RANDOM SUBGRAPHS OF CERTAIN GRAPH POWERS

LANE CLARK

Received 4 March 2002

We determine the limiting probability that a random subgraph of the Cartesian power $K_n^a$ or of $K_{n,a}$ is connected.

2000 Mathematics Subject Classification: 05C80.

1. Introduction. A finite, simple, undirected graph $G$ has vertex set $V(G)$ and edge set $E(G)$. The order of $G$ is $|V(G)|$ and the size of $G$ is $|E(G)|$. For $S \subseteq V(G)$, let $G[S]$ denote the subgraph of $G$ induced by $S$ and $G[S, V(G) - S]$ denote the spanning subgraph of $G$ with edges $xy$ where $x \in S$ and $y \in V(G) - S$. For $U \subseteq V(G)$, let $NG(U) = \{y \in V(G) : \exists x \in E(G) \text{ with } x \in U\}$ and $\tilde{NG}(U) = NG(U) \cup U$. Of course, $NG(v) = NG(\{v\})$ and the degree $d_G(v)$ of $v$ in $G$ is $|NG(v)|$ for $v \in V(G)$. For $S \subseteq V(G)$, let $b_G(S) = |\{xy \in E(G) : x \in S, y \in V(G) - S\}|$ and $b_G(s) = \min\{b_G(S) : S \subseteq V(G), |S| = s\}$ ($0 \leq s \leq |V(G)|$).

The Cartesian product $G \square H$ of graphs $G$ and $H$ is the graph with vertex set $V(G) \times V(H)$ where vertices $(g_1, h_1)$ and $(g_2, h_2)$ are adjacent if and only if $g_1 = g_2$ and $h_1 h_2 \in E(H)$, or, $h_1 = h_2$ and $g_1 g_2 \in E(G)$. For a graph $G$, define $G^1 = G$ and $G^n = G^{n-1} \square G$ for $n \geq 2$. We use the following recent isoperimetric result of Tillich [6]. Here $K_a$ denotes the complete graph of order $a$ and $K_{a,a}$ denotes the complete bipartite graph with parts of order $a$.

**Lemma 1.1 (see [6]).** For $G = K_n^a$ with $a \geq 2$ and $n \geq 1$,

$$b_G(s) \geq (a-1)s(n - \log_a s) \quad \text{for } 1 \leq s \leq a^n \quad (1.1)$$

and, for $G = K_{n,a}$ with $a \geq 1$ and $n \geq 1$,

$$b_G(s) \geq as(n - \log_{2a} s) \quad \text{for } 1 \leq s \leq (2a)^n. \quad (1.2)$$

Let $G$ be a graph of order $n$ and size $N$. The probability space $\mathcal{G}(G, p)$ consists of all spanning subgraphs $H$ of $G$ where edges of $G$ are chosen for $H$ independently with probability $0 \leq p = p(n) \leq 1$, so that, $\Pr(H) = p^{e(H)}q^{N-e(H)}$ where $q = q(n) = 1-p(n)$. (We denote the random graphs in $\mathcal{G}(G, p)$ generally by $G_p$.)

In this paper, we determine the limiting probability that $G_p$ is connected for $G = K_n^a$ and $K_{n,a}$. Specifically, we show that

$$\lim_{n \to \infty} \Pr(G_p \in \mathcal{G}(K_n^a, p) \text{ is connected}) = e^{-\lambda} \quad (1.3)$$
for fixed $a \geq 2$ where $p = p(n) = 1 - q(n)$ with $q(n) = \lfloor (\lambda(n))^{1/n}/a \rfloor^{1/(a-1)}$ and $\lim_{n \to \infty} \lambda(n) = \lambda \in (0, \infty)$. In addition, we show that

$$\lim_{n \to \infty} \Pr \left( G_p \in \mathcal{G}(K_{a,a}^n, p) \text{ is connected} \right) = e^{-\lambda} \tag{1.4}$$

for fixed $a \geq 1$ where $p = p(n) = 1 - q(n)$ with $q(n) = \lfloor (\lambda(n))^{1/n}/2a \rfloor^{1/a}$ and $\lim_{n \to \infty} \lambda(n) = \lambda \in (0, \infty)$. Our first result includes those of Burtin [3], Erdős and Spencer [5], and Bollobás [1] as a special case ($a = 2$). Our approach is similar to [1].

The $r$th factorial moment of a random variable (r.v.) $X$ is denoted by $E_r(X)$. We refer the reader to Bollobás [2] for random graph theory and Durrett [4] for probability.

2. Results. We use the following result from [1].

**Lemma 2.1** (see [1]). If $G$ is a simple graph having order $n \geq 1$, maximum degree $\Delta(G) \leq \Delta$, average degree $d = d(G) = 2e(G)/n$, and $\Delta + 1 < u < n - \Delta - 1$, then there exists a $u$-set $U \subseteq V(G)$ with

$$|\tilde{N}_G(U)| \geq n \frac{d}{\Delta} \left( 1 - \exp \left( - \frac{u(\Delta + 1)}{n} \right) \right). \tag{2.1}$$

Assume $n \geq 2\Delta + 4$, since the result is vacuously true otherwise, and $\Delta > 0$ (the right-hand side is defined to be 0 for $\Delta = 0$).

We first consider $G = K_a^n$ with $V(G) = [a]^n$ for fixed $a \geq 2$ and for $n \geq 3$. Note that $V(G)$ is totally ordered lexicographically which naturally extends to $u$-subsets of $V(G)$. In **Lemma 2.2** and **Theorem 2.5**, $\lambda(n) > 0$ for all $n$.

**Lemma 2.2.** For fixed $a \geq 2$, $q = q(n) = \lfloor (\lambda(n))^{1/n}/a \rfloor^{1/(a-1)}$ where $\lim_{n \to \infty} \lambda(n) = \lambda \in (0, \infty)$, and $p = p(n) = 1 - q(n)$, we have

$$\lim_{n \to \infty} \Pr \left( G_p \in \mathcal{G}(K_{a,a}^n, p) \text{ has no isolated vertices} \right) = e^{-\lambda} \tag{2.2}$$

**Proof.** Recall that $G = K_a^n$. Let $X_n(G_p)$ denote the number of isolated vertices in $G_p$. Fix $r \in \mathbb{P}$ and let $\mathcal{A}_r$ denote the set of $r$-tuples of $V$ with distinct coordinates; $\mathcal{B}_r = \{(v_1, \ldots, v_r) \in \mathcal{A}_r : e(G[\{v_1, \ldots, v_r\}] \neq 0)\}$ and $\mathcal{C}_r = \mathcal{A}_r - \mathcal{B}_r = \{(v_1, \ldots, v_r) \in \mathcal{A}_r : e(G[\{v_1, \ldots, v_r\}] = 0)\}$. Then $|\mathcal{B}_r| \leq (a^n)_{r-1} ran \leq a^{n(r-1)} ran$ and $|\mathcal{C}_r| = (a^n)_r - |\mathcal{B}_r| = a^n e^{-r^2/an} - a^{n(r-1)} ran$. Observe that the number of edges in $G$ incident with $\{v_1, \ldots, v_r\}$ is at least $(a - 1)r(n - r)$ for all $(v_1, \ldots, v_r) \in \mathcal{A}_r$. 
First,
\[ 0 \leq \sum_{(v_1, \ldots, v_r) \in \mathcal{F}_r} \Pr(d_{G_p}(v_1) = \cdots = d_{G_p}(v_r) = 0) \leq |\mathcal{B}_r| q^{(a-1)r^2/(n-r)} \]
\[ \leq an^{(r-1)}ran \frac{(\lambda(n))^{r-r^2/n}}{a^{r(n-r)}} = \frac{(\lambda(n))^{r-r^2/n} ran}{an^{r^2}}. \]
(2.3)

Next,
\[ \sum_{(v_1, \ldots, v_r) \in \mathcal{F}_r} \Pr(d_{G_p}(v_1) = \cdots = d_{G_p}(v_r) = 0) \]
\[ = |\mathcal{C}_r| q^{(a-1)nr} \]
\[ \geq \left[ a^{nr} e^{-r^2/an} - an^{r-1} ran \right] \frac{\lambda(r)(n)^r}{an} \]
\[ = \lambda^r(n)e^{-r^2/an} - \frac{\lambda^r(n)ran}{an}. \]
(2.4)
while,
\[ \sum_{(v_1, \ldots, v_r) \in \mathcal{F}_r} \Pr(d_{G_p}(v_1) = \cdots = d_{G_p}(v_r) = 0) \leq an^{rq}(a^{-1}n) = \lambda^r(n). \]
(2.5)

Hence,
\[ \lambda^r(n)e^{-r^2/an} - \frac{\lambda^r(n)ran}{an} \leq E_r(X_n) \leq \lambda^r(n) + \frac{(\lambda(n))^{r-r^2/n} ran}{an^{r^2}} \]
(2.6)
so that,
\[ \lim_{n \to \infty} E_r(X_n) = \lambda^r \]
(2.7)
and \(X_n \overset{d}{\to} P_\lambda\) (see [4]).

**Lemma 2.3.** For fixed \(a \geq 2\), \(q = q(n) = [(\ln n)^{1/a}]^{1/(a-1)}\), and \(p = p(n) = 1 - q(n)\), we have
\[ \Pr(G_p \in \mathcal{G}(K^a_n, p) has a component of order s with 2 \leq s \leq an^2/2) = o(1) \quad as \quad n \to \infty. \]
(2.8)

**Proof.** Recall that \(G = K^a_n\). Let \(\mathcal{A}_s = \{S \subseteq V(G) : |S| = s\} (1 \leq s \leq an)\). We consider four cases.

**Case 1** \((2 \leq s \leq s_1 = [an^2/n])\). We have
\[ |\{S \in \mathcal{A}_s : G[S] is connected\}| \leq an^{a-1} \cdot 2(a-1) \cdot n \cdot \cdots \cdot (s-1)(a-1)n \]
\[ \leq a^{n+s} n^{s-1}s^s, \]
(2.9)
so that **Lemma 1.1**
\[ \sum_{S \in \mathcal{A}_s} \Pr(G_p[S] is a component) \leq a^{n+s} n^{s-1}s^s q_{K^a_n(S)} \]
\[ = a^{n+s} n^{s-1}s^s \left[ \frac{\ln n}{a} \right]^{s(n-\log d_s)} \]
\[ = \frac{1}{n} \left[ \frac{ans^2 \ln n}{a^{n(1-1/s)}} \right]^s. \]
(2.10)
By examining the derivative \( f'(s) \ln(ce^2s^2/a^n) \) with respect to \( s \) of \( f(s) = c^s s^{2s}/a^{n(s-1)} \) with \( c = an \ln n \), we see that \( f(s) \) is decreasing for \( s \in [2, a^{n/2}/ec^{1/2}] \). Here \( f(s) \leq f(2) = 16a^2n^2 \ln^2 n/a^n \). Hence,

\[
\sum_{s=2}^{s_1} \sum_{S \in \mathcal{B}_s} \Pr(G_p[S] \text{ is a component}) \leq \sum_{s=2}^{s_1} \frac{16a^2n \ln^2 n}{a^n} = o(1) \quad \text{as } n \to \infty. \tag{2.11}
\]

**Case 2** \((s_1 + 1 \leq s \leq s_3 = \lfloor a^n/2 \rfloor)\). Let \( \mathcal{B}_s = \{S \in \mathcal{A}_s : b_G(S) \geq (a - 1)s(n - \log_a(s/n))\} \) and \( \mathcal{C}_s = \mathcal{A}_s - \mathcal{B}_s = \{S \in \mathcal{A}_s : b_G(S) < (a - 1)s(n - \log_a(s/n))\} \).

First,

\[
\sum_{S \in \mathcal{B}_s} \Pr(G_p[S] \text{ is a component}) \leq \left( \frac{a^n}{s} \right)^{(a-1)s(n-\log_a(s/n))} \leq \left( \frac{e^{an}}{s} \right)^{\frac{(\ln n)^{1/n}}{a} s(n-\log_a(s/n))} = \left[ \frac{e^{(\ln n)^{1-(1/n)\log_a(s/n)}}}{n} \right]^s \leq \left( \frac{e^{\ln n}}{n} \right)^s. \tag{2.12}
\]

Hence,

\[
\sum_{s=s_1+1}^{s_3} \sum_{S \in \mathcal{B}_s} \Pr(G_p[S] \text{ is a component}) \leq \sum_{s=s_1+1}^{s_3} \left( \frac{e^{\ln n}}{n} \right)^s = o(1) \quad \text{as } n \to \infty. \tag{2.13}
\]

Next, for \( S \in \mathcal{C}_s \), let \( H = G[S] \). Then

\[
(a - 1)s n = \sum_{v \in S} d_G(v) = 2e(H) + b_G(S) < 2e(H) + (a - 1)s \left( n - \log_a \frac{s}{n} \right), \tag{2.14}
\]

so that

\[
2e(H) \geq (a - 1)s \log_a \frac{s}{n} \tag{2.15}
\]

and the average degree \( d \) in \( H \) satisfies

\[
d > (a - 1) \log_a \frac{s}{n}. \tag{2.16}
\]

**Case 3** \((s_1 + 1 \leq s \leq s_2 = \lfloor a^n/\ln^2 n \rfloor)\). Let \( u = \lfloor s/n \rfloor \), so that \((a - 1)n + 1 < u < s - (a - 1)n - 1\), and by Lemma 2.1, for sufficiently large \( n \), there exists \( U \subseteq S, |U| = u \), and

\[
|\tilde{N}_H(U)| \geq \frac{s}{n} \log_a \frac{s}{n} \left\{ 1 - \exp \left( -\frac{u[(a - 1)n + 1]}{s} \right) \right\} \geq \frac{\delta s}{n} \log_a \frac{s}{n} \quad \text{with } \delta = 1 - e^{-1} = 0.631. \tag{2.17}
\]

Let \( t = \lfloor (\delta s/n) \log_a(s/n) \rfloor \), so that \( u < t < s \), and let \( w = s - t = s(1 - x) - \tau \) with \( x = (\delta/n) \log_a(s/n) \) and \( 0 \leq \tau < 1 \). Observe that \( \delta/4 \leq x \leq \delta \) here. For sufficiently
large \( n \), take the smallest such \( u \)-set \( U = \{d_1, \ldots, d_u\} \) in \( S (\subseteq V(G)) \) which is totally ordered; take the (uniquely determined) first \( t - u \) vertices of \((N_G(d_1) \cap (S - U)) \cup \cdots \cup (N_G(d_u) \cap (S - U)) \) \((\subseteq V(G))\); and add the remaining \( w \) vertices \( W \) of \( S \). Then

\[
S \mapsto (\{d_1, \ldots, d_u\}; N_G(d_1) \cap (S - U), \ldots, N_G(d_u) \cap (S - U); W)
\tag{2.18}
\]

is an injection. Hence,

\[
\left| \mathcal{E}_s \right| \leq \left( \frac{a^n}{u} \right)^{2(a-1)nu} \left( \frac{a^n}{w} \right)^w \leq \left( \frac{ea^n}{u} \right)^u \left( \frac{ea^n}{w} \right)^w \leq \left( \frac{ena^n}{s} \right)^{s/n} 2^{(a-1)s} \left( \frac{ea^n}{s(1-x)} \right)^{s(1-x)}.
\tag{2.19}
\]

Then (where \( x - 1/n \geq 0 \), Lemma 1.1)

\[
\sum_{S \in \mathcal{E}_s} \Pr (G_p[S] \text{ is a component}) \leq \left| \mathcal{E}_s \right| q^{b_G(s)} \leq \left( \frac{ena^n}{s} \right)^{s/n} 2^{(a-1)s} \left( \frac{ea^n}{s(1-x)} \right)^{s(1-x)} \left( \frac{(lnn)^{1/n} a}{a} \right) s^{(n-log_a s)}
\tag{2.20}
\]

Here

\[
2x + \frac{1}{n} \log a s - 1 - \frac{2}{n} \geq \delta - \frac{1}{2} - \frac{4}{n} \log a n - \frac{2}{n} \geq \frac{1}{10},
\tag{2.21}
\]

so that

\[
\sum_{S \in \mathcal{E}_s} \Pr (G_p[S] \text{ is a component}) \leq \left[ \left( en \right)^{1/n} 2^{a-1} \left( \frac{e}{1-x} \right)^{1-x} (lnn)^{-0.1} \right]^s.
\tag{2.22}
\]

Hence,

\[
\sum_{s=s_1}^{s_2} \sum_{S \in \mathcal{E}_s} \Pr (G_p[S] \text{ is a component}) \leq \sum_{s=s_1}^{s_2} \left[ \left( en \right)^{1/n} 2^{a-1} \left( \frac{e}{1-x} \right)^{1-x} (lnn)^{-0.1} \right]^s
\tag{2.23}
\]

\[
= o(1) \quad \text{as } n \to \infty.
\]
Case 4 \((s_2 + 1 \leq s \leq s_3)\). For \(S \in \mathcal{C}_k\) and \(H = G[S]\), let \(T = \{v \in S : d_H(v) \geq (a-1)n - \log_2^2 n\}\), \(t = |T|\) and \(H_1 = H[T] = G[T]\). Then

\[
2e(H_1) = 2e(H) - 2e(H[S - T, T]) - 2e(H[S - T])
\]

\[
> (a-1)s \log_a \frac{s}{n} - 2(a-1)n(s - t)
\]

\[
= (a-1)s \left[ \log_a \frac{s}{n} - \frac{2n}{s} (s - t) \right].
\]

Here

\[
\log_a \frac{s}{n} \geq n - 2 \log_a n,
\]

so that

\[
s(a-1)n - (s - t) \log^2_a n \geq \sum_{v \in T} d_H(v) + \sum_{v \in S - T} d_H(v) > (a-1)s \log_a \frac{s}{n}
\]

\[
\geq (a-1)s(n - 2 \log_a n),
\]

hence,

\[
t^* \geq s \left( 1 - \frac{2(a-1)}{\log_a n} \right).
\]

We take the first \(t\) vertices of \(T\) for \(H_1\) where \(t = s(1 - \epsilon)\) with \(s \epsilon = \lfloor 2(a-1)s/\log_a n \rfloor\) so that \(0 < (a-1)/\log_a n \leq \epsilon < 2(a-1)/\log_a n < 1/5\). Then

\[
2e(H_1)^* \geq (a-1)s[(1-2\epsilon)n - 2 \log_a n]
\]

and the average degree \(d_1\) in \(H_1\) satisfies

\[
d_1^* \geq (a-1)\left[ n - \frac{\epsilon}{1-\epsilon} n - \frac{2}{1-\epsilon} \log_a n \right] \geq (a-1)(1-3\epsilon)n.
\]

Let \(u = [a^n/\ln^6 n]\), so that \((a-1)n + 1 \leq u < t - (a-1)n - 1\), and by Lemma 2.1, for all sufficiently large \(n\), there exists \(U \subseteq T\), \(|U| = u\), and

\[
|\tilde{N}_H(U)| \geq |\tilde{N}_{H_1}(U)| \geq s(1-\epsilon)(1-3\epsilon)\left\{ 1 - \exp \left( - \frac{u[(a-1)n + 1]}{t} \right) \right\}
\]

\[
\geq s(1-\epsilon)^2(1-3\epsilon) \geq s(1-4\epsilon).
\]

Let \(t = s - \lfloor 4\epsilon s \rfloor\), so that \(u^* < t^* \leq s\), and \(w = \lfloor 4\epsilon s \rfloor\). For sufficiently large \(n\), take the smallest such \(u\)-set \(U = \{d_1, \ldots, d_u\}\) in \(S \subseteq V(G)\); take the (uniquely determined) first \(t - u\) vertices of \((N_G(d_1) \cap (S - U)) \cup \cdots \cup (N_G(d_u) \cap (S - U)) \subseteq V(G)\); and add the remaining \(w\) vertices \(W\) of \(S\). Then

\[
S \rightarrow \{d_1, \ldots, d_u; N_G(d_1) - S, \ldots, N_G(d_u) - S; W\}
\]
is an injection with \( |N_G(d_i) - S| \leq \lfloor \log_2 a n \rfloor \) (1 ≤ \( i \leq u \)). Hence, with \( y = \lfloor \log_2 a n \rfloor \),

\[
\begin{align*}
[\epsilon_s] & \leq \frac{a^n}{u} \sum_{(k_1, \ldots, k_u) \in \{0, \ldots, y\}^u} \binom{(a - 1)n}{k_i} a^n \\
& \leq \frac{a^n}{u} (y + 1)^u \left(\frac{(a - 1)n}{y + 1}\right)^u a^n,
\end{align*}
\]

(2.32)

since

\[
\binom{(a - 1)n}{k} \leq \binom{(a - 1)n}{y + 1}, \quad \forall k \in \{0, \ldots, y\}.
\]

(2.33)

Then

\[
\begin{align*}
[\epsilon_s] & \leq \left(\frac{e^a n}{u}\right)^u (y + 1)^u \left(\frac{e an}{y + 1}\right)^u \frac{e an}{w}^w \\
& \leq (e^2 an \ln^6 n)^u \left(\frac{e an}{\log_2 a n}\right)^u y^y \left(\frac{e an}{4 \epsilon s}\right)^4 e^\epsilon.
\end{align*}
\]

(2.34)

Hence, (Lemma 1.1)

\[
\sum_{S \in \epsilon_s} \Pr(G_p[S] \text{ is a component}) \leq [\epsilon_s] q^{b_G(s)}
\]

\[
\leq (e^2 an \ln^6 n)^u \left(\frac{e an}{\log_2 a n}\right)^u y^y \left(\frac{e an}{4 \epsilon s}\right)^4 e^\epsilon.
\]

(2.35)

Here

\[
1 \leq e an \ln^2 a \leq e^2 an \ln^6 n, \quad 0 < \frac{u}{s} \leq \frac{uy}{s} \leq \frac{5}{\ln^2 n},
\]

(2.36)

so that

\[
\sum_{S \in \epsilon_s} \Pr(G_p[S] \text{ is a component}) \leq \left[\left(e^3 a^2 n^2 \ln^2 a \ln^6 n\right)^{\frac{5}{\ln^2 n}} \left(\frac{e}{4 \epsilon}\right)^{4 e^\epsilon - 1}\right]^s
\]

\[
\leq \left(\frac{2}{3}\right)^s,
\]

(2.37)

since \( (e^3 a^2 n^2 \ln^2 a \ln^6 n)^{\frac{5}{\ln^2 n}} - 1, (e/4 \epsilon)^{4 e^\epsilon - 1} \) and \( \epsilon \to 0 \) as \( n \to \infty \). Hence,

\[
\sum_{s = s_2 + 1}^{s_3} \sum_{S \in \epsilon_s} \Pr(G_p[S] \text{ is a component}) \leq \sum_{s = s_2 + 1}^{s_3} \left(\frac{2}{3}\right)^s = o(1) \quad \text{as } n \to \infty.
\]

(2.38)
**Remark 2.4.** For all $a \geq 2$ and $n \geq 2$, $b_G(s) \geq 2$ when $2 \leq s \leq a^n/2$. Hence, $0 < \tilde{q}(n) \leq q(n)$ implies $(\tilde{q}(n))^{b_G(s)} \leq (q(n))^{b_G(s)}$ when $2 \leq s \leq a^n/2$. Then (2.10), (2.12), (2.20), and (2.35) hold for $G_{p(n)}$ where $\tilde{p}(n) = 1 - \tilde{q}(n)$ (the exponent in (2.12) is larger than $b_G(s)$). Hence, Lemma 2.3 holds for $G_{p(n)}$ as well. The inequalities in the proof of Lemma 2.3 hold for all sufficiently large $n$ which can be determined from nineteen appropriate inequalities there.

**Theorem 2.5.** For fixed $a \geq 2$, $q = q(n) = [(\lambda(n))^{1/n}/a]^{1/(a-1)}$ where $\lim_{n \to \infty} \lambda(n) = \lambda \in (0, \infty)$, and $p = p(n) = 1 - q(n)$, we have

$$\lim_{n \to \infty} \Pr(G_p \in \mathcal{G}(K^n_{a,a}) \text{ is connected}) = e^{-\lambda}. \quad (2.39)$$

**Proof.** We have

$$0 \leq \Pr(G_p \text{ is disconnected}) - \Pr(G_p \text{ has isolated vertices}) \leq \Pr(G_p \text{ has a component of order } s \text{ with } 2 \leq s \leq a^n/2) = o(1) \quad \text{as } n \to \infty,$$

by Remark 2.4. Hence, Lemma 2.2 gives

$$\lim_{n \to \infty} \Pr(G_p \text{ is disconnected}) = \lim_{n \to \infty} \Pr(G_p \text{ has isolated vertices}) = 1 - e^{-\lambda}. \quad (2.41)$$

We state the result for $G = K^n_{a,a}$ since its proof is similar to the proof of Theorem 2.5.

**Theorem 2.6.** For fixed $a \geq 1$, $q = q(n) = [(\lambda(n))^{1/n}/2a]^{1/a}$ where $\lim_{n \to \infty} \lambda(n) = \lambda \in (0, \infty)$, and $p = p(n) = 1 - q(n)$, we have

$$\lim_{n \to \infty} \Pr(G_p \in \mathcal{G}(K^n_{a,a}) \text{ is connected}) = e^{-\lambda}. \quad (2.42)$$

**References**


Lane Clark: Department of Mathematics, Southern Illinois University Carbondale, Carbondale, IL 62901-4408, USA

E-mail address: lclark@math.siu.edu
Special Issue on
Singular Boundary Value Problems for Ordinary Differential Equations

Call for Papers

The purpose of this special issue is to study singular boundary value problems arising in differential equations and dynamical systems. Survey articles dealing with interactions between different fields, applications, and approaches of boundary value problems and singular problems are welcome.

This Special Issue will focus on any type of singularities that appear in the study of boundary value problems. It includes:

- Theory and methods
- Mathematical Models
- Engineering applications
- Biological applications
- Medical Applications
- Finance applications
- Numerical and simulation applications

Before submission authors should carefully read over the journal's Author Guidelines, which are located at http://www.hindawi.com/journals/bvp/guidelines.html. Authors should follow the Boundary Value Problems manuscript format described at the journal site http://www.hindawi.com/journals/bvp/. Articles published in this Special Issue shall be subject to a reduced Article Processing Charge of €200 per article. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

<table>
<thead>
<tr>
<th>Manuscript Due</th>
<th>May 1, 2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Round of Reviews</td>
<td>August 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>November 1, 2009</td>
</tr>
</tbody>
</table>

Lead Guest Editor

Juan J. Nieto, Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Santiago de Compostela, Santiago de Compostela 15782, Spain; juanjose.nieto.roig@usc.es

Guest Editor

Donal O'Regan, Department of Mathematics, National University of Ireland, Galway, Ireland; donal.oregan@nuigalway.ie