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A NOTE ON COMPUTING THE GENERALIZED INVERSE $A^{(2)}_{T,S}$ OF A MATRIX A

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The generalized inverse $A^{(2)}_{T,S}$ of a matrix A is a $\{2\}$-inverse of A with the prescribed range T and null space S. A representation for the generalized inverse $A^{(2)}_{T,S}$ has been recently developed with the condition $\sigma(GA|_T) \subset (0,\infty)$, where G is a matrix with $R(G) = T$ and $N(G) = S$. In this note, we remove the above condition. Three types of iterative methods for $A^{(2)}_{T,S}$ are presented if $\sigma(GA|_T)$ is a subset of the open right half-plane and they are extensions of existing computational procedures of $A^{(2)}_{T,S}$, including special cases such as the weighted Moore-Penrose inverse $A^{\dagger}_{M,N}$ and the Drazin inverse A^D. Numerical examples are given to illustrate our results.

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1. Introduction. Given a complex matrix $A \in \mathbb{C}^{m\times n}$, any matrix $X \in \mathbb{C}^{n\times m}$ satisfying $XAX = X$ is called a $\{2\}$-inverse of A. Let T and S be subspaces of $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. A matrix $X \in \mathbb{C}^{n\times m}$ is called a $\{2\}$-inverse of A with the prescribed range T and null space S, denoted by $A^{(2)}_{T,S}$, if the following conditions are satisfied:

$$XAX = X, \qquad R(X) = T, \qquad N(X) = S, \tag{1.1}$$

where $R(X)$ is the range of X and $N(X)$ is the null space of X. It is a well-known fact [1] that if $\dim T = \dim S^{\perp} \le \operatorname{rank}(A)$, then there exists a unique $A^{(2)}_{T,S}$ if and only if $AT \oplus S = \mathbb{C}^m$. It is obvious from the definition above that $AA^{(2)}_{T,S} = P_{AT,S}$ and $A^{(2)}_{T,S}A = P_{T,(A^*S^{\perp})^{\perp}}$, where P_{S_1,S_2} is the projector on the subspace S_1 along the subspace S_2.

There are seven types of important $\{2\}$-inverses of A: the Moore-Penrose inverse $A^{\dagger}$, the weighted Moore-Penrose inverse $A^{\dagger}_{M,N}$, the W-weighed Drazin inverse $A_{d,w}$, the Drazin inverse A^D, the group inverse $A^{\#}$, the Bott-Duffin inverse $A^{(-1)}_{(L)}$, and the generalized Bott-Duffin inverse $A^{(\dagger)}_{(L)}$. All of them are the special cases of the generalized inverse $A^{(2)}_{T,S}$ of A for specific T and S.

Lemma 1.1. (a) *Let $A \in \mathbb{C}^{m\times n}$ [1]. Then, for the Moore-Penrose inverse $A^{\dagger}$ and the weighted Moore-Penrose inverse $A^{\dagger}_{MN}$,*

(i) $A^{\dagger} = A^{(2)}_{R(A^*),N(A^*)}$;

(ii) $A^{\dagger}_{M,N} = A^{(2)}_{R(N^{-1}A^*M),N(N^{-1}A^*M)}$, *where N and M are Hermitian positive definite matrices of order n and m, respectively;*

(iii) $A_{d,w} = (WAW)^{(2)}_{R(A(WA)^q),N(A(WA)^q)}$, *where $W \in \mathbb{C}^{n\times m}$ and $q = \operatorname{Ind}(WA)$, the index of WA.*

(b) [1, 2, 3] *Let* $A \in \mathbb{C}^{n\times n}$*. Then, for the Drazin inverse* A^D*, the group inverse* $A^{\#}$*, the Bott-Duffin inverse* $A^{(-1)}_{(L)}$*, and the generalized Bott-Duffin inverse* $A^{(\dagger)}_{(L)}$*,*

(iv) $A^D = A^{(2)}_{R(A^k),N(A^k)}$*, where* $k = \mathrm{Ind}(A)$*;*

(v) *in particular, when* $\mathrm{Ind}(A) = 1$*,* $A^{\#} = A^{(2)}_{R(A),N(A)}$*;*

(vi) $A^{(-1)}_{(L)} = P_L(AP_L + P_{L^\perp})^{-1} = A^{(2)}_{L,L^\perp}$*, where* L *is a subspace of* $\mathbb{C}^n$ *such that* $AL \oplus L^\perp = \mathbb{C}^n$ *and* P_L *is the orthogonal projector on* L*;*

(vii) $A^{(\dagger)}_{(L)} = A^{(-1)}_{(S)} = A^{(2)}_{S,S^\perp}$*, where* $S = R(P_LA)$*.*

The $\{2\}$-inverse has many applications, for example, the application in the iterative methods for solving nonlinear equations [1, 9] and the applications to statistics [6, 7]. In particular, $\{2\}$-inverse plays an important role in stable approximations of ill-posed problems and in linear and nonlinear problems involving rank-deficient generalized inverse [8, 12]. In literature, researchers have proposed many numerical methods for computing $A^{(2)}_{T,S}$, see [2, 3, 11, 13, 15, 16, 18].

As usual, we denote the spectrum and the spectral radius of A by $\sigma(A)$ and $\rho(A)$, respectively. The notation $\|\cdot\|$ stands for the spectral norm. The following theorem applied in this note is from the theory of semi-iterative method.

Theorem 1.2 (see [5]). *Let* $B \in \mathbb{C}^{n\times n}$ *be a nonsingular matrix and let* $\sigma(B) \subset \Omega$*, where* Ω *is a simply connected compact set excluding origin. If a sequence of polynomials* $\{s_m(z)\}_{m=0}^{\infty}$ *uniformly converges to* $1/z$ *on* Ω*, then* $\{s_m(B)\}$ *converges to* B^{-1}*.*

In this note, a representation for the generalized inverse $A^{(2)}_{T,S}$ with a condition $\sigma(GA|_T) \subset \{z : \mathrm{Re}(z) > 0\}$, where G is a matrix with $R(G) = T$ and $N(G) = S$ is presented in Section 2. Euler-Knopp iterative method and semi-iterative methods for $A^{(2)}_{T,S}$ with linear convergence are derived in Section 3. Quadratically convergent methods for $A^{(2)}_{T,S}$ are developed in Section 4. Finally, numerical examples are given to illustrate our results.

2. Representation.

In this section, we give a representation for the generalized inverse $A^{(2)}_{T,S}$, which may be viewed as an application of the classical theory summability to the representation of generalized inverse.

Lemma 2.1 (see [13]). *Suppose* $A \in \mathbb{C}^{m\times n}$*. Let* T *and* S *be subspaces of* $\mathbb{C}^n$ *and* $\mathbb{C}^m$*, respectively, such that* $AT \oplus S = \mathbb{C}^m$*. Suppose that* $G \in \mathbb{C}^{n\times m}$ *satisfies* $R(G) = T$ *and* $N(G) = S$*. Denote by* $\tilde{A} = (GA)|_T$ *the restriction of* GA *on* T*. Then* $\mathrm{Ind}(GA) = 1$ *and*

$$A^{(2)}_{T,S} = \tilde{A}^{-1}G. \tag{2.1}$$

It follows from Lemma 1.1 that the existence of G is assured for each of the common seven types of generalized inverses: A^*, $N^{-1}A^*M$, $A(WA)^q$, A^k, A, P_L, and P_S. Now we are in a position to establish a presentation theorem.

Theorem 2.2. *Let* A*,* T*,* S*,* G*, and* $\tilde{A}$ *be as in Lemma 2.1. If* $\sigma(\tilde{A})$ *is contained in a simply connected compact set* Ω *excluding origin and a polynomial sequence* $\{s_m(z)\}$ *uniformly converges to* $1/z$ *on* Ω*, then*

$$A^{(2)}_{T,S} = \lim_{m\to\infty} s_m(\tilde{A})G. \tag{2.2}$$

Furthermore,

$$\frac{||s_m(\tilde{A})G - A_{T,S}^{(2)}||_P}{||A_{T,S}^{(2)}||_P} \le \max_{z\in\sigma(\tilde{A})} |zs_m(z)-1| + O(\epsilon), \tag{2.3}$$

where P is invertible such that $P^{-1}GAP$ is the ϵ-Jordan canonical form of GA and $\|B\|_P = \|P^{-1}B\|$ for each $B \in \mathbb{C}^{n\times m}$.

Proof. Assume that $\sigma(\tilde{A}) \subset \Omega$. With applying Theorem 1.2, we get

$$\lim_{m\to\infty} s_m(\tilde{A}) = \tilde{A}^{-1} \tag{2.4}$$

uniformly on Ω. It follows from Lemma 2.1 that

$$\lim_{m\to\infty} s_m(\tilde{A})G = \tilde{A}^{-1}G = A_{T,S}^{(2)}. \tag{2.5}$$

The error can be written as

$$s_m(\tilde{A})G - A_{T,S}^{(2)} = (s_m(\tilde{A})\tilde{A} - I)A_{T,S}^{(2)}. \tag{2.6}$$

Since P is nonsingular such that $P^{-1}GAP$ is the ϵ-Jordan canonical form of GA, it is well known that

$$||P^{-1}GAP|| \le \rho(GA) + \epsilon. \tag{2.7}$$

Thus

$$\begin{aligned} ||s_m(\tilde{A})G - A_{T,S}^{(2)}||_P &= ||P^{-1}(s_m(\tilde{A})\tilde{A} - I)PP^{-1}A_{T,S}^{(2)}|| \\ &\le ||P^{-1}(s_m(\tilde{A})\tilde{A} - I)P||\ ||A_{T,S}^{(2)}||_P \\ &\le \left[\max_{z\in\sigma(\tilde{A})} |s_m(z)z - 1| + O(\epsilon)\right] ||A_{T,S}^{(2)}||_P. \end{aligned} \tag{2.8}$$

The last inequality is based on the spectrum mapping since $s_m(z)$ is a polynomial in z. This completes the proof. □

In order to make use of this general error estimate in Theorem 2.2 on specific approximation procedures, it will be convenient to have lower and upper bounds for $\sigma(\tilde{A})$. This is given in the next lemma.

Lemma 2.3. *Let A, T, S, G, and $\tilde{A}$ be as in Lemma 2.1. Then for each $\lambda \in \sigma(\tilde{A})$,*

$$\frac{1}{||(GA)^{\#}||} \le |\lambda| \le \|GA\|. \tag{2.9}$$

Proof. We only show the first inequality since the second is trivial. It follows from Lemma 2.1 that $\operatorname{Ind}(GA) = 1$. Then the Jordan canonical form of GA is

$$GA = P\begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}P^{-1}, \qquad (GA)^{\#} = P\begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix}P^{-1}, \tag{2.10}$$

where C is invertible. For each $\lambda \in \sigma(\tilde{A})$, $1/\lambda \in \sigma(\tilde{A}^{-1})$ since $\tilde{A}$ is invertible. Consequently, we have

$$\frac{1}{|\lambda|} \le \rho(\tilde{A}^{-1}) = \rho(C^{-1}) \le ||(GA)^{\#}||, \tag{2.11}$$

which leads to (2.9). This completes the proof. □

Remark 2.4. Theorem 2.2 extends the representation of $A^{(2)}_{T,S}$ in [15] in which $\sigma(GA|_T) \subset (0,\infty)$ is required. The theorem also recovers the representations of A^D in [16] and $A^{\dagger}_{M,N}$ in [17] as special cases.

3. Iterative methods for $A^{(2)}_{T,S}$.

In this section, we present applications of Theorem 2.2 and Lemma 2.3 in developing specific computational procedures for the generalized inverse $A^{(2)}_{T,S}$ and estimating corresponding error bounds.

A well-known summability method is called the Euler-Knopp method. A series $\sum_{m=0}^{\infty} a_m$ is said to be Euler-Knopp summable with parameter $\alpha > 0$ to the value a if the sequence defined by

$$s_m = \alpha \sum_{i=0}^{m} \sum_{j=0}^{i} \binom{i}{j} (1-\alpha)^{i-j} \alpha^j a_j \tag{3.1}$$

converges to a. If we choose $a_m = (1-z)^m$, $m \ge 0$, then as the Euler-Knopp transform of the series $\sum_{m=0}^{\infty}(1-z)^m$, we obtain a sequence $\{s_m(z)\}$, where

$$s_m(z) = \alpha \sum_{j=0}^{m} (1-\alpha z)^j. \tag{3.2}$$

Clearly, $\lim_{m\to\infty} s_m(z) = 1/z$ uniformly on any compact subset of an open set $E_\alpha := \{z : |1-\alpha z| < 1\}$. We assume that $\sigma(\tilde{A}) \subset \{z : \operatorname{Re}(z) > 0\}$. Denote

$$\phi := \max_{\lambda \in \sigma(\tilde{A})} \left\{ |\operatorname{Arg}\lambda| : -\frac{\pi}{2} < \operatorname{Arg}\lambda < \frac{\pi}{2} \right\}. \tag{3.3}$$

It follows from Lemma 2.3 that

$$\sigma(\tilde{A}) \subset \{z = re^{i\theta} : r_1 \le r \le r_2,\ -\phi \le \theta \le \phi\} =: F, \tag{3.4}$$

where $r_1 = 1/\|(GA)^{\#}\|$ and $r_2 = \|GA\|$. It can be shown with the law of Sines that

$$F \subset \{w : |w-g| \le g\}, \quad \text{for } g = \frac{\|GA\|}{2\cos\phi}. \tag{3.5}$$

If a parameter α satisfies

$$0 < \alpha < \frac{2\cos\phi}{\|GA\|}, \tag{3.6}$$

then $\sigma(\tilde{A}) \subset E_\alpha$. There is always a simply connected compact set Ω such that $\sigma(\tilde{A}) \subset \Omega \subset E_\alpha$. Hence $s_m(z)$ of (3.2) uniformly converges to $1/z$ on Ω. It follows from Theorem 2.2 that

$$A^{(2)}_{T,S} = \alpha \sum_{n=0}^{\infty} (I - \alpha GA)^n G. \tag{3.7}$$

Notice that if A_m is the mth partial sum, that is, $A_m = \alpha \sum_{j=0}^{m} (I - \alpha GA)^j G$, then an iteration form for $\{A_m\}$ is given by

$$A_0 = \alpha G, \qquad A_{m+1} = (I - \alpha GA)A_m + \alpha G, \quad m \geq 0. \tag{3.8}$$

For an error bound, we note that the sequence of polynomials $\{s_m(z)\}$ satisfies

$$z s_{m+1}(z) - 1 = (1 - \alpha z)(z s_m(z) - 1). \tag{3.9}$$

Thus

$$\left| z s_m(z) - 1 \right| = |1 - \alpha z|^{m+1} \leq \beta^{m+1} \longrightarrow 0, \quad (m \longrightarrow \infty), \tag{3.10}$$

where

$$\beta = \max_{z \in \sigma(\tilde{A})} |1 - \alpha z| \leq \max_{z \in F} |1 - \alpha z| < 1. \tag{3.11}$$

Actually, by the maximum modular theorem, $\max_{z \in F} |1 - \alpha z| = \max_{z \in \partial F} |1 - \alpha z|$. We denote four parts of ∂F as follows:

$$\begin{aligned} \Gamma_1 &= \{r_1 e^{i\theta} : -\phi \leq \theta \leq \phi\}, & \Gamma_2 &= \{r e^{i\phi} : r_1 \leq r \leq r_2\}, \\ \Gamma_3 &= \{r_2 e^{i\theta} : -\phi \leq \theta \leq \phi\}, & \Gamma_4 &= \{r e^{-i\phi} : r_1 \leq r \leq r_2\}. \end{aligned} \tag{3.12}$$

If $z \in \Gamma_1$, then $|1 - \alpha z|^2 = 1 - 2\alpha r_1 \cos\theta + \alpha^2 r_1^2$ and it is obvious that

$$\max_{z \in \Gamma_1} |1 - \alpha z| = \left| 1 - \alpha r_1 e^{i\phi} \right|. \tag{3.13}$$

With an analogous argument, we have

$$\max_{z \in \Gamma_3} |1 - \alpha z| = \left| 1 - \alpha r_2 e^{i\phi} \right|. \tag{3.14}$$

If $z \in \Gamma_2 \cup \Gamma_4$, then $|1 - \alpha z|^2 = 1 - 2\alpha r \cos\phi + \alpha^2 r^2$ is a quadratic function of r on $[r_1, r_2]$, which achieves its maximum at either $r = r_1$ or $r = r_2$. So

$$\max_{z \in \Gamma_2 \cup \Gamma_4} |1 - \alpha z| = \max\left\{ \left| 1 - \alpha r_1 e^{i\phi} \right|, \left| 1 - \alpha r_2 e^{i\phi} \right| \right\}. \tag{3.15}$$

It follows from Theorem 2.2 that an error bound is given by

$$\frac{\|A_m - A^{(2)}_{T,S}\|_P}{\|A^{(2)}_{T,S}\|_P} \leq \beta^{m+1} + O(\varepsilon), \tag{3.16}$$

where

$$\beta \le \max\{|1-\alpha e^{i\phi}/\|(GA)^{\#}\||, |1-\alpha e^{i\phi}\|GA\||\}. \quad (3.17)$$

Therefore, we have shown the following general convergence theorem.

Theorem 3.1. *Let A, T, S, and G be as in Lemma 2.1. Suppose the spectrum of $GA|_T$ is contained in the open right half-plane. Then the sequence $\{A_m\}$ of (3.8) linearly converges to $A_{T,S}^{(2)}$, if $0 < \alpha < 2\cos\phi/\|GA\|$, where ϕ is given by (3.3). Moreover, the relative error is bounded by (3.16).*

We remark that Theorem 3.1 is an extension of corresponding results in [15, 16].

The procedure of semi-iterative methods [5, 10] for solving a linear system can easily be extended to solve

$$X = HX + C, \quad \text{for } C \in \mathbb{C}^{n\times n}. \quad (3.18)$$

If $\rho(H) < 1$, then a sequence of matrices $\{X_m\}$, yielded by

$$X_0 = C; \qquad X_{m+1} = HX_m + C \quad (m \ge 0), \quad (3.19)$$

converges to $(I-H)^{-1}C$. In general, let $1 \notin \sigma(H)$. As usual, based on a sequence of polynomials $\{p_m(z)\}$ given by

$$p_m(z) = \sum_{i=0}^{m} \pi_{m,i} z^i, \quad \text{where } \sum_{i=0}^{m} \pi_{m,i} = 1, \quad (3.20)$$

the corresponding semi-iterative method induced by $\{p_m(z)\}$ for the computation of $(I-H)^{-1}C$ is defined as

$$Y_m = \sum_{i=0}^{m} \pi_{m,i} X_i, \quad m \ge 0. \quad (3.21)$$

Moreover, the matrices Y_m and the corresponding residual matrices R_m are given by

$$Y_m = p_m(H)Y_0 + q_{m-1}(H)C, \qquad R_m = p_m(H)(C-(I-H)Y_0), \quad (3.22)$$

where

$$q_{m-1}(z) = (1-p_m(z))/(1-z) \quad \text{with } q_{-1}(z) = 0. \quad (3.23)$$

If $\{q_m(H)\}$ converges to $(I-H)^{-1}$, or equivalently, if $\{p_m(H)\}$ converges to 0, then the sequence $\{Y_m\}$ of (3.21) converges to $(I-H)^{-1}C$. Especially, for $H = I - GA|_T$ and $C = G$, $\{Y_m\}$ converges to $A_{T,S}^{(2)}$. With an application of Theorem 1.2, we have the following corollary.

Corollary 3.2. *Let A, T, S, and G be as in Lemma 2.1 and let $H = I - GA|_T$. If $\sigma(H)$ is contained in Ω_1, a simply connected compact set excluding 1, and $\{q_m(z)\}$ of (3.23) uniformly converges to $1/(1-z)$ on Ω_1, then the sequence $\{Y_m\}$ of (3.21) converges to $A_{T,S}^{(2)}$ for $Y_0 = G$.*

Especially, Ω_1 is either a complex segment $[\alpha,\beta]$ excluding 1 or a closed ellipse in the left half-plane $\{z: \operatorname{Re}(z) < 1\}$ with foci α and β. Let a sequence of polynomials $\{p_m(z)\}$ given by

$$p_m(z) = \frac{T_m((z-\delta)/\xi)}{T_m((1-\delta)/\xi)}, \quad \left(\delta = \frac{\alpha+\beta}{2},\ \xi = \frac{\beta-\alpha}{2}\right), \tag{3.24}$$

where T_m is the mth Chebyshev polynomial. The semi-iterative method induced by $\{p_m(z)\}$ is the Chebyshev iterative method optimal for ellipse Ω_1. The corresponding two-step stationary method with the same asymptotically optimal convergence rate is given by

$$\begin{gathered} Y_0 = G; \qquad Y_1 = \mu(HY_0+G); \\ Y_{m+1} = \mu_0(HY_m+G)+\mu_1 Y_m+\mu_2 Y_{m-1}, \quad (m \ge 1), \end{gathered} \tag{3.25}$$

where

$$\mu_0 = \frac{4}{(\sqrt{1-\beta}+\sqrt{1-\alpha})^2}, \qquad \mu_1 = \frac{\alpha+\beta}{2}\mu_0, \qquad \mu_2 = 1-\mu_0-\mu_1. \tag{3.26}$$

The sequence $\{Y_m\}$ converges asymptotically optimally to $A^{(2)}_{T,S}$.

4. Quadratically convergent methods. Newton-Raphson method for finding the root $1/z$ of the function $s(w) = w^{-1}-z$ is given by

$$w_{m+1} = w_m(2-zw_m), \quad \text{for a suitable } w_0. \tag{4.1}$$

For $\alpha > 0$, a sequence of functions $\{s_m(z)\}$ is defined by

$$s_0(z) = \alpha, \qquad s_{m+1}(z) = s_m(z)[2-zs_m(z)]. \tag{4.2}$$

Let $z \in \sigma(GA|_T)$ and $0 < \alpha < 2\cos\phi/\|GA\|$. It follows from the recursive form $zs_{m+1}(z)-1 = -[zs_m(z)-1]^2$ that

$$\left|zs_m(z)-1\right| = |\alpha z-1|^{2^m} \le \beta^{2^m} \longrightarrow 0, \quad \text{as } m \longrightarrow \infty, \tag{4.3}$$

where an upper bound of β is given by (3.17).

The great attraction of the Newton-Raphson method is the generally quadratic nature of the convergence. Using the above facts in conjunction with Lemma 2.3, we see that a sequence $\{s_m(\tilde{A})\}$ defined by

$$s_0(\tilde{A}) = \alpha I, \qquad s_{m+1}(\tilde{A}) = s_m(\tilde{A})[2I-\tilde{A}s_m(\tilde{A})] \tag{4.4}$$

has the property that $\lim_{m\to\infty} s_m(\tilde{A})G = A^{(2)}_{T,S}$. If we set $A_m = s_m(\tilde{A})G$, then

$$A_0 = \alpha G, \qquad A_{m+1} = A_m(2I-AA_m). \tag{4.5}$$

Thus we have the following corollary.

Corollary 4.1. *Let A, T, S, and G be as in Lemma 2.1. Suppose that the spectrum of $\sigma(GA|_T) \subset \{z : \mathrm{Re}(z) > 0\}$. Then the sequence $\{A_m\}$ of (4.5) quadratically converges to $A^{(2)}_{T,S}$, for $0 < \alpha < 2\cos\phi/\|GA\|$. Furthermore, an error bound is given by*

$$||A_m - A^{(2)}_{T,S}||_P \leq (\beta^{2^m} + O(\epsilon))||A^{(2)}_{T,S}||_P, \tag{4.6}$$

where an upper bound of β is given in (3.17).

We remark that Corollary 4.1 is an extension of [4, 13, 15]. It covers iterative methods for $A^{\dagger}_{M,N}$ in [17].

The Newton-Raphson procedure can be speeded up by the successive matrix squaring technique in [14] if two parallel processors are available. In fact, the sequence in (4.5) is mathematically equivalent to

$$\begin{aligned} A_0 &= \alpha G, \qquad P_0 = I - \alpha GA, \\ A_{m+1} &= (I + P_m)A_m, \qquad P_{m+1} = P_m^2. \end{aligned} \tag{4.7}$$

There are two matrix multiplications each step both in (4.5) and (4.7). However, A_{m+1} and P_{m+1} in (4.7) can be calculated simultaneously.

Two algorithms given by (3.8) and (4.5) are also valid in the case when the spectrum of $\tilde{A}$ is contained in the left half-plane with slight modification.

Moreover, all results in the previous two sections are valid without the restriction on $\sigma(GA)$ if G is substituted by another matrix. This is stated as the following corollary.

Corollary 4.2. *Let A, T, S, and G be as in Lemma 2.1. Then Theorem 3.1 and Corollaries 3.2 and 4.1 are valid without any restriction on the spectrum of $GA|_T$ if G is substituted by*

$$G_0 = G(GAG)^*G. \tag{4.8}$$

Proof. It suffices to show that

$$R(G_0) = R(G), \qquad N(G_0) = N(G), \qquad \sigma(G_0A|_T) \subset (0, \infty). \tag{4.9}$$

As a matter of fact (4.9) is a direct result of [4, Lemma 3.4]. □

We remark a disadvantage of the choice G_0 of (4.8). In the case of computing A^D with $\mathrm{Ind}(A) = k \geq 3$, $G_0 = A^k(A^{2k+1})^*A^k$, the condition number of $G_0A|_T$ will be extremely large since $\mathrm{cond}(G_0A|_T) = \mathrm{cond}(A|_T)^{4k+2}$. An accurate numerical solution cannot be obtained if there is any round-off error in A.

5. Examples.

Three examples are given in this section to illustrate the computations of three types of $A^{(2)}_{T,S}$. All calculations were performed on a PC with MATLAB.

Example 5.1. Let A and W be 20 by 10 and 10 by 10 random matrices with entries on $[-1,1]$, respectively. We choose M and N as random symmetric and positive definite matrices of order 20 and 10, respectively. The stop criterion in (4.5) is

$\|A_m - A_{m-1}\|_\infty \le \epsilon = 10^{-10}$. Three special cases $A^\dagger$, $A_{M,N}^\dagger$, and $A_{d,w}$ are computed in this example. The choices of G, the number of iterations required and the norm of errors are listed in Table 5.1.

TABLE 5.1. Newton-Raphson method for $A^\dagger$, $A_{M,N}^\dagger$, and $A_{d,w}$.

$A_{T,S}^{(2)}$	G	m	$\|A_m - A_{m-1}\|_\infty$	$\|A_m - A_{T,S}^{(2)}\|_\infty$
$A^\dagger$	A^*	11	3.25E-14	2.56E-15
$A_{M,N}^\dagger$	$N^{-1}A^*M$	25	1.03E-15	3.09E-15
$A_{d,w}$	$AWA((WA)^*)^4WA$	36	1.96E-13	1.76E-07

It is remarked that the better accuracy of $A_{d,w}$ never be achieved and 1.7E-07 is the best error of $\|A_m - A_{d,w}\|_\infty$ even if $\|A_m - A_{m-1}\|_\infty \le \epsilon = 10^{-10}$ is used as a stop criterion. This is because the condition number of $GWAW|_T$ is as large as 10^{10}. If 2-step semi-iterative method of (3.25) is applied to compute $A_{T,S}^{(2)}$, then $\{Y_m\}$ converges to $A^\dagger$ after 54 iterations. However, the method fails to converge after 1500 iterations in other two cases because the segments $[\alpha,\beta]$ containing $\sigma(GA|_T)$ are $[-187970, 0.796]$ and $[-355800, 0.9997]$, respectively, so that the rate of asymptotic convergence is too slow.

EXAMPLE 5.2. Let A be 8 by 8 matrix with a complex spectrum given by

$$A = \begin{bmatrix} \frac{3}{2} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & \frac{3}{4} & -\frac{3}{4} & 0 & 0 & 0 & 0 \\ -1 & -1 & -\frac{3}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{3}{4} & -\frac{3}{4} & -1 & -1 \\ 0 & 0 & -1 & 0 & -\frac{3}{4} & \frac{3}{4} & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{3}{2} \end{bmatrix}. \tag{5.1}$$

In order to compute A^D, we choose $G = A^2$ since $\operatorname{Ind}(A) = 2$. The spectrum of $GA|_T$, $\sigma(GA|_T) = \{1.875 \pm 0.674i, 1.875 \pm 0.674i, 3.375, 3.375\}$, is located on the right half-plane. The foci $\alpha = -2.3$ and $\beta = -0.5$ of an ellipse containing $\sigma(I - GA|_T)$ is selected. It requires 28 iterations of 2-step method of (3.25) to compute A^D with the ∞-norm of the error less than 10^{-10}. As expected, Newton-Raphson algorithm of (4.5) converges much faster. It achieves the same accuracy with only 8 iterations.

TABLE 5.2. $A^{(2)}_{T,S}$ of a Toeplitz matrix.

$A^{(2)}_{T,S}$	G	No. of it. by Newton's	No. of it. by SIM
$A^{\dagger}$	A^*	10	63
$A^{\dagger}_{M,N}$	$N^{-1}A^*M$	11	75
$A_{d,w}$	$AWA((WA)^*)^4WA$	31	>1500

EXAMPLE 5.3. Let r and c be a row vector and column vector, respectively, such that

$$\begin{aligned} r_1 &= c_1 = 2.5, \\ r_j &= \frac{(-1)^j j}{16} + \frac{i(j-1)}{j}, \quad \text{for } j = 2,3,\dots,16, \\ c_k &= \frac{(-1)^k k}{10}, \quad \text{for } k = 2,3,\dots,10. \end{aligned} \tag{5.2}$$

A 10×16 complex Toeplitz matrix A is constructed by r and c. The stop criterion is the same as in Example 5.1. M and N are chosen positive definite diagonal matrix related to A, and W is a random matrix. The numbers of iterations by Newton's method and SIM method for $A^{(2)}_{T,S}$ are shown in Table 5.2.

The data shows that Newton's method is much faster than that of SIM.

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