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ESSENTIAL SUPREMUM NORM DIFFERENTIABILITY

I. E. LEONARD

Department of Mathematical Sciences
Northern Illinois University
DeKalb, IL 60115

K. F. TAYLOR

Department of Mathematics
University of Saskatchewan
Saskatoon, SK S7N 0W0

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ABSTRACT. The points of Gateaux and Fréchet differentiability in $L_\infty(\mu,X)$ are obtained, where (Ω,Σ,μ) is a finite measure space and X is a real Banach space. An application of these results is given to the space $B(L_1(\mu,\mathbb{R}),X)$ of all bounded linear operators from $L_1(\mu,\mathbb{R})$ into X.

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1. INTRODUCTION.

Let m be the restriction of Lebesgue measure to $[0,1]$ and $L_\infty(m,\mathbb{R})$ the Banach space of all measurable, essentially bounded, real-valued functions on $[0,1]$, equipped with the norm $\|f\| = \text{ess sup}\,\{|f(t)|: t \in [0,1]\}$ (as usual, identifying functions that agree a.e. on $[0,1]$).

In [4], Mazur proved that given any $f \in L_\infty(m,\mathbb{R})$, $f \neq 0$, there exists a $g \in L_\infty(m,\mathbb{R})$ such that

$$\lim_{\lambda\to 0} \frac{\|f+\lambda g\| - \|f\|}{\lambda}$$

does not exist. In other words, the closed unit ball in $L_\infty(m,\mathbb{R})$ has no smooth points.

In this note, we show that an analogous result holds for $L_\infty(\mu,X)$, the space of μ-measurable, essentially bounded functions, whose values lie in a Banach space X - provided that the underlying measure space (Ω,Σ,μ) is non-atomic. We then obtain a complete description of the smooth points of $L_\infty(\mu,X)$ in the general case. We show, in fact, that f is a smooth point of $L_\infty(\mu,X)$ if and only if f achieves its norm on a unique atom for μ, and its (μ-a.e. constant) value on this atom is a smooth point of X.

An application of this result is given to the space of all bounded linear operators from $L_1(\mu,\mathbb{R})$ into a Banach space X, when X has the Radon-Nikodým property with respect to μ.

2. PRELIMINARIES

Throughout this note, X denotes a real Banach space with dual X^*. A point $x \in X \sim \{0\}$ is a smooth point of X if there exists a unique $\phi \in X^*$ with $\|\phi\| = 1$ such that $\phi(x) = \|x\|$. The norm function on X is Gateaux differentiable at non-zero $x \in X$ if there exists a $\phi \in X^*$ such that

$$\lim_{\lambda\to 0}\left|\frac{\|x + \lambda h\| - \|x\|}{\lambda} - \phi(h)\right| = 0 \qquad (*)$$

for all $h \in X$. The functional ϕ is the Gateaux derivative of the norm at $x \in X$. Mazur, [4], has shown that the following are equivalent:

(i) x is a smooth point of X.

(ii) $\lim\limits_{\lambda\to 0}\frac{\|x + \lambda h\| - \|x\|}{\lambda}$ exists for all $h \in X$

(iii) the norm function on X is Gateaux differentiable at x.

The norm function on X is Fréchet differentiable, at a non-zero $x \in X$, if there exists a $\phi \in X^*$ such that

$$\lim_{\|h\|\to 0}\left|\frac{\|x + h\| - \|x\| - \phi(h)}{\|h\|}\right| = 0 . \qquad (**)$$

Of course, Fréchet differentiability at a point implies Gateaux differentiability at the point.

Let (Ω,Σ,μ) denote a finite measure space. A mapping $f\colon \Omega \to X$ is called μ- measurable (or strongly measurable) if

(i) $f^{-1}(V) \in \Sigma$ for each open set $V \subseteq X$, and

(ii) f is essentially separably valued; that is, there exists a set $N \in \Sigma$ with $\mu(N) = 0$, and a countable set $H \subseteq X$, such that $f(\Omega \sim N) \subseteq \overline{H}$.

The Lebesgue-Bochner function space $L_\infty(\mu,X)$ is the real vector space of all μ- measurable, essentially bounded, X-valued functions defined on Ω. $L_\infty(\mu,X)$ is a real Banach space when equipped with the norm
$\|f\| = \text{ess sup}\,\{\|f(\omega)\|: \omega \in \Omega\}$ (as usual, identifying functions which agree μ - a.e.).

A set $A \in \Sigma$ is an atom for the measure μ if and only if $\mu(A) > 0$, and for any $B \in \Sigma$, with $B \subseteq A$, either $\mu(B) = 0$ or $\mu(B) = \mu(A)$. The measure space (Ω,Σ,μ) is called non-atomic if there are no atoms for μ in Σ, and purely atomic if Ω can be expressed as a union of atoms for μ. We will write $\Omega = \Omega_c \cup \Omega_d$, with $\Omega_c, \Omega_d \in \Sigma$, for the (essentially unique) decomposition of Ω into its non-atomic and purely atomic parts. Since μ is a finite measure, there exists an at most countable pairwise disjoint collection $\{A_i : i \in I\}$ of atoms for μ such that $\Omega_d = \bigcup_{i\in I} A_i$. We note that if A is an atom for μ and $f \in L(\mu,X)$, then f is constant μ - a.e. on A, and this constant is called the essential value of f on A.

If $X_1, X_2, \ldots, X_n$ are Banach spaces, and $1 \leq p \leq \infty$, then the ℓ_p - product $(X_1 \oplus X_2 \oplus \ldots \oplus X_n)_p$ is the product space $X_1 \times X_2 \times \ldots \times X_n$ equipped with the norm

$$\|(x_1, x_2, \ldots, x_n)\|_p = (\|x_1\|^p + \|x_2\|^p + \ldots + \|x_n\|^p)^{1/p}$$

for $1 \leq p < \infty$, and

$$\|(x_1, x_2, \ldots, x_n)\|_\infty = \max (\|x_1\|, \|x_2\|, \ldots, \|x_n\|)$$

for $p = \infty$.

We will need the following lemmas in the discussion of the smooth points of $L_\infty(\mu,X)$.

LEMMA 2.1: If $X_1, X_2, \ldots, X_n$ are Banach spaces, then $(x_1, x_2, \ldots, x_n)$ is a smooth point of $(X_1 \oplus X_2 \oplus \ldots \oplus X_n)_\infty$ if and only if there exists a j_0, $1 \leq j_0 \leq n$, such that

(i) $\|x_{j_0}\| > \|x_j\|$ for $j \neq j_0$, and

(ii) x_{j_0} is a smooth point of X_{j_0}.

LEMMA 2.2: Let X be a Banach space, and (Ω,Σ,μ) a finite measure space. If $(\Omega_d,\Sigma_d,\mu_d)$ and $(\Omega_c,\Sigma_c,\mu_c)$ are the purely atomic and non-atomic measure spaces, respectively, in the decomposition of Ω; then $L_\infty(\mu,X)$ is isometrically isomorphic to $(L_\infty(\mu_d,X) \oplus L_\infty(\mu_c,X))_\infty$.

The proof of the second lemma is routine, while the proof of the first lemma uses the fact that $(X_1 \oplus X_2 \oplus .. \oplus X_n)_\infty^*$ is isometrically isomorphic to $(X_1^* \oplus X_2^* \oplus \ldots \oplus X_n^*)_1$, see [3].

The next two lemmas are straightforward generalizations of results in Kothe [2], we sketch the proof of the first lemma.

LEMMA 2.3: Let X be a Banach space and let $\ell_\infty(X)$ denote the space of bounded sequences in X with the supremum norm. If $x = \{x_n\}_{n>1} \in \ell_\infty(X)$, $x \neq 0$, then x is a smooth point of $\ell_\infty(X)$ if and only if there exists a positive integer n_0 such that

(i) $\|x_{n_0}\| > \sup \{\|x_n\| : n \neq n_0\}$, and

(ii) x_{n_0} is a smooth point of X.

PROOF.

Let $x = \{x_n\}_{n\geq 1} \in \ell_\infty(X)$ be a smooth point, we may assume that $\|x\| = \sup_{n\geq 1} \|x_n\| = 1$. If there exists a subsequence $\{x_{n_k}\}_{k\geq 1}$ such that $\lim_{k\to\infty} \|x_{n_k}\| = 1$, we can demonstrate the existence of distinct elements of $\ell_\infty(X)^*$ which support the unit ball at x, by the following modification of the argument given in Köthe [2] for $\ell_\infty(\mu)$.

We consider the disjoint sequences $\{x_{n_{2j}}\}_{j\geq 1}$ and $\{x_{n_{2j-1}}\}_{j\geq 1}$. For each

$j \geq 1$, let ϕ_j and ψ_j be elements of X^* such that $\|\phi_j\| = \|\psi_j\| = 1$ with $\phi_j(x_{n_{2j}}) = \|x_{n_{2j}}\|$ and $\psi_j(x_{n_{2j-1}}) = \|x_{n_{2j-1}}\|$.

Define Φ_j and Ψ_j on $\ell_\infty(X)$ by $\Phi_j(y) = \phi_j(y_{n_{2j}})$ and $\Psi_j(y) = \psi_j(y_{n_{2j-1}})$ for all $y = \{y_n\}_{n\geq 1} \in \ell_\infty(X)$ and $j \geq 1$; then $\Phi_j, \Psi_j \in \ell_\infty(X)^*$ and $\|\Phi_j\| = \|\Psi_j\| = 1$ for all $j \geq 1$.

Let Φ and Ψ be w*-accumulation points of the sequences $\{\Phi_j\}_{j\geq 1}$ and $\{\Psi_j\}_{j\geq 1}$ respectively, then by construction we have $\|\Phi\| = \|\Psi\| = 1$ and $\Phi \neq \Psi$, but $\Phi(x) = \Psi(x) = 1 = \|x\|$. This contradicts the fact that x is a smooth point of $\ell_\infty(X)$. Thus, we have shown that if $x = \{x_n\}_{n\geq 1}$ is a smooth point of $\ell_\infty(X)$, then $\overline{\lim_{n\to\infty}} \|x_n\| < \|x\|$, and therefore there must exist a positive integer n_0 such that $\|x_{n_0}\| = \|x\|$. If there exists another integer $m_0 \neq n_0$ with $\|x_{m_0}\| = \|x\|$, let ϕ , $\psi \in X^*$ with $\|\phi\| = \|\psi\| = 1$ and $\phi(x_{n_0}) = \psi(x_{m_0}) = \|x\|$. Now define Φ , $\Psi \in \ell_\infty(X)^*$ by $\Phi(y) = \phi(y_{n_0})$ and $\Psi(y) = \psi(y_{m_0})$ for $y = \{y_n\}_{n\geq 1} \in \ell_\infty(X)$, then Φ and Ψ are distinct support functionals to the ball in $\ell_\infty(X)$ at x . Again, a contradiction. We have established that if x is a smooth point of $\ell_\infty(X)$, then (i) must hold. A similar argument shows that (ii) must hold as well.

Conversely, if $x = \{x_n\}_{n\geq 1} \in \ell_\infty(X)$ and (i) and (ii) hold, then for any $y = \{y_n\}_{n\geq 1} \in \ell_\infty(X)$, $y \neq 0$, we have $\|x + \lambda y\| = \|x_{n_0} + \lambda y_{n_0}\|$ for all $\lambda \in \mathbb{R}$ satisfying $|\lambda| < \frac{1}{\|y\|}(\|x\| - \sup\{\|x_n\| : n \neq n_0\})$. Therefore,

$$\lim_{\lambda\to 0} \frac{\|x + \lambda y\| - \|x\|}{\lambda} = \lim_{\lambda\to 0} \frac{\|x_{n_0} + y_{n_0}\| - \|x_{n_0}\|}{\lambda}$$

which exists by (ii); thus, x is a smooth point of $\ell_\infty(X)$. This completes the proof of the lemma.

An argument similar to the above gives the following:

LEMMA 2.4: Let (Ω,Σ,μ) be a finite measure space which is purely non-atomic, and let X be a real Banach space, then $L_\infty(\mu,X)$ has no smooth points.

3. MAIN RESULT

In this section, we characterize the smooth points of the space $L_\infty(\mu,X)$.

THEOREM 3.1: Let (Ω,Σ,μ) be a finite measure space, X a Banach space, and $f \in L_\infty(\mu,X)$ with $f \neq 0$; then f is a smooth point of $L_\infty(\mu,X)$ if and only if there exists an atom A_0 for μ such that

(i) $\|f\| > \text{ess sup } \{\|f(\omega)\| : \omega \in \Omega \sim A_0\}$, and

(ii) x_0 is a smooth point of X , where x_0 is the essential value of f on A_0 .

PROOF.

Suppose $f \in L_\infty(\mu,X)$, $f \neq 0$, is a smooth point of $L_\infty(\mu,X)$, then Lemma 2.4 implies that Σ contains at least one atom for μ . Let $\Omega = \Omega_c \cup \Omega_d$ be the decomposition of Ω into its non-atomic and purely atomic parts. Since, by Lemma 2.2, $L_\infty(\mu,X)$ is isometrically isomorphic to $(L_\infty(\mu_c,X) \oplus L_\infty(\mu_d,X))_\infty$, then Lemma 2.1 and the fact that f is a smooth point of $L_\infty(\mu,X)$ imply that either

1°. $\|f|_{\Omega_c}\| > \text{ess sup } \{\|f(\omega)\| : \omega \in \Omega_d\}$, and

$f|_{\Omega_c}$ is a smooth point of $L_\infty(\mu_c,X)$

or

2°. $\|f|_{\Omega_d}\| > \text{ess sup } \{\|f(\omega)\| : \omega \in \Omega_c\}$, and

$f|_{\Omega_d}$ is a smooth point of $L_\infty(\mu_d,X)$.

Now, case 1° is ruled out by Lemma 2.4, since $(\Omega_c,\Sigma_c,\mu_c)$ is a finite non-atomic measure space. Therefore, $\|f|_{\Omega_d}\| > \text{ess sup } \{\|f(\omega)\| : \omega \in \Omega_c\}$, and $f|_{\Omega_d}$ is a smooth point of $L_\infty(\mu_d,X)$.

Let $\Omega_d = \bigcup_{i\in I} A_i$, where $\{A_i : i \in I\}$ is a pairwise disjoint collection of atoms for μ , since μ is finite, then either I is finite or countably infinite. If I is finite, then $L_\infty(\mu_d,X)$ is isometrically isomorphic to $(X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\infty$, with $X_j = X$ for $j = 1, 2, \ldots, n$: while if I is countably infinite, then $L_\infty(\mu_d,X)$ is isometrically isomorphic to $\ell_\infty(X)$. In either case, it is easily seen (from Lemma 2.1 or Lemma 2.3) that there exists an atom A_0 for μ with

(i) $\|f\| > \text{ess sup } \{\|f(\omega)\| : \omega \in \Omega \sim A_0\}$, and

(ii) x_0 is a smooth point of X , where x_0 is the essential value of f on A_0 .

Conversely, suppose that $f \in L_\infty(\mu,X)$ and there exists an atom A_0 for μ in Σ such that (i) and (ii) hold. Let $\omega_0 \in A_0$ with $f(\omega_0) = x_0$; then from (i) we have $\|f\| = \|f(\omega_0)\|$. Let $\delta = \|f(\omega_0)\| - \text{ess sup } \{\|f(\omega)\| : \omega \in \Omega \sim A_0\} > 0$, and let $g \in L_\infty(\mu,X)$. If $\lambda \in \mathbb{R}$ with $0 < |\lambda| < \frac{\delta}{2\|g\|}$; then

$$\|f(\omega) + \lambda g(\omega)\| \leq \|f(\omega)\| + |\lambda| \, \|g(\omega)\| < \|f(\omega_0)\| + |\lambda| \, \|g(\omega)\| - \delta$$

μ - a.e. on $\Omega \sim A_0$, and hence

$$\|f(\omega) + \lambda g(\omega)\| \leq \|f(\omega_0)\| + |\lambda| \, \|g\| - \delta$$

μ a.e. on $\Omega \sim A_0$.

Therefore,

$$\|f(\omega) + \lambda g(\omega)\| < \|f(\omega_0)\| - \frac{\delta}{2}$$

μ - a.e. on $\Omega \sim A_0$, whenever $0 < |\lambda| < \frac{\delta}{2\|g\|}$. This implies that

$$\text{ess sup } \{\|f(\omega) + \lambda g(\omega)\| : \omega \in \Omega \sim A_0\} \leq \|f(\omega_0)\| - \frac{\delta}{2}$$

whenever $0 < |\lambda| < \frac{\delta}{2\|g\|}$. On the other hand,

$$\|f + \lambda g\| \geq \|f\| - |\lambda| \, \|g\| > \|f(\omega_0)\| - \frac{\delta}{2}$$

whenever $0 < |\lambda| < \frac{\delta}{2\|g\|}$.

Therefore,

$$\|f + \lambda g\| = \text{ess sup } \{\|f(\omega) + \lambda g(\omega)\| : \omega \in A_0\}$$

whenever $0 < |\lambda| < \frac{\delta}{2\|g\|}$.

Now, f and g are constant μ - a.e. on A_0 , so there exists an $\omega_1 \in A_0$ such that $f(\omega) + \lambda g(\omega) = f(\omega_0) + \lambda g(\omega_1)$ μ - a.e. on A_0 ; and hence $\|f + \lambda g\| = \|f(\omega_0) + \lambda g(\omega_1)\|$ when $0 < |\lambda| < \frac{\delta}{2\|g\|}$. Therefore,

$$\lim_{\lambda\to 0} \frac{\|f + \lambda g\| - \|f\|}{\lambda} = \lim_{\lambda\to 0} \frac{\|f(\omega_0) + \lambda g(\omega_1)\| - \|f(\omega_0)\|}{\lambda} ,$$

and the latter limit exists since $f(\omega_0)$ is a smooth point of X ; hence f is a smooth point of $L_\infty(\mu,X)$. This completes the proof of the Theorem.

COROLLARY 3.2: Let (Ω,Σ,μ) be a finite measure space, X a Banach space and $f \in L_\infty(\mu,X)$ with $f \neq 0$; then the norm function on $L_\infty(\mu,X)$ is Fréchet differentiable at f if and only if there exists an atom A_0 for μ such that

(i) $\|f\| > \text{ess sup } \{\|f(\omega)\| : \omega \in \Omega \sim A_0\}$, and

(ii) the norm function on X is Fréchet differentiable at x_0 , where x_0 is the essential value of f on A_0 .

This follows immediately from the proof of Theorem 3.1.

4. REPRESENTABLE OPERATORS ON $L_1(\mu,\mathbb{R})$

If X is a Banach space and (Ω,Σ,μ) is a finite measure space, then X is said to have the Radon-Nikodým property with respect to μ if and only if for every countably additive X-valued measure $m: \Sigma \to X$ which is of bounded variation and absolutely continuous with respect to μ , there exists a $g \in L_1(\mu,X)$ such that $m(E) = \int_E g(\omega)d\mu(\omega)$, for $E \in \Sigma$.

A bounded linear operator $T : L_1(\mu) \to X$ is said to be representable if and only if there exists a $g \in L_\infty(\mu,X)$ such that

$$T(f) = \int_\Omega f(\omega)g(\omega)d\mu(\omega)$$

for all $f \in L_1(\mu,\mathbb{R})$.

Let $B(L_1(\mu,\mathbb{R}),X)$ denote the Banach space of all bounded linear operators from $L_1(\mu,\mathbb{R})$ into X . For each $g \in L_\infty(\mu,X)$, define $\sigma(g) \in B(L_1(\mu,\mathbb{R}),X)$ by

$$\sigma(g)(f) = \int_\Omega f(\omega)g(\omega)d\mu(\omega) , \quad f \in L_1(\mu,\mathbb{R}) .$$

It follows from the results in Diestel and Uhl [1, p. 63], that if X has the Radon-Nikodým property with respect to μ , then σ is a linear isometry of $L_\infty(\mu,X)$ onto $B(L_1(\mu,\mathbb{R}),X)$. Using this fact and Theorem 3.1 , we get the following characterization of the points of Gateaux and Fréchet differentiability of the norm function on $B(L_1(\mu,\mathbb{R}), X)$.

THEOREM 4.1: Let X be a real Banach space and (Ω,Σ,μ) a finite measure space such that X has the Radon-Nikodým property with respect to μ. Let $T \in B(L_1(\mu,\mathbb{R}),X)$ with $T \neq 0$. The norm function on $B(L_1(\mu,\mathbb{R}),X)$ is Gateaux (Fréchet) differentiable at T if and only if there exists an atom A_0 for μ such that $0 < \mu(A_0) < \mu(\Omega)$, and

(i) $\|T\| = \frac{1}{\mu(A_0)}\|T(\chi_{A_0})\| > \frac{1}{\mu(\Omega\sim A_0)}\|T(\chi_{\Omega\sim A_0})\|$, and

(ii) $T(\chi_{A_0})$ is a point of Gateaux (Fréchet) differentiability of the norm of X

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