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A NOTE ON GENERALIZED HARMONIC-CESARO SUMMABILITY

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ABSTRACT. This note shows that conjectures proposed by G. Das and P.C. Mohapatra [1] on inclusion relations between two generalized Harmonic-Cesàro methods of summability, are true.

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1. INTRODUCTION

For any real numbers α and β, let

$$(1-z)^{-\alpha-1}\left(\log\frac{e}{1-z}\right)^{\beta}=\sum_{n=0}^{\infty}A_n^{\alpha,\beta}z^n.$$

Then A. Zygmund [2: p. 192] shows that

$$A_n^{\alpha,\beta}\simeq\frac{n^{\alpha}}{\Gamma(\alpha+1)}(\log n)^{\beta}\quad\text{if } \alpha\neq -1,-2,\dots . \tag{1.1}$$

The (Z,α,β) mean $\sigma^{\alpha,\beta}$ of a sequence $\{s_n\}$ is defined as the Nörlund mean of the sequence $\{s_n\}$, associated with the sequence $\{A_n^{\alpha-1,\beta}\}$: that is,

$$\sigma_n^{\alpha,\beta}=\sum_{k=0}^{n}A_{n-k}^{\alpha-1,\beta}s_k/A_n^{\alpha,\beta}.$$

This summability (Z,α,β) was introduced by A. Zygmund [3] and was studied recently, by G. Das and P.C. Mohapatra [1]. They have named it the generalized Harmonic-Cesàro summability, and have proposed the following conjectures in their paper [1; p.43]:

(I) If $\alpha<-1$ and $\alpha<\alpha'$, then there exists a sequence which is summable (Z,α,β) but not summable (Z,α',β').

(II) If $-1<\alpha'$ and $\beta<0$, then there exists a sequence which is summable $(Z,-1,\beta)$ but not summable (Z,α',β').

The purpose of this note is to prove that the above conjectures are true. For the proof we need the following lemma.

LEMMA 1. For any α,β,α' and β',

$$\sigma_n^{\alpha',\beta'}=\sum_{k=0}^{n}A_{n-k}^{\alpha'-\alpha-1,\beta'-\beta}A_k^{\alpha,\beta}\sigma_k^{\alpha,\beta}/A_n^{\alpha',\beta'}.$$

This is Lemma 1 of G. Das and P.C. Mohapatra [1].

2. PROOF OF CONJECTURES

PROOF OF (I). Suppose that a sequence $\{s_n\}$ summable (Z,α,β) is also summable (Z,α',β'): that is,

$$\sigma^{\alpha,\beta} \to s \quad \text{implies} \quad \sigma_n^{\alpha',\beta'} \to s.$$

Then we have by Lemma 1 and the Toeplitz Theorem [4; Theorem 2],

$$\sum_{k=0}^{n} |A_{n-k}^{\alpha'-\alpha-1,\beta'-\beta} A_k^{\alpha,\beta}/A_n^{\alpha',\beta'}| = O(1) \quad \text{as} \quad n \to \infty.$$

Hence

$$A_n^{\alpha'-\alpha-1,\beta'-\beta}\, A_0^{\alpha,\beta}/A_n^{\alpha',\beta'} = O(1) \quad \text{as } n \to \infty. \tag{2.1}$$

On the other hand, by (1.1) there exists a constant $c > 0$ such that

$$|A_n^{\alpha'-\alpha-1,\beta'-\beta} A_0^{\alpha-\beta}/A_n^{\alpha',\beta'}| \geq c\, n^{\alpha'-\alpha-1}(\log n)^{\beta'-\beta}.n^{-\alpha'}(\log n)^{-\beta'}$$

$$= c\, n^{-\alpha-1}(\log n)^{-\beta} \to \infty \quad \text{as } n \to \infty.$$

This contradicts (2.1) and the proof is complete.

PROOF OF (II). The method of the proof is similar to that of (I). If we suppose that a sequence summable $(Z,-1,\beta)$ is also summable (Z,α',β'), then, as before, we have

$$A_n^{\alpha',\beta'-\beta} A_0^{-1,\beta}/A_n^{\alpha',\beta'} = O(1) \quad \text{as } n \to \infty. \tag{2.2}$$

But, using (1.1) there exists a constant $c > 0$ such that

$$|A_n^{\alpha',\beta'-\beta} A_0^{-1,\beta}/A_n^{\alpha',\beta'}| \geq c\, n^{\alpha'}(\log n)^{\beta'-\beta}.n^{-\alpha'}(\log n)^{-\beta'}$$

$$= c\, (\log n)^{-\beta} \to \infty \quad \text{as } n \to \infty.$$

This contradicts (2.2) and the proof is complete.

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