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THE DUAL OF THE MULTIPLIER ALGEBRA OF PEDERSEN'S IDEAL

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ABSTRACT. It is shown that the dual of the multiplier algebra of Pederson's ideal is not always spanned by its positive elements.

KEY WORDS AND PHRASES. Multiplier algebra, Pederson's ideal.

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1. INTRODUCTION.

In [1] Lazar and Taylor study the multiplier algebra $\Gamma(K)$ of Pedersen's minimal dense ideal K of a C*-algebra A. Equipped with its canonical strict topology, $\Gamma(K)$ is a locally convex space, and Lazar and Taylor have demonstrated that the dual $\Gamma(K)'$ can be identified with the set of all linear functionals $A \cdot F + G \cdot A$ where $A \in K^+$ and $F, G \in A'$ ([1] 6.1). They have also shown that, under this identification each positive element of $\Gamma(K)'$ is of the form $A^* \cdot F \cdot A$ for some $A \in K$ and $F \in A'$ ([1] 6.5). This note answers negatively their question, whether or not $\Gamma(K)'$ is the span of its positive elements.

2. MAIN RESULTS.

Let H denote the Hilbert space $\ell_2(\mathbf{Z})$, $<,>$ its inner product, and $\{b_n\}_{n \in \mathbf{Z}}$ its canonical Hilbert basis. For vectors $v, w \in H$, let $v \otimes w^*$ denote the linear operator sending each $x \in H$ to $<x,w>v$. Denote by $(B, *, \| \|_B)$ the C*-algebra of all bounded linear transformations of H. The identity transformation I has a decomposition P + Q where P is the orthogonal projection of H onto $\ell_2(\mathbf{N})$ and Q the orthogonal projection onto $\ell_2(Z/\mathbf{N})$.

Let A be the C*-algebra of all bounded sequences $A \mid \mathbf{N} \to B$ such that

$$\lim_{n\to\infty} \|P A_n\|_B + \|A_n P\|_B = 0. \tag{2.1}$$

We write $\| \ \|$ for the norm on A:

$$\|A\| = \sup_{n \in \mathbf{N}} \|A_n\|_B \qquad (\forall\ A \in A).$$

Let K be the set of all $A \in A$ such that

$$\{n \in \mathbf{N} : \|P A_n\| + \|A_n P\| \neq 0\} \text{ is finite.} \tag{2.2}$$

PROPOSITION. Pedersen's ideal in A is just K.

PROOF. That K is an ideal is trivial.

For each $A \in A$ and $n \in \mathbf{N}$, let $A^{(n)}$ be the element of K defined by

$A_m{}^{(n)} = A_m$ $(\forall m = 1,2,\ldots,n)$, $A_k{}^{(n)} = QA_kQ$ $(\forall k = n + 1,\ n + 2,\ \ldots)$.

Then, for each $A \in A$,

$$\overline{\lim_n}\ \|A - A^{(n)}\| = \overline{\lim_n}\ \sup_{k>n}\ \|A_k - QA_kQ\|_B \leq$$

$$\overline{\lim_n}\ \sup_{k>n}\ \|PA_kQ\|_B + \|QA_kP\|_B + \|PA_kP\|_B = 0$$

by (2.1), which proves that K is dense in A.

Since the minimal dense ideal of A contains all positive elements $A \in A$ such that $AB = A$ for some $B \in A^+$, it will suffice in showing K is Pedersen's ideal to demonstrate that K is spanned by elements of this sort. Since K is evidently spanned by its positive elements, it will be sufficient to examine an arbitrary positive element A of K. For such $A \in K^+$, there exists $n \in N$ such that $PA_m = 0 = A_mP$ for all $m > n$. Let $B \in A^+$ be defined by

$$B_k \equiv I \quad (\forall k = 1,2,\ldots,n),\ B_m \equiv Q \quad (\forall m = n + 1,\ n + 2,\ \ldots).$$

Then, since $A_m = QA_mQ$ for all $m > n$, $AB = A$. Q.E.D.

Let F be the bounded linear functional on A defined by

$$F(A) \equiv \sum_{n=1}^{\infty} < A_n(b_1),\ b_{-1} > \cdot\ 2^{-n} \qquad (\forall A \in A).$$

Let $D \in K$ be the partial unitary operator defined by

$$D_n \equiv Q \qquad (\forall n \in \mathbf{N}).$$

For $G \in A'$ and $A \in A$, the linear functionals $A \cdot G$ and $A \cdot G \cdot A$ are defined by

$$A \cdot G(B) \equiv G(AB) \text{ and } A \cdot G \cdot A(B) \equiv G(ABA) \qquad (\forall B \in A).$$

Let S be the linear span of the set of all elements of A' of the form $A^* \cdot G \cdot A$ such that $A \in K^+$ and $G \in A'$.

THEOREM. The linear functional $D \cdot F$ is not in S.

PROOF. Assume false. Then there exists a finite subset F of K^+ and a map $G|F \to A'$ such that

$$D \cdot F = \sum_{A \in F} A^* \cdot G_A \cdot A. \tag{2.3}$$

Choose $n \in \mathbb{N}$ such that

$$PA_m = A_m P = 0 \qquad (\forall A \in F;\ m = n, n + 1, \ldots).$$

Let $B \in A$ be defined by

$$B_n \equiv b_{-1} \otimes b_1^* \quad \text{and} \quad B_k \equiv 0 \qquad (\forall k \in \mathbb{N} : k \neq n),$$

and note that

$$QB_n = B_n \quad \text{and} \quad QB_nQ = 0.$$

We have

$$D \cdot F(B) = F(DB) = \langle (DB)_n(b_1), b_{-1} \rangle 2^{-n} = 2^{-n} \neq 0. \tag{2.4}$$

For each $A \in F$, we have $A_n = QA_nQ$; therefore, since $(b_{-1} \otimes b_1^*)Q = 0$, it follows that $B_nA_n = 0$. Thus

$$BA = 0 \quad \text{and}$$

$$A^* \cdot G_A \cdot A(B) = G_A(A^*BA) = 0 \qquad (\forall A \in F). \tag{2.5}$$

But (2.3), (2.4), and (2.5) are incompatible. Q.E.D.

REFERENCE

[1] Lazar, A. J., and Taylor, D. C., Multipliers of Pedersen's Ideal, Mem. Am. Math. Soc. 169, (1976).