Internat. J. Math. & Math. Sci.
Vol. 3 No. 3 (1980) 423-432

GENERALIZED KÖTHE-TOEPLITZ DUALS

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(Received November 9, 1979)

ABSTRACT. The α and β-duals spaces of generalized ℓ_p spaces are characterized, where $0 < p \leq \infty$. The question of when the α and β dual spaces coincide is also considered.

KEY WORDS AND PHRASES. Generalized Köthe-Toeplitz dual spaces, Sequences of linear operators, Generalized ℓ_p spaces.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. 40C05, 40J05.

1. INTRODUCTION.

X and Y denote complex Banach spaces with zero elements Θ, and $||.||$ denotes the norm in either X or Y. The continuous dual of X is written X*. By s(X) we mean the space of all X-valued sequences $x = (x_k)$, where $x_k \in X$ for $k \in N = \{1,2,3,...\}$.

If $0 < p < \infty$, we mean by $\ell_p(X)$ the space of all X-valued sequences $x = (x_k)$ such that $\Sigma||x_k||^p < \infty$. Sums are over $k \in N$, unless otherwise indicated.

By $\ell_\infty(X)$ we denote the space of all $x = (x_k)$ such that $\sup ||x_k|| < \infty$.

In case $X = C$, the space of complex numbers, we write ℓ_p instead of $\ell_p(C)$.

Let $A = (A_k)$ denote a sequence of linear, but not necessarily bounded, operators on X into Y. If E is any nonempty subset of $s(X)$ then the α-dual of E is defined to be

$$E^\alpha = \{A : \Sigma||A_k x_k|| < \infty, \text{ for all } x \in E\}.$$

The β-dual of E is defined to be

$$E^\beta = \{A : \Sigma A_k x_k \text{ converges, for all } x \in E\}.$$

Since Y is complete, we have $E^\alpha \subset E^\beta$. The α and β duals of E may be regarded as generalized Köthe-Toeplitz duals, since in case $X = Y = C$, when the A_k may be identified with complex numbers a_k, the duals reduce to the classical spaces first considered by Köthe and Toeplitz [1].

Using the notation $(1/p) + (1/q) = 1$, where $1 \le p \le \infty$, with the convention that $q = \infty$ when $p = 1$, and $q = 1$ when $p = \infty$, it is well-known that

$$\ell_p^\alpha = \ell_p^\beta = \ell_q. \tag{1.1}$$

We shall see that, in general, $\ell_p^\alpha(X) \subset \ell_p^\beta(X)$, where the inclusion may be strict. However, when $0 < p \le 1$ the α and β duals coincide. Also, when $1 < p \le \infty$, the α and β duals coincide provided that Y is finite dimensional.

2. CHARACTERIZATION OF THE DUALS.

THEOREM 1. *Let* $0 < p \le 1$. *Then* $A \in \ell_p^\beta(X)$ *if and only if there exists* $m \in N$ *such that* A_k *is bounded, for all* $k \ge m$, *and*

$$H = \sup_{k\ge m} ||A_k|| < \infty. \tag{2.1}$$

PROOF. *Sufficiency*. Let (2.1) hold and $\Sigma||x_k||^p < \infty$. By a familiar inequality, see for example Maddox [2], page 22,

$$\left(\sum_{k=m}^{\infty} ||A_k x_k||\right)^p \le \sum_{k=m}^{\infty} ||A_k x_k||^p$$

$$\le \sum_{k=m}^{\infty} ||A_k||^p ||x_k||^p$$

$$\le H^p \; \Sigma||x_k||^p.$$

Hence $\Sigma A_k x_k$ is absolutely convergent, and so convergent.

Necessity. Let $A \in \ell_p^\beta(X)$ and suppose, if possible, that no such m exists. Then there are natural numbers $k(1) < k(2) < \dots$ and $z_i \in X$, $||z_i|| \le 1$, such that for $i \in N$,

$$||A_{k(i)} z_i|| > i^{2/p}. \tag{2.2}$$

Define $x_k = z_i/i^{2/p}$ for $k = k(i)$ and $x_k = \theta$ otherwise. Then $x \in \ell_p(X)$ since $\Sigma||x_k||^p \le \pi^2/6$, but $||A_k x_k|| > 1$ for infinitely many k, contrary to the fact that $\Sigma A_k x_k$ converges.

Now suppose, if possible, that $\sup_{k\ge m}||A_k|| = \infty$. Then there are natural numbers $k(1) < k(2) < \dots$ with $k(1) \ge m$ such that for $i \in N$,

$$||A_{k(i)}|| > 2i^{2/p}. \tag{2.3}$$

Choose $z_i \in X$ with $||z_i|| \le 1$ such that $2||A_{k(i)} z_i|| \ge ||A_{k(i)}||$, so by (2.3) we see that (2.2) holds with the new $k(i)$ and z_i. We may define $x \in \ell_p(X)$ as above and obtain a contradiction. Hence (2.1) must hold, and the proof is complete.

If we examine the proof of Theorem 1 we see that in the sufficiency we had $\Sigma||A_k x_k|| < \infty$, so that $A \in \ell_p^\alpha(X)$. Also, in the necessity, the constructions involved $x \in \ell_p(X)$ such that $\Sigma||A_k x_k||$ was divergent. Hence we have:

THEOREM 2. _If_ $0 < p \le 1$ _then_

$$\ell_p^\alpha(X) = \ell_p^\beta(X).$$

Next we consider the case $1 < p < \infty$.

THEOREM 3. _Let_ $1 < p < \infty$. _Then_ $A \in \ell_p^\alpha(X)$ _if and only if there exists_ $m \in N$ _such that_ A_k _is bounded for all_ $k \ge m$, _and_

$$M = \sum_{k=m}^{\infty} ||A_k||^q < \infty. \tag{2.4}$$

PROOF. _Sufficiency_. Let (2.4) hold and $x \in \ell_p(X)$. By Hölder's inequality,

$$\sum_{k=m}^{\infty} ||A_k x_k|| \le M^{1/q} \left(\Sigma||x_k||^p\right)^{1/p} < \infty.$$

Necessity. Since $\ell_p^\alpha(X) \subset \ell_1^\alpha(X)$ when $p > 1$, the existence of the m in the theorem follows from Theorems 1 and 2.

Now for $k \ge m$ we may choose $z_k \in X$ with $||z_k|| \le 1$ such that $2||A_k z_k|| \ge ||A_k||$.

For all $\lambda \in \ell_p$ we have $(\lambda_k z_k) \in \ell_p(X)$, so

$$\sum_{k=m}^{\infty} |\lambda_k| \, ||A_k z_k|| < \infty$$

for all $\lambda \in \ell_p$. By (1.1) it follows that

$$H = \sum_{k=m}^{\infty} ||A_k z_k||^q < \infty,$$

whence $M \leq 2^q H$, so (2.4) holds, and the proof is complete.

THEOREM 4. *Let* $1 < p < \infty$. *Then* $A \in \ell_p^{\beta}(X)$ *if and only if there exists* $m \in N$ *such that* A_k *is bounded for all* $k \geq m$, *and*

$$\sup \sum_{k=m}^{\infty} ||A_k^* f||^q < \infty, \tag{2.5}$$

where the supremum is over all $f \in Y^*$ *with* $||f|| \leq 1$.

PROOF. With the restriction that all the A_k are bounded, and with different notation, this result was proved by Thorp [3]. Only the existence of m in the necessity needs attention, and this follows from Theorems 1 and 2, and the fact that $\ell_p^{\beta}(X) \subset \ell_1^{\beta}(X)$.

Finally, we examine the case $p = \infty$. The proofs are left to the reader. We remark that with the restriction that all the A_k are bounded, the result concerning $\ell_\infty^{\beta}(X)$ was given by Maddox [4].

THEOREM 5. $A \in \ell_\infty^{\alpha}(X)$ *if and only if there exists* $m \in N$ *such that* A_k *is bounded for all* $k \geq m$, *and*

$$\sum_{k=m}^{\infty} ||A_k|| < \infty . \tag{2.6}$$

THEOREM 6. $A \in \ell_\infty^{\beta}(X)$ *if and only if there exists* $m \in N$ *such that* A_k *is bounded for all* $k \geq m$, *and*

$$\sup \left\| \sum_{k=m}^{m+n} A_k x_k \right\| < \infty, \tag{2.7}$$

$$\sup \left\| \sum_{k=m}^{m+n} A_k x_k \right\| \to 0 \ (m \to \infty), \tag{2.8}$$

where the suprema are over all $n \geq 0$ *and all* $x_k \in X$ *with* $\|x_k\| \leq 1$.

3. COINCIDENCE OF DUALS.

It was shown in Theorem 2 that, when $0 < p \leq 1$, $\ell_p^\alpha(X) = \ell_p^\beta(X)$ for any Banach spaces X and Y.

We next shown that, when $1 < p < \infty$, the inclusion $\ell_p^\alpha(X) \subset \ell_p^\beta(X)$ may be strict.

THEOREM 7. *If* $1 < p < \infty$ *then there are Banach spaces* X *and* Y *such that* $\ell_p^\alpha(X) \subset \ell_p^\beta(X)$ *with strict inclusion*.

PROOF. Take $X = Y = \ell_p$ and write

$$e_k = (0,0,\ldots,1,0,0,\ldots)$$

where 1 is in the k-place and there are zeros elsewhere. Define bounded linear operators A_k on ℓ_p into itself by

$$A_k x = x_k e_k$$

for each $x = (x_k) \in \ell_p$. Then $\|A_k\| = 1$ for all $k \in N$, so A is not in $\ell_p^\alpha(X)$ by Theorem 3.

Let us now show that (2.5) holds. Take any $f \in \ell_p^*$ with $\|f\| \leq 1$. Then for $x \in \ell_p$ we have

$$f(x) = \Sigma f_i x_i$$

for some (f_i) such that $\Sigma|f_i|^q \leq 1$. Hence, by definition of A_k^*,

$$(A_k^* f)(x) = f(A_k x) = f_k x_k$$

and so $||A_k^* f|| = |f_k|$. Hence

$$\Sigma||A_k^* f||^q = \Sigma|f_k|^q \leq 1,$$

so by Theorem 4 we have $A \in \ell_p^\beta(X)$.

Still with the case $1 < p < \infty$ we have:

THEOREM 8. *If* $1 < p < \infty$ *and* Y *is finite dimensional then for any* X *we have*

$$\ell_p^\alpha(X) = \ell_p^\beta(X).$$

PROOF. We have to show that $A \in \ell_p^\beta(X)$ implies $A \in \ell_p^\alpha(X)$. Now if $A \in \ell_p^\beta(X)$ then by Theorem 4 there exists $m \in N$ such that A_k is bounded for all $k \geq m$. Suppose Y has finite dimension n and that $(b_1, b_2, \ldots, b_n)$ is a Hamel base for Y. Then $y \in Y$ implies

$$y = \sum_{i=1}^{n} \lambda_i(y) b_i$$

where each $\lambda_i \in Y^*$. Take $z \in X$ and $k \geq M$. Then

$$A_k z = \sum_{i=1}^{n} \lambda_i(A_k z) b_i \tag{2.9}$$

and $\lambda_i \bullet A_k \in X^*$. Since $\sum_{k=m}^{\infty} A_k x_k$ converges for all $x \in \ell_p(X)$ we have

$$\sum_{k=m}^{\infty} (\lambda_i \circ A_k) x_k$$

convergent for all $x \in \ell_p(X)$ and each i.

Choose $z_k \in X$, $||z_k|| \leq 1$ such that $2|(\lambda_i \circ A_k)z_k| \geq ||\lambda_i \bullet A_k||$. If $t \in \ell_p$ then $(t_k z_k) \in \ell_p(X)$ so that

$$\sum_{k=m}^{\infty} t_k(\lambda_i \bullet A_k)z_k$$

converges for all $t \in \ell_p$, whence for each i,

$$\sum_{k=m}^{\infty} ||\lambda_i \bullet A_k||^q < \infty. \tag{2.10}$$

By (2.9) and Hölder's inequality,

$$||A_k||^q \leq \sum_{i=1}^{n} ||\lambda_i \bullet A_k||^q . \quad (\sum_{i=1}^{n} ||b_i||^p)^{q/p}. \tag{2.11}$$

Denoting the final term in (2.11) by H,

$$\sum_{k=m}^{\infty} ||A_k||^q \leq H \sum_{i=1}^{n} \sum_{k=m}^{\infty} ||\lambda_i \circ A_k||^q. \tag{2.12}$$

It follows from (2.10) and (2.12) that (2.4) holds, so by Theorem 3 we have $A \in \ell_p^\alpha(X)$.

For certain values of p, and any X, the next result is the converse of Theorem 8.

THEOREM 9. _If_ $2 < p < \infty$ _and_ $\ell_p^\alpha(X) = \ell_p^\beta(X)$ _then_ Y _must be finite dimensional_.

PROOF. Suppose, if possible, that Y is infinite dimensional. Since $q < 2$, if $c_k = k^{-2/q}$ then $\Sigma c_k < \infty$. By the Dvoretzky-Rogers theorem [5], there exists an unconditionally convergent series Σy_k in Y such that $||y_k||^2 = c_k$ for $k \in N$. Hence

$$\Sigma ||y_k||^q \text{ diverges.} \tag{2.13}$$

Take $f \in X^*$ with $||f|| = 1$ and define rank one operators $A_k = y_k \otimes f$. Then $||A_k|| = ||y_k||$, so by (2.13) and Theorem 3, A is not in $\ell_p^\alpha(X)$.

Now if $x \in \ell_p(X)$ then

$$\Sigma A_k x_k = \Sigma f(x_k) y_k.$$

But $(f(x_k)) \in \ell_\infty$ and Σy_k is unconditionally convergent, so that $\Sigma f(x_k) y_k$ converges, whence $A \in \ell_p^\beta(X)$, which gives a contradiction.

We remark that it would appear that the argument of Theorem 9 cannot be used in the case p = 2, since in a general Hilbert space Y the unconditional convergence of Σy_k implies that $\Sigma ||y_k||^2$.

However, we can deal with the case p = 2 of Theorem 9 when Y is a Hilbert space:

THEOREM 10. Let Y be a Hilbert space and suppose $\ell_2^\alpha(X) = \ell_2^\beta(X)$. Then Y must be finite dimensional.

PROOF. Suppose, if possible, that Y is infinite dimensional. Choose an orthonormal sequence (e_k) in Y and denote the inner product in Y by (y_1, y_2). Take $g \in X^*$, $||g|| = 1$ and define rank one operators $A_k = e_k \otimes g$, so that $||A_k|| = 1$. Now let $f \in Y^*$ with $||f|| \le 1$. Then there exists $y \in Y$ such that

$$f(z) = (z, y)$$

for all $z \in Y$, with $||y|| = ||f|| \le 1$. Then for $x \in X$,

$$(A_k^* f)(x) = (g(x)e_k, y) = g(x)(e_k, y).$$

Hence $||A_k^* f|| \le |(e_k, y)|$, so by Bessel's inequality,

$$\Sigma ||A_k^* f||^2 \le ||y||^2 \le 1.$$

Thus (2.5) holds with q = 2, and so $A \in \ell_2^\beta(X)$. But $A \notin \ell_2^\alpha(X)$ since $||A_k|| = 1$ for all k. This contradiction implies our result.

The case $p = \infty$ is due essentially to Thorp [3], who shows that $\ell_\infty^\alpha(X) = \ell_\infty^\beta(X)$ if and only if Y is finite dimensional.

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