VARIATIONAL-LIKE INEQUALITIES FOR PSEUDOMONOTONE OPERATORS

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ABSTRACT. The aim of this note is to use a fixed point theorem to prove results for variational-like inequalities for pseudomonotone operators.

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1. Introduction. Recently, Singh et al. [10] studied pseudomonotone operators and derived interesting results in variational inequality and complementarity problems using a recent fixed point theorem of Tarafdar [13], which is equivalent to F-KKM theorem [13]. They derived a few interesting results as corollaries and gave an application in minimization problems. Earlier, Parida et al. [7] studied a variational-like inequality problem and developed a theory for the existence of its solution using Kakutani’s fixed point theorem, and also established the relationship between the variational-like inequality problem and some mathematical programming problems. Further results on existence theorem for variational-like inequality problems were obtained by Wadhwa and Ganguly [14] using Tarafdar’s fixed point theorem [11], which is equivalent to the KKM fixed point theorem [13].

In this note, we use Tarafdar’s result [13] and prove an existence theorem for variational-like inequality problem for \( g \)-pseudomonotone operators and then derive some interesting results and corollaries.

We need the following definitions:

Let \( E \) stand for a real locally convex Hausdorff topological vector space and \( X \) a nonempty convex subset of \( E \) with \( E^* \neq \{0\} \), being the continuous dual of \( E \). Let \( T : X \rightarrow E^* \) be a nonlinear map. The mapping \( T : X \rightarrow E^* \) is hemicontinuous if \( T \) is continuous from the line segment of \( X \) to the weak topology of \( E^* \). A point \( y \in X \) is said to be a solution of the variational inequality if

\[
\langle Ty, x - y \rangle \geq 0 \quad \forall x \in X. \tag{1.1}
\]

Let \( g \) be a continuous map, \( g : X \times X \rightarrow E \). A point \( y \in X \) is said to be a solution of the variational-like inequality problems if

\[
\langle Ty, g(x,y) \rangle \geq 0 \quad \forall x \in X. \tag{1.2}
\]

If \( g(x,y) = x - y \), (1.2) reduces to (1.1) [7].

A map \( T : X \rightarrow E^* \) is said to be monotone if

\[
\langle Ty - Tx, y - x \rangle \geq 0 \quad \forall x, y \in X. \tag{1.3}
\]
Here, $(\cdot, \cdot)$ denotes the pairing between $E^*$ and $E$.

The map $T$ is called pseudomonotone if

$$
(Ty, y - x) \geq 0 \quad \text{whenever} \quad (Tx, y - x) \geq 0 \quad \forall x, y \in X. \tag{1.4}
$$

**Definition 1.1.** A map $T : X \to E^*$ is said to be $g$-monotone on $X$ if

$$
(Tx, g(y, x)) + (Ty, g(x, y)) \leq 0 \quad \forall x, y \in X. \tag{1.5}
$$

For $g(y, x) = y - x$, we get the definition of monotone operators.

**Definition 1.2.** A map $T : X \to E^*$ is said to be $g$-pseudomonotone if

$$
(Tx, g(y, x)) \geq 0 \quad \text{whenever} \quad (Ty, g(x, y)) \geq 0 \quad \forall x, y \in X. \tag{1.6}
$$

For $g(y, x) = y - x$, we get the definition of pseudomonotone operators.

We are interested in the following:

Find $x \in X$ such that

$$
(Tx, g(y, x)) + hy - hx \geq 0 \quad \forall y \in X, \tag{1.7}
$$

where $T : X \to E^*$ is a nonlinear mapping and $h : X \to \mathbb{R}$ is a low semi-continuous and convex functional.

We need the following fixed point theorem [13].

**Theorem 1.3.** Let $X$ be a nonempty, convex subset of a Hausdorff topological vector space $E$. Let $F : X \to 2^X$ be a set-valued mapping such that

(i) for each $x \in X$, $f(x)$ is a nonempty, convex subset of $X$;
(ii) for each $y \in X$, $F^{-1}(y) = \{x \in X : y \in f(x)\}$ contains a relatively open subset $O_y$ of $X$ ($O_y$ may be empty for some $y$);
(iii) $U_{x \in X} O_x = X$; and
(iv) $X$ contains a nonempty subset $X_0$ contained in a compact convex subset $X_1$ of $X$ such that the set $D = \bigcap_{x \in X_0} O_x$ is compact ($D$ may be empty and $O_x$ denotes the complement of $O_x$ in $X$).

Then there exists a point $x_0 \in X$ such that $x_0 \in F(x_0)$.

We make the following hypothesis.

**Condition 1.4.** For $X \subset E$, let $T : X \to E^*$ and $g : X \times X \to E$ satisfy the following:

(i) for each $x \in X$, $g(y, x)$ is convex $y \in X$;
(ii) $g(x, y) + g(y, z) = g(x, z)$ for all $x, y, z \in X$;
(iii) $g(x, x) = 0$;
(iv) for every $x \in E^*$, $(Tx, y)$ is monotone increasing in $y \in E^*$.

2. Main results. First, we give the following result.

**Lemma 2.1.** If $X$ is a nonempty convex subset of a topological vector space $E$ and $T : X \to E^*$ is a $g$-pseudomonotone and hemicontinuous, then $x \in X$ is a solution of

$$
(Tx, g(y, x)) + hy - hx \geq 0 \quad \forall y \in X \tag{2.1}
$$
if and only if \( x \in X \) is a solution of

\[
\langle Ty, g(y, x) \rangle + hy - hx \geq 0 \quad \forall y \in X,
\]

(2.2)

where \( h : X \to \mathbb{R} \) is a convex function and \( g : X \times X \to E \) is such that it satisfies Condition 1.4.

**Proof.** Let \( x \in X \) be a solution of (2.1). Then, by Condition 1.4(i), (ii) and the \( g \)-pseudomonotonicity of \( T \), we have

\[
\langle Ty, g(y, x) \rangle + hy - hx \geq 0 \quad \forall y \in X.
\]

(2.3)

Now, assume that \( x \) satisfies (2.2) and let \( y \in X \) be arbitrary. Then, using Minty’s technique [5],

\[
yt = (1 - t)x + ty \in X \quad \forall t \in (0, 1)
\]

(2.4)

since \( X \) is convex. Hence, we have

\[
\langle Ty_t, g(y_t, x) \rangle + hy_t - hx \geq 0.
\]

(2.5)

So, by Condition 1.4(ii), (iii),

\[
t \langle Ty_t, g(y, x) \rangle + t(hy - hx) \geq 0
\]

(2.6)

since \( T \) is hemicontinuous. Letting \( t \to 0 \), we get

\[
\langle Tx, g(y, x) \rangle + hy - hx \geq 0.
\]

(2.7)

Now, we state the following result.

**Theorem 2.2.** Let \( X \) be a nonempty closed convex subset of a real Hausdorff topological vector space \( E \) with \( E^* \neq \{0\} \). Let \( T : X \to E^* \) be \( g \)-pseudomonotone and hemicontinuous map such that Condition 1.4 is satisfied, and \( h : X \to \mathbb{R} \) is a lower semicontinuous and convex function. Further, assume that there exists a nonempty set \( X_0 \) contained in a compact convex subset \( X_1 \) of \( X \) such that the set

\[
D = \bigcap_{x \in X_0} \{ y \in X : \langle Tx, g(x, y) \rangle + hx - hy \geq 0 \}
\]

(2.8)

is either empty or compact.

Then, there exists an \( x_0 \in X \) such that

\[
\langle Tx_0, g(y, x_0) \rangle + hy - hx_0 \geq 0 \quad \forall y \in X.
\]

(2.9)

**Proof.** Suppose that, for each \( y \in X \), there exists an \( x \in X \) such that

\[
\langle Tx, g(y, x) \rangle + hx - hy < 0.
\]

(2.10)
First, suppose that (2.10) does not hold. This means that there exists at least one $y_0 \in X$ such that
\[ \langle Tx, g(y_0, x) \rangle + hx - hy \geq 0 \quad \forall x \geq X, \tag{2.11} \]
that is, $y_0 \geq X$ is a solution of (2.2). Then, by Lemma 2.1, $y_0 \in X$ is a solution of (2.1).

Next, assume that there is no solution of (2.1) under condition (2.10) given that (2.10) holds. Then, for each $x \in X$, the set
\[ F(x) = \{ y \in X : \langle Tx, g(y, x) \rangle + hy - hx < 0 \} \tag{2.12} \]
must be nonempty. It also follows from the convexity of $h$ and by Condition 1.4 that the set $F(x)$ is convex for each $x \in X$. Thus, $F : X \to 2^X$ is a set-valued map with $F(x)$ nonempty and convex for each $x \in X$.

Now, for each $x \in X$,
\[ F^{-1}(x) = \{ y \in X : x \in (y) \} = \{ y \in X : \langle Ty, g(x, y) \rangle + hx - hy < 0 \}. \tag{2.13} \]
For each $x \in X$,
\[ \{ F^{-1}(x) \}^c = \text{complement of } F^{-1}(x) \text{ in } X \]
\[ = \{ y \in X : \langle Ty, g(x, y) \rangle + hx - hy \geq 0 \} \tag{2.14} \]
by the $g$-pseudomonotonicity of $T = G(x)$.

Again, using Condition 1.4 and the convexity of $h$, we can show that $G(x)$ is convex for each $x \in X$. Since $g$ is continuous and $h$ is lower semi-continuous, $G(x)$ is a relatively closed subset of $X$.

Hence, for each $x \in X$,
\[ F^{-1}(x) \supset [G(x)]^c = 0_x \quad \text{is a relatively open subset of } X. \tag{2.15} \]

Now, by condition (2.10), we can easily see that $\bigcup_{x \in X} O_x = X$. (Indeed, if $y \in X$, by (2.10), there exists an $x \in X$ such that $y \in [G(x)]^c = O_x$. Thus, $y \in \bigcup_{x \in X} O_x$. Hence, $\bigcup_{x \in X} O_x = X$.)

Finally, $D = \bigcap_{x \in X} G(x) = \bigcap_{x \in X} O_x^c$ is compact or empty by the given condition. Hence, by Theorem 1.3, there exists an $x \in X$ such that $\langle Tx, g(x, x) \rangle + hx - hx < 0$, which is impossible. Hence, there is a solution in this case as well.

Here, we give a few results that are special cases of Theorem 2.2.

**Corollary 2.3.** Let $T : X \to E^*$ be $g$-monotone and hemicontinuous, where $g$-satisfies Condition 1.4, $h : X \to \mathbb{R}$ is convex and lower semi-continuous. Further, assume that there exists a nonempty set $X_0$ contained in a compact convex subset $X_1$ of $X$ such that $D = \bigcap_{x \in X_0} \{ y \in X : \langle Tx, g(x, y) \rangle + hx - hy \geq 0 \}$ is either empty or compact. Then there is an $x \in X$ satisfying (2.1).

**Remark 2.4.** For $g(x, y) = x - y$, Corollary 2.3 implies Corollary 1.2 of Singh et al. [10] which, in turn, implies a well-known result of Tarafdar [12].
**Corollary 2.5.** Let $X$ be a compact convex subset of $E$ and $T : X \to E^*$ be $g$-pseudomonotone and hemicontinuous where $g$ satisfies Condition 1.4. Suppose that $h : X \to \mathbb{R}$ is lower semicontinuous and convex. Then there is an $x \in X$ satisfying (2.1).

**Remark 2.6.** For $g(x,y) = x - y$,

(i) Corollary 2.5 implies [10, Corollary 1.3].

(ii) If we take $T = A - B$, where $A$ is a monotone map and $B$ is antimonotone and both are hemicontinuous, then we derive a result due to Siddiqui et al. [8]. Here, we need only two conditions, the lower semicontinuity, and the convexity of the function $h$.

**Remark 2.7.** For $h = 0$, Corollary 2.5 implies Theorem 2 and Corollary 1 of Wadhwa and Ganguly [14] which implies, respectively, Theorem 2 and Corollary of Tarafdar [11]. Tarafdar’s result covered the result of Browder [1] and Theorem 1.1 of Hartman and Stampacchia [3].

Now, we prove a result similar to Theorem 2.1 of Singh et al. [9]. For $A \subset E$, $\text{int}(A)$ and $\partial(A)$ denote, respectively, the interior and the boundary of $A$, while for $A, X \subset E$, $\text{int}_X(A)$ and $\partial(A)$ denote, respectively, the relative interior and the relative boundary of $A$ in $X$. A subset of a Banach space is said to be solid if it has a nonempty interior.

**Theorem 2.8.** Let $X$ be a closed convex subset of a reflexive Banach space $E$ and $T : X \to E^*$ a $g$-pseudomonotone and hemicontinuous mapping, $g : X \times X \to E$ satisfy Condition 1.4, and $h$ is convex and lower semicontinuous. Then the following conditions are equivalent:

(i) There exists $\hat{x} \in X$ such that $\langle T\hat{x},g(x,\hat{x}) \rangle + hx - h\hat{x} \geq 0$ for all $x \in X$, that is, $x$ is a solution of (2.1).

(ii) There exists a $u \in X$ and a constant $r > \|u\|$ such that $X\langle T(x),g(x,u) \rangle + hx - hu \geq 0$ for all $x \in X$ with $\|x\| = r$.

(iii) There exists $r > 0$ such that the set $\{x \in X : \|x\| \leq r\}$ is nonempty with the property that, for each $x \in X$ with $\|x\| = r$, there exists a $u \in X$ with $\|u\| < r$ and $\langle T(x),g(x,u) \rangle hxhu \geq 0$.

**Proof.** This can be proved following Cottle and Yao [2, Theorem 2.2] as well as Parida et al. [7, Theorem 3.4].

**Remark 2.9.** For a monotone $T$ operator and $h = 0$:

(1) Theorem 2.8(i), (ii), and (iii) were obtained by Parida et al. [7].

(2) For $g(x,\hat{x}) = x - \hat{x}$, Theorem 2.8(ii) and (iii) reduce to the results of Theorems 2.3 and 2.4 of Moré [6], respectively.

**Remark 2.10.** For $g(x,x) = x - \hat{x}$ and $h = 0$, Theorem 2.8(i), (ii), and (iii) were obtained as Theorem 2.1(i), (ii), and (iii) by Singh et al. [9] and, in Hilbert spaces, similar results were obtained by Cottle and Yao (see [1, Theorem 2.2]).

Let $H,K$ be nonempty, closed subsets of $\mathbb{R}^n$, then we denote, by $B_H(K)$, the set of $z \in K$ such that $U(z) \cap (H - K) \neq \emptyset$ and, by $I_H(K)$, the set of $z \in K$ such that $U(z) \cap (H - K) = \emptyset$, for some neighbourhood $U(z)$ of $z$. 


Finally, we present a result similar to Hirano and Takahashi [4] for unbounded subsets in \( \mathbb{R}^n \). Before that, we present the following result of Singh et al. [9, Corollary 1.12].

**Corollary 2.11.** Let \( X \) be a closed bounded convex subset of a reflexive Banach space \( E \) and \( T : X \to E^* \) a pseudomonotone and hemicontinuous mapping. Then the set of solutions of variational inequality for a point \( x_0 \in X \), \( \langle Tx_0, y - x_0 \rangle \geq 0 \) for all \( y \in X; y \in x; \) is a nonempty weakly compact convex subset of \( X \).

**Theorem 2.12.** Let \( X \) be a nonempty closed convex subset of \( \mathbb{R}^n \) and \( T : X \to \mathbb{R}^n \) be \( g \)-pseudomonotone such that Condition 1.4 is satisfied; \( h : X \to \mathbb{R} \) a lower semicontinuous and convex function. Then there exists a solution of (2.1) in \( X \) if and only if there exists a bounded closed convex subset \( K \) of \( X \) such that, for each \( z \in B_x(K) \), there exists \( y \in I_x(K) \) such that

\[
\langle Tz, g(y^*, z) \rangle + hz - hy \to 0. \tag{2.16}
\]

**Proof.** Using Corollary 2.11, with little modification, it can be shown that if there exists a solution of (2.1), then there exists a weakly compact convex subset \( K \) of \( X \) such that (2.16) is satisfied. Conversely, let \( K \) be a weakly compact convex subset and there exists \( x^* \in K \) such that

\[
\langle Tx^*, g(x, x^*) \rangle \geq 0 \quad \forall x \geq K, \tag{2.17}
\]

where \( T \) is a \( g \)-pseudomonotone operator. The rest of the proof is similar to that of Theorem 3 of Wadhwa and Ganguly [14].

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**References**


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