CHARACTERIZATIONS OF SOME NEAR-CONTINUOUS FUNCTIONS AND NEAR-OPEN FUNCTIONS

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(Received April 7, 1986)

ABSTRACT. A subset N of a topological space is defined to be a \( \theta \)-neighborhood of x if there exists an open set U such that \( x \in U \subseteq \text{Cl} U \subseteq N \). This concept is used to characterize the following types of functions: weakly continuous, \( \theta \)-continuous, strongly \( \theta \)-continuous, almost strongly \( \theta \)-continuous, weakly \( \delta \)-continuous, weakly open and almost open functions. Additional characterizations are given for weakly \( \delta \)-continuous functions. The concept of \( \theta \)-neighborhood is also used to define the following types of open maps: \( \theta \)-open, strongly \( \theta \)-open, almost strongly \( \theta \)-open, and weakly \( \delta \)-open functions.

KEY WORDS AND PHRASES. \( \theta \)-neighborhood, weakly continuous function, \( \theta \)-continuous function, strongly \( \theta \)-continuous function, almost strongly \( \theta \)-continuous function, weakly \( \delta \)-continuous function, weakly open function, almost open function, \( \theta \)-open function, strongly \( \theta \)-open function, almost strongly \( \theta \)-open function, weakly \( \delta \)-open function.

1980 AMS SUBJECT CLASSIFICATION CODE. 54C10.

1. INTRODUCTION.

Near-continuity has been investigated by many authors including Levine [1], Long and Herrington [2], Noiri [3], and Rose [4]. Near-openness has been developed by Rose [5] and Singal and Singal [6]. The purpose of this note is to characterize several types of near-continuity and near-openness in terms of the concept of \( \theta \)-neighborhood. These characterizations clarify both the nature of these functions and the relationships among them. Additional characterizations of weak \( \delta \)-continuity are given. The concept of \( \theta \)-neighborhood also leads to the definition of several new types of near-open functions.

2. DEFINITIONS AND NOTATION.

The symbols X and Y denote topological spaces with no separation axioms assumed unless explicitly stated. Let U be a subset of a space X. The closure of U and the interior of U are denoted by \( \text{Cl} U \) and \( \text{Int} U \) respectively. The set U is said to be regular open (regular closed) if \( U = \text{Int} \text{Cl} U \) (\( U = \text{Cl} \text{Int} U \)). The \( \theta \)-closure (\( \delta \)-closure) (Velicko [7]) of U is the set of all x in X such that every closed neighborhood (the interior of every closed neighborhood) of x intersects
The $\theta$-closure and the $\delta$-closure of $U$ are denoted by $\text{Cl}_\theta U$ and $\text{Cl}_\delta U$ respectively. The set $U$ is called $\theta$-closed ($\delta$-closed) if $U = \text{Cl}_\theta U$ ($U = \text{Cl}_\delta U$). A set is said to be $\theta$-open ($\delta$-open) if its complement is $\theta$-closed ($\delta$-closed). For a given space $X$ the collection of all $\theta$-open sets and the collection of all $\delta$-open sets both form topologies. The space $X$ with the $\theta$-open ($\delta$-open) topology will be signified by $X_\theta$ ($X_\delta$).

**Definition 1.** A function $f: X \to Y$ is said to be weakly continuous (Levine [1]) ($\theta$-continuous (Fomin [8]), strongly $\theta$-continuous (Long and Herrington [2]), almost strongly $\theta$-continuous (Noiri and Kang [9]), weakly $\delta$-continuous (Baker [10])) if for each $x \in X$ and each open neighborhood $V$ of $f(x)$, there exists an open neighborhood $U$ of $x$ such that $f(U) \subseteq \text{Cl}_\theta V$ ($f(\text{Cl}_\theta U) \subseteq \text{Cl}_\delta V$, $f(\text{Cl}_\delta U) \subseteq V$, $f(\text{Cl}_\delta U) \subseteq \text{Int} \text{Cl}_\delta V$, $f(\text{Int} \text{Cl}_\theta U) \subseteq \text{Cl}_\delta V$).

**Definition 2.** A function $f: X \to Y$ is said to be weakly open (Rose [5]) (almost open (Rose [5])) provided that for each open subset $U$ of $X$, $f(U) \subseteq \text{Int} f(\text{Cl}_\theta U)$ ($f(U) \subseteq \text{Int} f(\text{Cl}_\delta U)$).

**Definition 3.** A subset $N$ of a space $X$ is said to be a $\theta$-neighborhood ($\delta$-neighborhood) of a point $x$ in $X$ if there exists an open set $U$ such that $x \in U \subseteq \text{Cl}_\theta U \subseteq N$ ($x \in U \subseteq \text{Int} \text{Cl}_\delta U \subseteq N$).

Note that a $\theta$-neighborhood is not necessarily a neighborhood in the $\theta$-topology, but a $\delta$-neighborhood is a neighborhood in the $\delta$-topology.

3. NEAR-CONTINUOUS FUNCTIONS.

The main results can be paraphrased as follows: weak continuity corresponds to "$f^{-1}(\theta$-neighborhood) = neighborhood"; $\theta$-continuity corresponds to "$f^{-1}(\theta$-neighborhood) = $\theta$-neighborhood"; strong $\theta$-continuity corresponds to "$f^{-1}(neighborhood) = \theta$-neighborhood"; almost strong $\theta$-continuity corresponds to "$f^{-1}(\delta$-neighborhood) = $\theta$-neighborhood", and weak $\delta$-continuity corresponds to "$f^{-1}(\theta$-neighborhood) = $\delta$-neighborhood".

**Theorem 1.** A function $f: X \to Y$ is weakly continuous if and only if for each $x$ in $X$ and each $\theta$-neighborhood $N$ of $f(x)$, $f^{-1}(N)$ is a neighborhood of $x$.

**Proof.** Assume $f$ is weakly continuous. Let $x \in X$ and let $N$ be a $\theta$-neighborhood of $f(x)$. Then there exists an open set $V$ such that $f(x) \in V \subseteq \text{Cl}_\theta V \subseteq N$. Since $f$ is weakly continuous, there exists an open neighborhood $U$ of $x$ such that $f(U) \subseteq \text{Cl}_\theta V \subseteq N$. Thus $x \in U \subseteq f^{-1}(N)$ and hence $f^{-1}(N)$ is a neighborhood of $x$.

Assume for each $x \in X$ and each $\theta$-neighborhood $N$ of $x$ that $f^{-1}(N)$ is a neighborhood of $x$. Let $x \in X$ and let $V$ be an open neighborhood of $f(x)$. Since $\text{Cl}_\theta V$ is a $\theta$-neighborhood of $f(x)$, $f^{-1}(\text{Cl}_\theta V)$ is a neighborhood of $x$. Thus there is an open set $U$ for which $x \in U \subseteq f^{-1}(\text{Cl}_\theta V)$ and $f(U) \subseteq \text{Cl}_\theta V$ which proves $f$ is weakly continuous.

**Theorem 2.** A function $f: X \to Y$ is $\theta$-continuous if and only if for each $x$ in $X$ and each $\theta$-neighborhood $N$ of $f(x)$, $f^{-1}(N)$ is a $\theta$-neighborhood of $x$. 
PROOF. Assume $f: X \to Y$ is $\theta$-continuous. Let $x \in X$ and let $N$ be a $\theta$-neighborhood of $f(x)$. Then there exists an open set $V$ for which $f(x) \in V \subseteq C_1 V \subseteq N$. By the $\theta$-continuity of $f$, there exists an open neighborhood $U$ of $x$ such that $f(C_1 U) \subseteq C_1 V \subseteq N$. Thus $x \in U \subseteq C_1 U \subseteq f^{-1}(N)$ and hence $f^{-1}(N)$ is a $\theta$-neighborhood of $x$.

Assume for each $x$ in $X$ and for each $\theta$-neighborhood $N$ of $f(x)$ that $f^{-1}(N)$ is a $\theta$-neighborhood of $x$. Let $x \in X$ and let $V$ be an open neighborhood of $f(x)$. Since $C_1 V$ is a $\theta$-neighborhood of $f(x)$, $f^{-1}(C_1 V)$ is a $\theta$-neighborhood of $x$. Hence there exists an open set $U$ for which $x \in U \subseteq C_1 U \subseteq f^{-1}(C_1 V)$. That is, $f(C_1 U) \subseteq C_1 V$ and thus $f$ is $\theta$-continuous.

The proof of the following theorem is similar to that of Theorem 2 and is omitted.

**THEOREM 3.** A function $f: X \to Y$ is strongly $\theta$-continuous if and only if for each $x$ in $X$ and each neighborhood $N$ of $f(x)$, $f^{-1}(N)$ is a $\theta$-neighborhood of $x$.

**THEOREM 4.** A function $f: X \to Y$ is almost strongly $\theta$-continuous if and only if for each $x$ in $X$ and each $\delta$-neighborhood $N$ of $f(x)$, $f^{-1}(N)$ is a $\theta$-neighborhood of $x$.

PROOF. Assume $f: X \to Y$ is almost strongly $\theta$-continuous. Let $x \in X$ and let $N$ be a $\delta$-neighborhood of $f(x)$. Then there exists an open set $V$ such that $f(x) \in V \subseteq Int C_1 V \subseteq N$. Since $f$ is almost strongly $\theta$-continuous, there exists an open neighborhood $U$ of $x$ for which $f(C_1 U) \subseteq Int C_1 V \subseteq N$. Then $x \in U \subseteq C_1 U \subseteq f^{-1}(N)$ which proves that $f^{-1}(N)$ is a $\theta$-neighborhood of $x$.

Assume for each $x \in X$ and each $\delta$-neighborhood $N$ of $f(x)$ that $f^{-1}(N)$ is a $\theta$-neighborhood of $x$. Let $x \in X$ and let $V$ be an open neighborhood of $f(x)$. Since $Int C_1 V$ is a $\delta$-neighborhood of $f(x)$, $f^{-1}(Int C_1 V)$ is a $\theta$-neighborhood of $x$. Hence there is an open set $U$ such that $x \in U \subseteq C_1 U \subseteq f^{-1}(Int C_1 V)$. That is, $f(C_1 U) \subseteq Int C_1 V$ and hence $f$ is almost strongly $\theta$-continuous.

**THEOREM 5.** A function $f: X \to Y$ is weakly $\delta$-continuous if and only if for each $x \in X$ and each $\theta$-neighborhood $N$ of $f(x)$, $f^{-1}(N)$ is a $\delta$-neighborhood of $x$.

The proof of this theorem is similar to that of Theorem 4. The following theorem gives additional characterizations of weak $\delta$-continuity. These results are analogous to those obtained by Noiri and Kang in [9] for almost strongly $\theta$-continuous functions.

**LEMMA.** Let $X$ be a space and $H \subseteq X$. Then

(a) $C_1 H = \{x \in X: \text{every } \theta\text{-neighborhood of } x \text{ intersects } H\}$ and
(b) $C_1 H = \{x \in X: \text{every } \delta\text{-neighborhood of } x \text{ intersects } H\}$.

The proof follows easily from the definitions.

**THEOREM 6.** For $f: X \to Y$ the following statements are equivalent:

(a) $f: X \to Y$ is weakly $\delta$-continuous.

(b) For each $H \subseteq X$, $f(C_1 H) \subseteq C_1 f(H)$.

(c) For each $K \subseteq Y$, $C_1 f^{-1}(K) \subseteq f^{-1}(C_1 K)$.

(d) $f: X \to Y$ is weakly continuous.
PROOF. (a) \(\Rightarrow\) (b). Let \(H \subseteq X\) and let \(y \in f(\text{Cl}_\delta H)\). Then there exists an \(x\) in \(\text{Cl}_\delta H\) such that \(y = f(x)\). Let \(N\) be a \(\theta\)-neighborhood of \(f(x)\). By Theorem 5 \(f^{-1}(N)\) is a \(\delta\)-neighborhood of \(x\). Since \(x \in \text{Cl}_\delta H\), \(f^{-1}(N) \cap H \neq \emptyset\). That is, \(N \cap f(H) \neq \emptyset\). Hence \(y \in \text{Cl}_\theta f(H)\). Thus \(f(\text{Cl}_\delta H) \subseteq \text{Cl}_\theta f(H)\).

(b) \(\Rightarrow\) (c). Let \(K \subseteq Y\). By (b) \(f(\text{Cl}_\delta f^{-1}(K)) \subseteq \text{Cl}_\theta f(f^{-1}(K)) \subseteq \text{Cl}_\theta K\). Thus \(\text{Cl}_\delta f^{-1}(K) \subseteq f^{-1}(\text{Cl}_\theta K)\).

(c) \(\Rightarrow\) (d). Let \(x \in X\) and let \(V\) be an open neighborhood of \(f(x)\). Since \(\text{Cl} V\) is a \(\theta\)-neighborhood of \(f(x)\), \(f(x) \notin \text{Cl}_\theta (Y \setminus \text{Cl} V)\). Hence \(x \notin f^{-1}(\text{Cl}_\theta (Y \setminus \text{Cl} V))\).

By (c) \(x \notin \text{Cl}_\delta f^{-1}(Y \setminus \text{Cl} V)\). Thus there is a neighborhood \(U\) of \(x\) such that \((\text{Int} \text{Cl} U) \cap f^{-1}(Y \setminus \text{Cl} V) = \emptyset\). Then \(f(\text{Int} \text{Cl} U) \subseteq \text{Cl} V\). Since \(\text{Int} \text{Cl} U\) is a regular open, \(f: X_\delta \to Y\) is weakly continuous.

(d) \(\Rightarrow\) (a). Let \(x \in X\) and let \(V\) be an open neighborhood of \(f(x)\). Since \(f: X_\delta \to Y\) is weakly continuous, there exists a \(\delta\)-open set \(W\) containing \(x\) such that \(f(W) \subseteq \text{Cl} V\). Then there is a regular open set \(U\) for which \(x \in U \subseteq W\). Then \(f(\text{Int} \text{Cl} U) = f(U) \subseteq f(W) \subseteq \text{Cl} V\) and hence \(f\) is weakly \(\delta\)-continuous.

4. NEAR-OPEN FUNCTIONS.

In this section weak openness and almost openness are characterized in terms of the concept of \(\theta\)-neighborhood.

THEOREM 7. A function \(f: X \to Y\) is weakly open if and only if for each \(x \in X\) and each \(\theta\)-neighborhood \(N\) of \(x\), \(f(N)\) is a neighborhood of \(f(x)\).

PROOF. Assume \(f\) is weakly open. Let \(x \in X\) and let \(N\) be a \(\theta\)-neighborhood of \(x\). Then there is an open set \(U\) such that \(x \in U \subseteq \text{Cl} U \subseteq N\). Since \(f\) is weakly open \(f(x) \in f(U) \subseteq \text{Int} f(\text{Cl} U) \subseteq \text{Int} f(N)\). Hence \(f(N)\) is a neighborhood of \(f(x)\).

Assume for each \(x\) in \(X\) and each \(\theta\)-neighborhood \(N\) of \(x\) that \(f(N)\) is a neighborhood of \(f(x)\). Let \(U\) be an open set in \(X\). Suppose \(x \in U\). Since \(\text{Cl} U\) is a \(\theta\)-neighborhood of \(x\), \(f(\text{Cl} U)\) is a neighborhood of \(f(x)\). Hence \(f(x) \in \text{Int} f(\text{Cl} U)\). Thus \(f(U) \subseteq \text{Int} f(\text{Cl} U)\) and \(f\) is weakly open.

The proof of the following theorem is similar and is omitted.

THEOREM 8. A function \(f: X \to Y\) is almost open if and only if for each \(x \in X\) and each neighborhood \(N\) of \(x\), \(\text{Cl} f(N)\) is a \(\theta\)-neighborhood of \(f(x)\).

Theorem 7 and the characterizations of near-continuous functions in Section 3 suggest the following definitions of near-open functions.

DEFINITION 4. A function \(f: X \to Y\) is said to be \(\theta\)-open (strongly \(\theta\)-open, almost strongly \(\theta\)-open, weakly \(\delta\)-open) if for each \(x \in X\) and each \(\theta\)-neighborhood \(N\) of \(x\), \(f(N)\) is a \(\theta\)-neighborhood of \(f(x)\).

The following theorems characterize these near-open functions in terms of the closure and interior operators. Since the proofs are all similar, only the first theorem is proved.

THEOREM 9. A function \(f: X \to Y\) is \(\theta\)-open if and only if for each \(x \in X\) and each open neighborhood \(U\) of \(x\), there exists an open neighborhood \(V\) of \(f(x)\) such that \(\text{Cl} V \subseteq f(\text{Cl} U)\).
PROOF. Assume $f: X \to Y$ is $\theta$-open. Let $x \in X$ and let $U$ be an open neighborhood of $x$. Since $f(Cl U)$ is a $\theta$-neighborhood of $f(x)$, there exists an open set $V$ such that $f(x) \in V \subseteq Cl V \subseteq f(Cl U)$.

Assume that for each $x \in X$ and each open neighborhood $U$ of $x$ there exists an open neighborhood $V$ of $f(x)$ for which $Cl V \subseteq f(Cl U)$. Let $x \in X$ and let $N$ be a $\theta$-neighborhood of $f(x)$. Then there is an open set $U$ for which $x \in U \subseteq Cl U \subseteq N$. There exists an open set $V$ such that $f(x) \in V \subseteq Cl V \subseteq f(Cl U) \subseteq f(N)$. Hence $f(N)$ is a $\theta$-neighborhood of $f(x)$ and $f$ is $\theta$-open.

**THEOREM 10.** A function $f: X \to Y$ is strongly $\theta$-open if and only if for each $x \in X$ and each open neighborhood $U$ of $x$, there exists an open neighborhood $V$ of $f(x)$ such that $Cl V \subseteq f(U)$.

**THEOREM 11.** A function $f: X \to Y$ is almost strongly $\theta$-open if and only if for each $x \in X$ and each open neighborhood $U$ of $x$ there exists an open neighborhood $V$ of $f(x)$ such that $Cl V \subseteq f(Int Cl U)$.

**THEOREM 12.** A function $f: X \to Y$ is weakly $\delta$-open if and only if for each $x \in X$ and each open neighborhood $U$ of $x$, there exists an open neighborhood $V$ of $f(x)$ such that $Int Cl V \subseteq f(Cl U)$.

We have the following implications: almost open $\Rightarrow$ st. $\theta$-open $\Rightarrow$ almost st. $\theta$-open $\Rightarrow$ $\theta$-open $\Rightarrow$ weak $\delta$-open $\Rightarrow$ weak open. The following examples show that these implications are not reversible.

**EXAMPLE 1.** Let $X = \{a, b\}$, $T_1 = \{X, \emptyset, \{a\}\}$, $Y = \{a, b, c\}$, and $T_2 = \{Y, \emptyset, \{a\}, \{a, b\}\}$. The inclusion mapping: $(X, T_1) \to (Y, T_2)$ is weak open but not weak $\delta$-open.

In the next example the space $(Y, T_2)$ is from Example 2.2 in Noiri and Kang [9].

**EXAMPLE 2.** Let $(X, T_1)$ be as in Example 1. Let $Y = \{a, b, c, d\}$, and $T_2 = \{Y, \emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. The inclusion mapping: $(X, T_1) \to (Y, T_2)$ is weak $\delta$-open, but not $\theta$-open.

**EXAMPLE 3.** Let $(Y, T_2)$ be as in Example 2. The identity mapping: $(Y, T_2) \to (Y, T_2)$ is $\theta$-open but not almost strongly $\theta$-open.

**EXAMPLE 4.** Let $X = \{a, b, c\}$, $T_1 = \{X, \emptyset, \{a\}, \{a, c\}\}$ and $T_2 = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}, \{a, b\}\}$. The identity mapping: $(X, T_1) \to (X, T_2)$ is almost strongly $\theta$-open and almost open, but not strongly $\theta$-open.

**REFERENCES**

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<th>Event</th>
<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manuscript Due</td>
<td>March 1, 2009</td>
</tr>
<tr>
<td>First Round of Reviews</td>
<td>June 1, 2009</td>
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<tr>
<td>Publication Date</td>
<td>September 1, 2009</td>
</tr>
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</table>

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