the representation of a matrix as used in [2, 8, 16].

7. Numerical results

As the computation of the state probabilities at various epochs is heavily dependent on algebraic manipulation of matrices, we have carried out extensive numerical work using Matlab, but only a few selected results are presented here. To achieve the generic solution using Matlab for multiple inputs, the programming language PERL has been used. The initial PERL script is executed with desired inputs and it produces an output file with Matlab format which has been used to produce numerical results. These results for certain cases have been presented in self-explanatory Tables 7.1 and 7.2. We have presented state probabilities only for a few values of n $(0\le n\le N)$ in various columns of the tables. Also, various performance measures, such as average queue lengths L_q, L_{q2}, L_{q1}, L_{q0}, the probability that the server is busy (ρ'), and the loss probability (P_{loss}), are presented below the tables. In Table 7.1, results are given for variable batch size, whereas for fixed batch size, that is, $a=b$, results are given in Table 7.2.

7.1. Numerical results for variable batch size. In Table 7.1, the results of $MAP/PH^{(4,7)}/1/70$ (vacation time follows E_2 distribution) queue are given for the following input parameters. The MAP representation is taken as $\mathbf{C}=\begin{bmatrix}-1.625 & 0.250\\ 0.875 & -1.375\end{bmatrix}$ and $\mathbf{D}=\begin{bmatrix}0.875 & 0.500\\ 0.125 & 0.375\end{bmatrix}$. For service time and vacation time, PH representation is taken as $\boldsymbol{\beta}=[\,0.3\;\;0.7\,]$, $\mathbf{S}=\begin{bmatrix}-0.1 & 0.074\\ 0.0575 & -0.25\end{bmatrix}$, and E_2 as $\boldsymbol{\alpha}=[\,1.0\;\;0.0\,]$, $\mathbf{T}=\begin{bmatrix}-\gamma & \gamma\\ 0.0 & -\gamma\end{bmatrix}$, where $\gamma=1.23$.

7.2. Numerical results for fixed batch size. In Table 7.2, the results of $MAP/E_2^{(8,8)}/1/60$ (vacation time follows E_2 distribution) queue are given for the same following parameters. The MAP representation is taken as $\mathbf{C}=\begin{bmatrix}-1.625 & 0.250\\ 0.875 & -1.375\end{bmatrix}$ and $\mathbf{D}=\begin{bmatrix}0.875 & 0.500\\ 0.125 & 0.375\end{bmatrix}$. For service time and vacation time, PH representation is taken as $\boldsymbol{\beta}=[\,0.3\;\;0.7\,]$, $\mathbf{S}=\begin{bmatrix}-0.1 & 0.074\\ 0.0575 & -0.25\end{bmatrix}$, and E_2 as $\boldsymbol{\alpha}=[\,1.0\;\;0.0\,]$, $\mathbf{T}=\begin{bmatrix}-\gamma & \gamma\\ 0.0 & -\gamma\end{bmatrix}$, where $\gamma=1.23$.

It can be seen from Table 7.1 that ρ and ρ' are different. Further, it is observed from Table 7.2 that for fixed batch size and $\rho<1$, ρ and ρ' are equal when N is large. This confirms the validity of analytic analysis and accuracy of our results. It may be remarked here that such an observation holds in case of nonvacation $M/G^{(b,b)}/1/\infty$ queue; see Chaudhry and Templeton [3, page 224].

Table 7.1. Distributions of number of customers in the queue at various epochs for $MAP/PH^{(4,7)}/1/70$ (vacation time follows E_2 distribution) queue with $m = 2$, $\lambda^* = 1.0$, $\theta_s = 10$, $\theta_v = 1.6260$, and $\rho = 1.4286$.

n	$\mathbf{p}^+(n)$ $\sum_{k=1}^m$	$\boldsymbol{\pi}^+(n)$ $\sum_{k=1}^m$	$\boldsymbol{\omega}^+(n)$ $\sum_{k=1}^m$	$\boldsymbol{\pi}(n)$ $\sum_{k=1}^m$	$\boldsymbol{\omega}(n)$ $\sum_{k=1}^m$	$\mathbf{p}(n)$ $\sum_{k=1}^m$	$\mathbf{p}^-(n)$ $\sum_{k=1}^m$
0	0.0009	0.0009	0.0003	0.0006	0.0001	0.0007	0.0007
1	0.0009	0.0009	0.0005	0.0007	0.0001	0.0008	0.0008
2	0.0009	0.0009	0.0007	0.0007	0.0001	0.0009	0.0009
5	0.0009	0.0009	0.0003	0.0008	0.0000	0.0008	0.0008
6	0.0010	0.0010	0.0002	0.0009	0.0000	0.0009	0.0009
7	0.0010	0.0010	0.0001	0.0009	0.0000	0.0009	0.0009
8	0.0011	0.0011	0.0001	0.0010	0.0000	0.0010	0.0010
9	0.0011	0.0011	0.0000	0.0010	0.0000	0.0010	0.0010
10	0.0012	0.0012	0.0000	0.0011	0.0000	0.0011	0.0011
20	0.0022	0.0022	0.0000	0.0020	0.0000	0.0020	0.0020
30	0.0042	0.0042	0.0000	0.0039	0.0000	0.0039	0.0039
40	0.0080	0.0080	0.0000	0.0073	0.0000	0.0073	0.0073
50	0.0154	0.0153	0.0000	0.0140	0.0000	0.0140	0.0140
60	0.0269	0.0268	0.0000	0.0249	0.0000	0.0249	0.0249
70	0.2446	0.2437	0.0000	0.3024	0.0000	0.3024	0.3022
Sum	1.0000	0.9966	0.0034	0.9990	0.0006	1.0000	1.0000

$\rho' = 0.9990$, $\boldsymbol{\nu}(0)\mathbf{e} = 0.0000$, $\boldsymbol{\nu}(1)\mathbf{e} = 0.0000$, $\boldsymbol{\nu}(2)\mathbf{e} = 0.0001$, $\boldsymbol{\nu}(3)\mathbf{e} = 0.0002$, $Lq = 58.5464$, $Lq_2 = 58.5439$, $Lq_1 = 0.0015$, $Lq_0 = 9.9803e-004$, and $P_{\text{loss}} = 0.3022$.

Conclusions and future scope

In this paper, we have analyzed bulk service $MAP/G^{(a,b)}/1$ queue with finite buffer and single vacation. The analysis of the more general model $BMAP/G^{(a,b)}/1/N$ with single and multiple vacations can be carried out using the procedure discussed in this paper. The analysis of waiting time of bulk service queues with vacations is, in general, a difficult task and is left for future investigation.

Appendix

Lemma A.1.

$$\sum_{n=0}^{N} \boldsymbol{\pi}(n)\mathbf{e} = \theta_s \sum_{n=0}^{N} \boldsymbol{\pi}(n,0)\mathbf{e}, \qquad \sum_{n=0}^{N} \boldsymbol{\omega}(n)\mathbf{e} = \theta_v \sum_{n=0}^{N} \boldsymbol{\omega}(n,0)\mathbf{e}. \tag{A.1}$$

Here, $\sum_{n=0}^{N} \boldsymbol{\pi}(n,0)\mathbf{e}$ is the number of service completion per unit of time, then multiplying this by θ_s will give the probability that the server is busy, which must be equal to $\sum_{n=0}^{N} \boldsymbol{\pi}(n)\mathbf{e} = \rho'$. The second identity can also be interpreted similarly.

Table 7.2. Distributions of number of customers in the queue at various epochs for $MAP/E_2^{(8,8)}/1/60$ (vacation time follows E_2 distribution) queue with $m = 2$, $\lambda^* = 4.0$, $\theta_s = 1.3333$, $\theta_v = 0.2857$, and $\rho = 0.6667$.

n	$\mathbf{p}^+(n)$ $\sum_{k=1}^m$	$\boldsymbol{\pi}^+(n)$ $\sum_{k=1}^m$	$\boldsymbol{\omega}^+(n)$ $\sum_{k=1}^m$	$\boldsymbol{\pi}(n)$ $\sum_{k=1}^m$	$\boldsymbol{\omega}(n)$ $\sum_{k=1}^m$	$\mathbf{p}(n)$ $\sum_{k=1}^m$	$\mathbf{p}^-(n)$ $\sum_{k=1}^m$
0	0.0454	0.0285	0.0115	0.0709	0.0034	0.0765	0.0766
1	0.0716	0.0450	0.0266	0.0710	0.0070	0.0855	0.0856
2	0.0842	0.0528	0.0392	0.0672	0.0097	0.0922	0.0923
5	0.0802	0.0503	0.0512	0.0487	0.0117	0.1054	0.1055
10	0.0439	0.0276	0.0062	0.0233	0.0009	0.0243	0.0242
15	0.0202	0.0127	0.0001	0.0104	0.0000	0.0104	0.0104
20	0.0089	0.0056	0.0000	0.0045	0.0000	0.0045	0.0045
25	0.0038	0.0024	0.0000	0.0019	0.0000	0.0019	0.0019
30	0.0016	0.0010	0.0000	0.0008	0.0000	0.0008	0.0008
35	0.0007	0.0004	0.0000	0.0004	0.0000	0.0004	0.0004
40	0.0003	0.0002	0.0000	0.0002	0.0000	0.0002	0.0002
45	0.0001	0.0001	0.0000	0.0001	0.0000	0.0001	0.0001
50	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
51	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Sum	1.0000	0.6276	0.3724	0.6667	0.0848	1.0000	1.0000

$\rho' = 0.6667$, $\boldsymbol{\nu}(0)\mathbf{e} = 0.0022$, $\boldsymbol{\nu}(1)\mathbf{e} = 0.0076$, $\boldsymbol{\nu}(2)\mathbf{e} = 0.0153$, $\boldsymbol{\nu}(3)\mathbf{e} = 0.0247$, $\boldsymbol{\nu}(4)\mathbf{e} = 0.0348$, $\boldsymbol{\nu}(5)\mathbf{e} = 0.0451$, $\boldsymbol{\nu}(6)\mathbf{e} = 0.0548$, $\boldsymbol{\nu}(7)\mathbf{e} = 0.0639$, $Lq = 5.8806$, $Lq_2 = 4.2476$, $Lq_1 = 0.3790$, $Lq_0 = 1.2541$, and $P_{\text{loss}} = 1.2952e-005$.

Proof. Postmultiplying (3.15), (3.16), (3.17), and (3.18) by $\mathbf{e}$, differentiating these with respect to s, and using $(\mathbf{C}+\mathbf{D})\mathbf{e} = \mathbf{0}$, we get

$$-\boldsymbol{\pi}^*(0,s)\mathbf{e} - s\boldsymbol{\pi}^{*(1)}(0,s)\mathbf{e} = \boldsymbol{\pi}^{*(1)}(0,s)\mathbf{C}\mathbf{e} + S^{*(1)}(s)\sum_{n=a}^{b}\big(\boldsymbol{\pi}(n,0)+\boldsymbol{\omega}(n,0)\big)\mathbf{e} + S^{*(1)}(s)\boldsymbol{\nu}(a-1)\mathbf{D}\mathbf{e}, \tag{A.2}$$

$$-\boldsymbol{\pi}^*(n,s)\mathbf{e} - s\boldsymbol{\pi}^{*(1)}(n,s)\mathbf{e} = \boldsymbol{\pi}^{*(1)}(n,s)\mathbf{C}\mathbf{e} + \boldsymbol{\pi}^{*(1)}(n-1,s)\mathbf{D}\mathbf{e} + S^{*(1)}(s)\big(\boldsymbol{\pi}(n+b,0)+\boldsymbol{\omega}(n+b,0)\big)\mathbf{e}, \quad 1 \le n \le N-b, \tag{A.3}$$

$$-\boldsymbol{\pi}^*(n,s)\mathbf{e} - s\boldsymbol{\pi}^{*(1)}(n,s)\mathbf{e} = \boldsymbol{\pi}^{*(1)}(n,s)\mathbf{C}\mathbf{e} + \boldsymbol{\pi}^{*(1)}(n-1,s)\mathbf{D}\mathbf{e}, \quad N-b+1 \le n \le N-1, \tag{A.4}$$

$$-\boldsymbol{\pi}^*(N,s)\mathbf{e} - s\boldsymbol{\pi}^{*(1)}(N,s)\mathbf{e} = \boldsymbol{\pi}^{*(1)}(N-1,s)\mathbf{D}\mathbf{e}. \tag{A.5}$$

Setting $s = 0$ in (A.2), (A.3), (A.4), and (A.5), adding them, and using Lemmas 3.1, 3.2 and $(\mathbf{C}+\mathbf{D})\mathbf{e} = \mathbf{0}$, after simplification, we obtain $\sum_{n=0}^{N}\boldsymbol{\pi}(n)\mathbf{e} = \theta_s\sum_{n=0}^{N}\boldsymbol{\pi}(n,0)\mathbf{e}$. Similarly, postmultiplying (3.19), (3.20), (3.21), and (3.22) by $\mathbf{e}$, differentiating these equations with respect to s, setting $s=0$, and using Lemma 3.2, we get $\sum_{n=0}^{N}\boldsymbol{\omega}(n)\mathbf{e}=\theta_v\sum_{n=0}^{N}\boldsymbol{\omega}(n,0)\mathbf{e}$. □

Lemma A.2.

$$\theta_i=\theta_v+\frac{\sum_{n=0}^{a-1}\sum_{j=0}^{n}\sum_{k=0}^{j}\mathbf{p}^+(j-k)\mathbf{V}(k)\overline{\mathbf{D}}^{n-j}(-\mathbf{C})^{-1}\mathbf{e}}{\sum_{n=0}^{a-1}\mathbf{p}^+(n)\mathbf{e}}. \tag{A.6}$$

Proof. Since $\theta_b/\theta_i=\sum_{n=0}^{N}\boldsymbol{\pi}(n)\mathbf{e}/(\sum_{n=0}^{N}\boldsymbol{\omega}(n)\mathbf{e}+\sum_{n=0}^{a-1}\boldsymbol{\nu}(n)\mathbf{e})$ (see [3, page 324]), using (A.1) and (4.25), we get

$$\frac{\theta_b}{\theta_i}=\frac{\theta_s\sum_{n=0}^{N}\boldsymbol{\pi}(n,0)\mathbf{e}}{\theta_v\sum_{n=0}^{N}\boldsymbol{\omega}(n,0)\mathbf{e}+\sigma\sum_{n=0}^{a-1}\sum_{j=0}^{n}\boldsymbol{\omega}^+(j)\overline{\mathbf{D}}^{n-j}(-\mathbf{C})^{-1}\mathbf{e}}. \tag{A.7}$$

Dividing numerator and denominator by σ and using (4.1), we get

$$\frac{\theta_b}{\theta_i}=\frac{\theta_s\sum_{n=0}^{N}\boldsymbol{\pi}^+(n)\mathbf{e}}{\theta_v\sum_{n=0}^{N}\boldsymbol{\omega}^+(n)\mathbf{e}+\sum_{n=0}^{a-1}\sum_{j=0}^{n}\boldsymbol{\omega}^+(j)\overline{\mathbf{D}}^{n-j}(-\mathbf{C})^{-1}\mathbf{e}}. \tag{A.8}$$

Substituting $\sum_{n=0}^{N}\boldsymbol{\omega}^+(n)\mathbf{e}=\sum_{n=0}^{a-1}\boldsymbol{\pi}^+(n)\mathbf{e}$ (from Lemma 3.2, after dividing both sides by σ) and $\theta_b=\theta_s/\sum_{n=0}^{a-1}\mathbf{p}^+(n)\mathbf{e}=\theta_s\sum_{n=0}^{N}\boldsymbol{\pi}^+(n)\mathbf{e}/\sum_{n=0}^{a-1}\boldsymbol{\pi}^+(n)\mathbf{e}$ (using (4.20)) in (A.8), we get

$$\theta_i=\theta_v+\frac{\sum_{n=0}^{a-1}\sum_{j=0}^{n}\boldsymbol{\omega}^+(j)\overline{\mathbf{D}}^{n-j}(-\mathbf{C})^{-1}\mathbf{e}}{\sum_{n=0}^{a-1}\boldsymbol{\pi}^+(n)\mathbf{e}}. \tag{A.9}$$

Now, using (4.16) and (4.20) in (A.9), after simplification, we get the result. □

Acknowledgment

The authors are thankful to the referee for valuable comments and suggestions which have led to the presentation of this paper in its current form.

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