ON IMAGINABLE $T$-FUZZY SUBALGEBRAS AND IMAGINABLE $T$-FUZZY CLOSED IDEALS IN BCH-ALGEBRAS

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ABSTRACT. We inquire further into the properties on fuzzy closed ideals. We give a characterization of a fuzzy closed ideal using its level set, and establish some conditions for a fuzzy set to be a fuzzy closed ideal. We describe the fuzzy closed ideal generated by a fuzzy set, and give a characterization of a finite-valued fuzzy closed ideal. Using a $t$-norm $T$, we introduce the notion of (imaginable) $T$-fuzzy subalgebras and (imaginable) $T$-fuzzy closed ideals, and obtain some related results. We give relations between an imaginable $T$-fuzzy subalgebra and an imaginable $T$-fuzzy closed ideal. We discuss the direct product and $T$-product of $T$-fuzzy subalgebras. We show that the family of $T$-fuzzy closed ideals is a completely distributive lattice.

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1. Introduction. In 1983, Hu et al. introduced the notion of a BCH-algebra which is a generalization of a BCK/BCI-algebra (see [6, 7]). In [4], Chaudhry et al. stated ideals and filters in BCH-algebras, and studied their properties. For further properties on BCH-algebras, we refer to [2, 3, 5]. In [8], the first author considered the fuzzification of ideals and filters in BCH-algebras, and then described the relation among fuzzy subalgebras, fuzzy closed ideals and fuzzy filters in BCH-algebras. In this paper, we inquire further into the properties on fuzzy closed ideals. We give a characterization of a fuzzy closed ideal using its level set, and establish some conditions for a fuzzy set to be a fuzzy closed ideal. We describe the fuzzy closed ideal generated by a fuzzy set, and give a characterization of a finite-valued fuzzy closed ideal. Using a $t$-norm $T$, we introduce the notion of (imaginable) $T$-fuzzy subalgebras and (imaginable) $T$-fuzzy closed ideals, and obtain some related results. We give relations between an imaginable $T$-fuzzy subalgebra and an imaginable $T$-fuzzy closed ideal. We discuss the direct product and $T$-product of $T$-fuzzy subalgebras. We show that the family of $T$-fuzzy closed ideals is a completely distributive lattice.

2. Preliminaries. By a BCH-algebra we mean an algebra $(X, *, 0)$ of type $(2,0)$ satisfying the following axioms:

(H1) $x * x = 0$,

(H2) $x * y = 0$ and $y * x = 0$ imply $x = y$,

(H3) $(x * y) * z = (x * z) * y$,

for all $x, y, z \in X$.

In a BCH-algebra $X$, the following statements hold:

(P1) $x * 0 = x$. 

by

\[ U(\mu \in X \in [0,1]) \]

In what follows, let \( X \) denote a BCH-algebra unless otherwise specified. A fuzzy set in \( X \) is a function \( \mu : X \to [0,1] \). Let \( \mu \) be a fuzzy set in \( X \). For \( \alpha \in [0,1] \), the set \( U(\mu;\alpha) = \{ x \in X \mid \mu(x) \geq \alpha \} \) is called a level set of \( \mu \).

A fuzzy set \( \mu \) in \( X \) is called a fuzzy subalgebra of \( X \) if

\[
\mu(x \ast y) \geq \min\{\mu(x),\mu(y)\}, \quad \forall x,y \in X. \tag{2.1}
\]

**Definition 2.1** (see [1]). By a \( t \)-norm \( T \) on \([0,1]\), we mean a function \( T : [0,1] \times [0,1] \to [0,1] \) satisfying the following conditions:

(T1) \( T(x,1) = x \),
(T2) \( T(x,y) \leq T(x,z) \) if \( y \leq z \),
(T3) \( T(x,y) = T(y,x) \),
(T4) \( T(x,T(y,z)) = T(T(x,y),z) \), for all \( x,y,z \in [0,1] \).

In what follows, let \( T \) denote a \( t \)-norm on \([0,1]\) unless otherwise specified. Denote by \( \Delta_T \) the set of elements \( \alpha \in [0,1] \) such that \( T(\alpha,\alpha) = \alpha \), that is,

\[
\Delta_T := \{ \alpha \in [0,1] \mid T(\alpha,\alpha) = \alpha \}. \tag{2.2}
\]

Note that every \( t \)-norm \( T \) has a useful property:

(P4) \( T(\alpha,\beta) \leq \min(\alpha,\beta) \) for all \( \alpha,\beta \in [0,1] \).

3. Fuzzy closed ideals

**Definition 3.1** (see [8]). A fuzzy set \( \mu \) in \( X \) is called a fuzzy closed ideal of \( X \) if

(F1) \( \mu(0 \ast x) \geq \mu(x) \) for all \( x \in X \),
(F2) \( \mu(x) \geq \min\{\mu(x \ast y),\mu(y)\} \) for all \( x,y \in X \).

**Theorem 3.2.** Let \( D \) be a subset of \( X \) and let \( \mu_D \) be a fuzzy set in \( X \) defined by

\[
\mu_D(x) = \begin{cases} 
\alpha_1 & \text{if } x \in D, \\
\alpha_2 & \text{if } x \notin D,
\end{cases} \tag{3.1}
\]

for all \( x \in X \) and \( \alpha_1 > \alpha_2 \). Then \( \mu_D \) is a fuzzy closed ideal of \( X \) if and only if \( D \) is a closed ideal of \( X \).

**Proof.** Assume that \( \mu_D \) is a fuzzy closed ideal of \( X \). Let \( x \in D \). Then, by (F1), we have \( \mu(0 \ast x) \geq \mu(x) = \alpha_1 \) and so \( \mu(0 \ast x) = \alpha_1 \). It follows that \( 0 \ast x \in D \). Let \( x,y \in X \) be such that \( x \ast y \in D \) and \( y \in D \). Then \( \mu_D(x \ast y) = \alpha_1 = \mu_D(y) \), and hence

\[
\mu_D(x) \geq \min\{\mu_D(x \ast y),\mu_D(y)\} = \alpha_1. \tag{3.2}
\]

Thus \( \mu_D(x) = \alpha_1 \), that is, \( x \in D \). Therefore \( D \) is a closed ideal of \( X \).
Conversely, suppose that $D$ is a closed ideal of $X$. Let $x \in X$. If $x \in D$, then $0 \ast x \in D$ and thus $\mu_D(0 \ast x) = \alpha_1 = \mu_D(x)$. If $x \notin D$, then $\mu_D(x) = \alpha_2 \leq \mu_D(0 \ast x)$. Let $x, y \in X$. If $x \ast y \in D$ and $y \in D$, then $x \in D$. Hence

$$\mu_D(x) = \alpha_1 = \min \{\mu_D(x \ast y), \mu_D(y)\}. \quad (3.3)$$

If $x \ast y \notin D$ and $y \notin D$, then clearly $\mu_D(x) \geq \min \{\mu_D(x \ast y), \mu_D(y)\}$. If exactly one of $x \ast y$ and $y$ belong to $D$, then exactly one of $\mu_D(x \ast y)$ and $\mu_D(y)$ is equal to $\alpha_2$. Therefore, $\mu_D(x) \geq \alpha_2 = \min \{\mu_D(x \ast y), \mu_D(y)\}$. Consequently, $\mu_D$ is a fuzzy closed ideal of $X$.

Using the notion of level sets, we give a characterization of a fuzzy closed ideal.

**Theorem 3.3.** A fuzzy set $\mu$ in $X$ is a fuzzy closed ideal of $X$ if and only if the nonempty level set $U(\mu; \alpha)$ of $\mu$ is a closed ideal of $X$ for all $\alpha \in [0, 1]$.

We then call $U(\mu; \alpha)$ a level closed ideal of $\mu$.

**Proof.** Assume that $\mu$ is a fuzzy closed ideal of $X$ and $U(\mu; \alpha) \neq \emptyset$ for all $\alpha \in [0, 1]$. Let $x \in U(\mu; \alpha)$. Then $\mu(0 \ast x) \geq \mu(x) \geq \alpha$, and so $0 \ast x \in U(\mu; \alpha)$. Let $x, y \in X$ be such that $x \ast y \in U(\mu; \alpha)$ and $y \in U(\mu; \alpha)$. Then

$$\mu(x) \geq \min \{\mu(x \ast y), \mu(y)\} \geq \min \{\alpha, \alpha\} = \alpha, \quad (3.4)$$

and thus $x \in U(\mu; \alpha)$. Therefore $U(\mu; \alpha)$ is a closed ideal of $X$. Conversely, suppose that $U(\mu; \alpha) \neq \emptyset$ is a closed ideal of $X$. If $\mu(0 \ast a) < \mu(a)$ for some $a \in X$, then $\mu(0 \ast a) < \alpha_0 < \mu(a)$ by taking $\alpha_0 := 1/2(\mu(0 \ast a) + \mu(a))$. It follows that $a \in U(\mu; \alpha_0)$ and $0 \ast a \notin U(\mu; \alpha_0)$, which is a contradiction. Hence $\mu(0 \ast x) \geq \mu(x)$ for all $x \in X$.

Assume that there exist $x_0, y_0 \in X$ such that

$$\mu(x_0) < \min \{\mu(x_0 \ast y_0), \mu(y_0)\}. \quad (3.5)$$

Taking $\beta_0 := 1/2(\mu(x_0) + \min\{\mu(x_0 \ast y_0), \mu(y_0)\})$, we get $\mu(x_0) < \beta_0 < \mu(x_0 \ast y_0)$ and $\mu(x_0) < \beta_0 \leq \mu(y_0)$. Thus $x_0 \ast y_0 \in U(\mu; \beta_0)$ and $y_0 \in U(\mu; \beta_0)$, but $x_0 \notin U(\mu; \beta_0)$. This is impossible. Hence $\mu$ is a fuzzy closed ideal of $X$. \hfill $\Box$

**Theorem 3.4.** Let $\mu$ be a fuzzy set in $X$ and $\text{Im}(\mu) = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$, where $\alpha_i < \alpha_j$ whenever $i > j$. Let $\{D_k \mid k = 0, 1, 2, \ldots, n\}$ be a family of closed ideals of $X$ such that

(i) $D_0 \subseteq D_1 \subseteq \cdots \subseteq D_n = X$,

(ii) $\mu(D^+_k) = \alpha_k$, where $D^+_k = D_k \setminus D_{k-1}$ and $D_{-1} = \emptyset$ for $k = 0, 1, \ldots, n$.

Then $\mu$ is a fuzzy closed ideal of $X$.

**Proof.** For any $x \in X$ there exists $k \in \{0, 1, \ldots, n\}$ such that $x \in D^+_k$. Since $D_k$ is a closed ideal of $X$, it follows that $0 \ast x \in D_k$. Thus $\mu(0 \ast x) \geq \alpha_k = \mu(x)$. To prove that $\mu$ satisfies condition (F2), we discuss the following cases: if $x \ast y \in D^+_k$ and $y \in D^+_k$, then $x \in D_k$ because $D_k$ is a closed ideal of $X$. Hence

$$\mu(x) \geq \alpha_k = \min \{\mu(x \ast y), \mu(y)\}. \quad (3.6)$$
If \( x \ast y \notin D_k^\ast \) and \( y \notin D_k^\ast \), then the following four cases arise:

(i) \( x \ast y \in X \setminus D_k \) and \( y \in X \setminus D_k \),
(ii) \( x \ast y \in D_{k-1} \) and \( y \in D_{k-1} \),
(iii) \( x \ast y \in X \setminus D_k \) and \( y \in D_{k-1} \),
(iv) \( x \ast y \in D_{k-1} \) and \( y \in X \setminus D_k \).

But, in either case, we know that \( \mu(x) \geq \min \{\mu(x \ast y), \mu(y)\} \). If \( x \ast y \in D_k^\ast \) and \( y \notin D_k^\ast \), then either \( y \in D_{k-1} \) or \( y \in X \setminus D_k \). It follows that either \( x \in D_k \) or \( x \in X \setminus D_k \). Thus \( \mu(x) \geq \min \{\mu(x \ast y), \mu(y)\} \). Similarly for the case \( x \ast y \notin D_k^\ast \) and \( y \in D_k^\ast \), we have the same result. This completes the proof.

**Theorem 3.5.** Let \( \Lambda \) be a subset of \([0, 1]\) and let \( \{D_\lambda \mid \lambda \in \Lambda\} \) be a collection of closed ideals of \( X \) such that

(i) \( X = \bigcup_{\lambda \in \Lambda} D_\lambda \),
(ii) \( \alpha > \beta \) if and only if \( D_\alpha \subseteq D_\beta \) for all \( \alpha, \beta \in \Lambda \).

Define a fuzzy set \( \mu \) in \( X \) by \( \mu(x) = \sup \{\lambda \in \Lambda \mid x \in D_\lambda\} \) for all \( x \in X \). Then \( \mu \) is a fuzzy closed ideal of \( X \).

**Proof.** Let \( x \in X \). Then there exists \( \alpha_i \in \Lambda \) such that \( x \in D_{\alpha_i} \). It follows that \( 0 \ast x \in D_{\alpha_j} \) for some \( \alpha_j \geq \alpha_i \). Hence

\[
\mu(x) = \sup \{\alpha_k \in \Lambda \mid \alpha_k \leq \alpha_i\} \leq \sup \{\alpha_k \in \Lambda \mid \alpha_k \leq \alpha_j\} = (0 \ast x).
\]

Let \( x, y \in X \) be such that \( \mu(x \ast y) = m \) and \( \mu(y) = n \), where \( m, n \in [0, 1] \). Without loss of generality we may assume that \( m \leq n \). To prove \( \mu \) satisfies condition (F2), we consider the following three cases:

\[
(1^\ast) \lambda \leq m, \quad (2^\ast) \lambda \geq n, \quad (3^\ast) \lambda > n.
\]

Case \((1^\ast)\) implies that \( x \ast y \in D_\lambda \) and \( y \in D_\lambda \). It follows that \( x \in D_\lambda \) so that

\[
\mu(x) = \sup \{\lambda \in \Lambda \mid x \in D_\lambda\} \geq m = \min \{\mu(x \ast y), \mu(y)\}.
\]

For the case \((2^\ast)\), we have \( x \ast y \notin D_\lambda \) and \( y \in D_\lambda \). Then either \( x \in D_\lambda \) or \( x \notin D_\lambda \). If \( x \in D_\lambda \), then \( \mu(x) = n \geq \min \{\mu(x \ast y), \mu(y)\} \). If \( x \notin D_\lambda \), then \( x \in D_\lambda \) for some \( \delta < \lambda \), and so \( \mu(x) > m = \min \{\mu(x \ast y), \mu(y)\} \). Finally, case \((3^\ast)\) implies \( x \ast y \notin D_\lambda \) and \( y \notin D_\lambda \). Thus we have that either \( x \in D_\lambda \) or \( x \notin D_\lambda \). If \( x \in D_\lambda \), then obviously \( \mu(x) \geq \min \{\mu(x \ast y), \mu(y)\} \). If \( x \notin D_\lambda \), then \( x \in D_{\lambda - \delta} \) for some \( \epsilon < \lambda \), and thus \( \mu(x) \geq m = \min \{\mu(x \ast y), \mu(y)\} \). This completes the proof.

Let \( D \) be a subset of \( X \). The least closed ideal of \( X \) containing \( D \) is called the closed ideal generated by \( D \), denoted by \( \langle D \rangle \). Note that if \( C \) and \( D \) are subsets of \( X \) and \( C \subseteq D \), then \( \langle C \rangle \subseteq \langle D \rangle \). Let \( \mu \) be a fuzzy set in \( X \). The least fuzzy closed ideal of \( X \) containing \( \mu \) is called a fuzzy closed ideal of \( X \) generated by \( \mu \), denoted by \( \langle \mu \rangle \).

**Lemma 3.6.** For a fuzzy set \( \mu \) in \( X \), then

\[
\mu(x) = \sup \{\alpha \in [0, 1] \mid x \in U(\mu; \alpha)\}, \quad \forall x \in X.
\]

**Proof.** Let \( \delta := \sup \{\alpha \in [0, 1] \mid x \in U(\mu; \alpha)\} \) and let \( \varepsilon > 0 \) be given. Then \( \delta - \varepsilon < \alpha \) for some \( \alpha \in [0, 1] \) such that \( x \in U(\mu; \alpha) \), and so \( \delta - \varepsilon < \mu(x) \). Since \( \varepsilon \) is arbitrary, it
follows that $\mu(x) \leq \delta$. Now let $\mu(x) = \beta$. Then $x \in U(\mu; \beta)$ and hence $\beta \in \{ \alpha \in [0,1] \mid x \in U(\mu; \alpha) \}$. Therefore
\[
\mu(x) = \beta \leq \sup \{ \alpha \in [0,1] \mid x \in U(\mu; \alpha) \} = \delta,
\]
and consequently $\mu(x) = \delta$, as desired.

**Theorem 3.7.** Let $\mu$ be a fuzzy set in $X$. Then the fuzzy set $\mu^*$ in $X$ defined by
\[
\mu^*(x) = \sup \{ \alpha \in [0,1] \mid x \in \langle U(\mu; \alpha) \rangle \}
\]
for all $x \in X$ is the fuzzy closed ideal $(\mu)$ generated by $\mu$.

**Proof.** We first show that $\mu^*$ is a fuzzy closed ideal of $X$. For any $\gamma \in \text{Im}(\mu^*)$, let $y_n = \gamma - 1/n$ for any $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of all positive integers, and let $x \in U(\mu^*; y)$. Then $\mu^*(x) \geq \gamma$, and so
\[
\sup \{ \alpha \in [0,1] \mid x \in \langle U(\mu; \alpha) \rangle \} \geq \gamma > y_n,
\]
for all $n \in \mathbb{N}$. Hence there exists $\beta \in [0,1]$ such that $\beta > y_n$ and $x \in \langle U(\mu; \beta) \rangle$. It follows that $U(\mu; \beta) \subseteq U(\mu; y_n)$ so that $x \in \langle U(\mu; \beta) \rangle \subseteq \langle U(\mu; y_n) \rangle$ for all $n \in \mathbb{N}$. Consequently, $x \in \cap_{n \in \mathbb{N}} \langle U(\mu; y_n) \rangle$. On the other hand, if $x \in \cap_{n \in \mathbb{N}} \langle U(\mu; y_n) \rangle$, then $y_n \in \{ \alpha \in [0,1] \mid x \in \langle U(\mu; \alpha) \rangle \}$ for any $n \in \mathbb{N}$. Therefore
\[
y_n = \frac{1}{n} = \sup \{ \alpha \in [0,1] \mid x \in \langle U(\mu; \alpha) \rangle \} = \mu^*(x),
\]
for all $n \in \mathbb{N}$. Since $n$ is an arbitrary positive integer, it follows that $\gamma \leq \mu^*(x)$ so that $x \in U(\mu^*; \gamma)$. Hence $U(\mu^*; \gamma) = \cap_{n \in \mathbb{N}} \langle U(\mu; y_n) \rangle$, which is a closed ideal of $X$. Using Theorem 3.3, we know that $\mu^*$ is a fuzzy closed ideal of $X$. We now prove that $\mu^*$ contains $\mu$. For any $x \in X$, let $\beta \in \{ \alpha \in [0,1] \mid x \in \langle U(\mu; \alpha) \rangle \}$. Then $x \in U(\mu; \beta)$ and so $x \in \langle U(\mu; \beta) \rangle$. Thus we get $\beta \in \{ \alpha \in [0,1] \mid x \in \langle U(\mu; \alpha) \rangle \}$, and so
\[
\{ \alpha \in [0,1] \mid x \in U(\mu; \alpha) \} \subseteq \{ \alpha \in [0,1] \mid x \in \langle U(\mu; \alpha) \rangle \}.
\]
It follows from Lemma 3.6 that
\[
\mu(x) = \sup \{ \alpha \in [0,1] \mid x \in U(\mu; \alpha) \}
\leq \sup \{ \alpha \in [0,1] \mid x \in \langle U(\mu; \alpha) \rangle \}
\leq \mu^*(x).
\]
Hence $\mu \subseteq \mu^*$. Finally let $\nu$ be a fuzzy closed ideal of $X$ containing $\mu$ and let $x \in X$. If $\mu^*(x) = 0$, then clearly $\mu^*(x) \leq \nu(x)$. Assume that $\mu^*(x) = \gamma \neq 0$. Then $x \in U(\mu^*; \gamma) = \cap_{n \in \mathbb{N}} \langle U(\mu; y_n) \rangle$, that is, $x \in U(\mu; y_n)$ for all $n \in \mathbb{N}$. It follows that $\nu(x) \geq \mu(x) \geq y_n = \gamma - 1/n$ for all $n \in \mathbb{N}$ so that $\nu(x) \geq \gamma = \mu^*(x)$ since $n$ is arbitrary. This shows that $\mu^* \subseteq \mu$, completing the proof.

**Definition 3.8.** A fuzzy closed ideal $\mu$ of $X$ is said to be $n$-valued if $\text{Im}(\mu)$ is a finite set of $n$ elements. When no specific $n$ is intended, we call $\mu$ a finite-valued fuzzy closed ideal.
**Theorem 3.9.** Let $\mu$ be a fuzzy closed ideal of $X$. Then $\mu$ is finite valued if and only if there exists a finite-valued fuzzy set $\nu$ in $X$ which generates $\mu$. In this case, the range sets of $\mu$ and $\nu$ are identical.

**Proof.** If $\mu : X \rightarrow [0, 1]$ is a finite-valued fuzzy closed ideal of $X$, then we may choose $\nu = \mu$. Conversely, assume that $\nu : X \rightarrow [0, 1]$ is a finite-valued fuzzy set. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be distinct elements of $\nu(X)$ such that $\alpha_1 > \alpha_2 > \cdots > \alpha_n$, and let $C_i = \nu^{-1}(\alpha_i)$ for $i = 1, 2, \ldots, n$. Clearly, $\bigcup_{i=1}^j C_i \subseteq \bigcup_{i=1}^k C_i$ whenever $j < k \leq n$. Hence if we let $D_j = \langle \bigcup_{i=1}^j C_i \rangle$, then we have the following chain:

$$D_1 \subseteq D_2 \subseteq \cdots \subseteq D_n = X.$$  

(3.17)

Define a fuzzy set $\mu : X \rightarrow [0, 1]$ as follows:

$$\mu(x) = \begin{cases} 
\alpha_1 & \text{if } x \in D_1, \\
\alpha_j & \text{if } x \in D_j \setminus D_{j-1}.
\end{cases}$$  

(3.18)

We claim that $\mu$ is a fuzzy closed ideal of $X$ generated by $\nu$. Clearly $\mu(0 \ast x) \geq \mu(x)$ for all $x \in X$. Let $x, y \in X$. Then there exist $i$ and $j$ in $\{1, 2, \ldots, n\}$ such that $x \ast y \in D_i$ and $y \in D_j$. Without loss of generality, we may assume that $i$ and $j$ are the smallest integers such that $i \geq j$, $x \ast y \in D_i$, and $y \in D_j$. Since $D_i$ is a closed ideal of $X$, it follows from $D_j \subseteq D_i$ that $x \in D_i$. Hence $\mu(x) \geq \alpha_i = \min\{\mu(x \ast y), \mu(y)\}$, and so $\mu$ is a fuzzy closed ideal of $X$. If $\nu(x) = \alpha_j$ for every $x \in X$, then $x \in C_j$ and thus $x \in D_j$. But we have $\mu(x) \geq \alpha_j = \nu(x)$. Therefore $\mu$ contains $\nu$. Let $\delta : X \rightarrow [0, 1]$ be a fuzzy closed ideal of $X$ containing $\nu$. Then $U(\nu; \alpha_j) \subseteq U(\delta; \alpha_j)$ for every $j$. Hence $U(\nu; \alpha_j)$, being a closed ideal, contains the closed ideal generated by $U(\nu; \alpha_j) = \bigcup_{i=1}^j C_i$. Consequently, $D_j \subseteq U(\delta; \alpha_j)$. It follows that $\mu$ is contained in $\delta$ and that $\mu$ is generated by $\nu$. Finally, note that $|\text{Im}(\mu)| = n = |\text{Im}(\nu)|$. This completes the proof.

**Theorem 3.10.** Let $D_1 \supseteq D_2 \supseteq \cdots$ be a descending chain of closed ideals of $X$ which terminates at finite step. For a fuzzy closed ideal $\mu$ of $X$, if a sequence of elements of $\text{Im}(\mu)$ is strictly increasing, then $\mu$ is finite valued.

**Proof.** Suppose that $\mu$ is infinite valued. Let $\{\alpha_n\}$ be a strictly increasing sequence of elements of $\text{Im}(\mu)$. Then $0 \leq \alpha_1 < \alpha_2 < \cdots \leq 1$. Note that $U(\mu; \alpha_t)$ is a closed ideal of $X$ for $t = 1, 2, 3, \ldots$. Let $x \in U(\mu; \alpha_t)$ for $t = 2, 3, \ldots$. Then $\mu(x) \geq \alpha_t > \alpha_{t-1}$, which implies that $x \in U(\mu; \alpha_{t-1})$. Hence $U(\mu; \alpha_t) \subseteq U(\mu; \alpha_{t-1})$ for $t = 2, 3, \ldots$. Since $\alpha_{t-1} \in \text{Im}(\mu)$, there exists $x_{t-1} \in X$ such that $\mu(x_{t-1}) = \alpha_{t-1}$. It follows that $x_{t-1} \in U(\mu; \alpha_{t-1})$, but $x_{t-1} \notin U(\mu; \alpha_t)$. Thus $U(\mu; \alpha_t) \subseteq U(\mu; \alpha_{t-1})$, and so we obtain a strictly descending chain $U(\mu; \alpha_1) \supseteq U(\mu; \alpha_2) \supseteq \cdots$ of closed ideals of $X$ which is not terminating. This is impossible and the proof is complete.

Now we consider the converse of Theorem 3.10.

**Theorem 3.11.** Let $\mu$ be a finite-valued fuzzy closed ideal of $X$. Then every descending chain of closed ideals of $X$ terminates at finite step.
**Proof.** Suppose there exists a strictly descending chain $D_0 \supseteq D_1 \supseteq D_2 \supseteq \cdots$ of closed ideals of $X$ which does not terminate at finite step. Define a fuzzy set $\mu$ in $X$ by

$$
\mu(x) = \begin{cases} 
\frac{n}{n+1} & \text{if } x \in D_n \setminus D_{n+1}, \ n = 0, 1, 2, \ldots, \\
1 & \text{if } x \in \cap_{n=0}^{\infty} D_n,
\end{cases}
$$

where $D_0$ stands for $X$. Clearly, $\mu(0 \ast x) \geq \mu(x)$ for all $x \in X$. Let $x, y \in X$. Assume that $x \ast y \in D_n \setminus D_{n+1}$ and $y \in D_k \setminus D_{k+1}$ for $n = 0, 1, 2, \ldots; k = 0, 1, 2, \ldots$. Without loss of generality, we may assume that $n \leq k$. Then clearly $y \in D_n$, and so $x \in D_n$ because $D_n$ is a closed ideal of $X$. Hence

$$
\mu(x) \geq \frac{n}{n+1} = \min \{ \mu(x \ast y), \mu(y) \}.
$$

(3.19)

If $x \ast y \notin \cap_{n=0}^{\infty} D_n$ and $y \in \cap_{n=0}^{\infty} D_n$, then $x \in \cap_{n=0}^{\infty} D_n$. Thus $\mu(x) = 1 = \min \{ \mu(x \ast y), \mu(y) \}$. If $x \ast y \notin \cap_{n=0}^{\infty} D_n$ and $y \in \cap_{n=0}^{\infty} D_n$, then there exists a positive integer $k$ such that $x \ast y \in D_k \setminus D_{k+1}$. It follows that $x \in D_k$ so that

$$
\mu(x) \geq \frac{k}{k+1} = \min \{ \mu(x \ast y), \mu(y) \}.
$$

(3.20)

Finally suppose that $x \ast y \notin \cap_{n=0}^{\infty} D_n$ and $y \notin \cap_{n=0}^{\infty} D_n$. Then $y \in D_r \setminus D_{r+1}$ for some positive integer $r$. It follows that $x \in D_r$, and hence

$$
\mu(x) \geq \frac{r}{r+1} = \min \{ \mu(x \ast y), \mu(y) \}.
$$

(3.21)

Consequently, we conclude that $\mu$ is a fuzzy closed ideal of $X$ and $\mu$ has an infinite number of different values. This is a contradiction, and the proof is complete. \hfill \Box

**Theorem 3.12.** The following are equivalent:

(i) Every ascending chain of closed ideals of $X$ terminates at finite step.

(ii) The set of values of any fuzzy closed ideal of $X$ is a well-ordered subset of $[0, 1]$.

**Proof.** (i)$\Rightarrow$(ii). Let $\mu$ be a fuzzy closed ideal of $X$. Suppose that the set of values of $\mu$ is not a well-ordered subset of $[0, 1]$. Then there exists a strictly decreasing sequence $\{\alpha_n\}$ such that $\mu(x_n) = \alpha_n$. It follows that

$$
U(\mu; \alpha_1) \subset U(\mu; \alpha_2) \subset U(\mu; \alpha_3) \subset \cdots
$$

(3.22)

is a strictly ascending chain of closed ideals of $X$. This is impossible.

(ii)$\Rightarrow$(i). Assume that there exists a strictly ascending chain

$$
D_1 \subset D_2 \subset D_3 \subset \cdots
$$

(3.23)

of closed ideals of $X$. Note that $D := \cup_{n \in N} D_n$ is a closed ideal of $X$. Define a fuzzy set $\mu$ in $X$ by

$$
\mu(x) = \begin{cases} 
0 & \text{if } x \notin D_n, \\
1/k & \text{where } k = \min \{ n \in N \mid x \in D_n \},
\end{cases}
$$

(3.24)

where $k = \min \{ n \in N \mid x \in D_n \}$. **Proof.**
We claim that $\mu$ is a fuzzy closed ideal of $X$. Let $x \in X$. If $x \notin D_n$, then obviously $\mu(0 \ast x) \geq 0 = \mu(x)$. If $x \in D_n \setminus D_{n-1}$ for $n = 2, 3, \ldots$, then $0 \ast x \in D_n$. Hence $\mu(0 \ast x) \geq 1/n = \mu(x)$. Let $x, y \in X$. If $x \ast y \in D_n \setminus D_{n-1}$ and $y \in D_n \setminus D_{n-1}$ for $n = 2, 3, \ldots$, then $x \in D_n$. It follows that

$$\mu(x) \geq \frac{1}{n} = \min \{\mu(x \ast y), \mu(y)\}. \quad (3.26)$$

Suppose that $x \ast y \in D_n$ and $y \in D_n \setminus D_m$ for all $m < n$. Then $x \in D_n$, and so $\mu(x) \geq 1/n \geq 1/m + 1 \geq \mu(y)$. Hence $\mu(x) \geq \min \{\mu(x \ast y), \mu(y)\}$. Similarly for the case $x \ast y \in D_n \setminus D_m$ and $y \in D_n$, we get $\mu(x) \geq \min \{\mu(x \ast y), \mu(y)\}$. Therefore $\mu$ is a fuzzy closed ideal of $X$. Since the chain (3.24) is not terminating, $\mu$ has a strictly descending sequence of values. This contradicts that the value set of any fuzzy closed ideal is well ordered. This completes the proof. 

4. $T$-fuzzy subalgebras and $T$-fuzzy closed ideals

**Definition 4.1.** A fuzzy set $\mu$ in $X$ is said to satisfy imaginable property if $\text{Im}(\mu) \subseteq \Delta_T$.

**Definition 4.2.** A fuzzy set $\mu$ in $X$ is called a fuzzy subalgebra of $X$ with respect to a $t$-norm $T$ (briefly, $T$-fuzzy subalgebra of $X$) if $\mu(x \ast y) \geq T(\mu(x), \mu(y))$ for all $x, y \in X$. A $T$-fuzzy subalgebra of $X$ is said to be imaginable if it satisfies the imaginable property.

**Example 4.3.** Let $T_m$ be a $t$-norm defined by $T_m(\alpha, \beta) = \max(\alpha + \beta - 1, 0)$ for all $\alpha, \beta \in [0, 1]$ and let $X = \{0, a, b, c, d\}$ be a BCH-algebra with the following Cayley table:

<table>
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<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>d</td>
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<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
<td>a</td>
<td>d</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>0</td>
<td>0</td>
<td>d</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>c</td>
<td>0</td>
<td>d</td>
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<td>d</td>
<td>d</td>
<td>d</td>
<td>d</td>
<td>0</td>
</tr>
</tbody>
</table>

(1) Define a fuzzy set $\mu : X \to [0, 1]$ by

$$\mu(x) = \begin{cases} 
0.9 & \text{if } x \in \{0, d\}, \\
0.09 & \text{otherwise.}
\end{cases} \quad (4.1)$$

Then $\mu$ is a $T_m$-fuzzy subalgebra of $X$, which is not imaginable.

(2) Let $\nu$ be a fuzzy set in $X$ defined by

$$\nu(x) = \begin{cases} 
1 & \text{if } x \in \{0, d\}, \\
0 & \text{otherwise.}
\end{cases} \quad (4.2)$$

Then $\nu$ is an imaginable $T_m$-fuzzy subalgebra of $X$. 

Proposition 4.4. Let $A$ be a subalgebra of $X$ and let $\mu$ be a fuzzy set in $X$ defined by

$$
\mu(x) := \begin{cases} 
\alpha_1 & \text{if } x \in A, \\
\alpha_2 & \text{otherwise,}
\end{cases}
$$

(4.3)

for all $x \in X$, where $\alpha_1, \alpha_2 \in [0,1]$ with $\alpha_1 > \alpha_2$. Then $\mu$ is a $T_m$-fuzzy subalgebra of $X$. In particular, if $\alpha_1 = 1$ and $\alpha_2 = 0$ then $\mu$ is an imaginable $T_m$-fuzzy subalgebra of $X$, where $T_m$ is the t-norm in Example 4.3.

Proof. Let $x, y \in X$. If $x \in A$ and $y \in A$ then

$$
T_m(\mu(x), \mu(y)) = T_m(\alpha_1, \alpha_1) = \max(2\alpha_1 - 1, 0)
$$

$$
= \begin{cases} 
2\alpha_1 - 1 & \text{if } \alpha_1 \geq \frac{1}{2} \\
0 & \text{if } \alpha_1 < \frac{1}{2}
\end{cases}
$$

(4.4)

$$
\leq \alpha_1 = \mu(x \ast y).
$$

If $x \in A$ and $y \notin A$ (or, $x \notin A$ and $y \in A$) then

$$
T_m(\mu(x), \mu(y)) = T_m(\alpha_1, \alpha_2) = \max(\alpha_1 + \alpha_2 - 1, 0)
$$

$$
= \begin{cases} 
\alpha_1 + \alpha_2 - 1 & \text{if } \alpha_1 + \alpha_2 \geq 1 \\
0 & \text{otherwise}
\end{cases}
$$

(4.5)

$$
\leq \alpha_2 = \mu(x \ast y).
$$

If $x, y \notin A$ then

$$
T_m(\mu(x), \mu(y)) = T_m(2\alpha_2 - 1, 0)
$$

$$
= \begin{cases} 
2\alpha_2 - 1 & \text{if } \alpha_2 \geq \frac{1}{2} \\
0 & \text{if } \alpha_2 < \frac{1}{2}
\end{cases}
$$

(4.6)

$$
\leq \alpha_2 = \mu(x \ast y).
$$

Hence $\mu$ is a $T_m$-fuzzy subalgebra of $X$. Assume that $\alpha_1 = 1$ and $\alpha_2 = 0$. Then

$$
T_m(\alpha_1, \alpha_1) = \max(\alpha_1 + \alpha_1 - 1, 0) = 1 = \alpha_1,
$$

$$
T_m(\alpha_2, \alpha_2) = \max(\alpha_2 + \alpha_2 - 1, 0) = 0 = \alpha_2.
$$

(4.7)

Thus $\alpha_1, \alpha_2 \in \Delta_{T_m}$, that is, $\operatorname{Im}(\mu) \subseteq \Delta_{T_m}$ and so $\mu$ is imaginable. This completes the proof. 

Proposition 4.5. If $\mu$ is an imaginable $T$-fuzzy subalgebra of $X$, then $\mu(0 \ast x) \geq \mu(x)$ for all $x \in X$. 

Proof. For any \( x \in X \) we have
\[
\mu(0 \ast x) \geq T(\mu(0), \mu(x)) = T(\mu(x \ast x), \mu(x)) \quad \text{[by (H1)]}
\]
\[
\geq T(T(\mu(x), \mu(x)), \mu(x)) \quad \text{[by (T2) and (T3)]}
\]
\[
= \mu(x), \quad \text{[since \( \mu \) satisfies the imaginable property].}
\]
This completes the proof. \( \Box \)

Theorem 4.6. Let \( \mu \) be a \( T \)-fuzzy subalgebra of \( X \) and let \( \alpha \in [0,1] \) be such that \( T(\alpha, \alpha) = \alpha \). Then \( U(\mu; \alpha) \) is either empty or a subalgebra of \( X \), and moreover \( \mu(0) \geq \mu(x) \) for all \( x \in X \).

Proof. Let \( x, y \in U(\mu; \alpha) \). Then
\[
\mu(x \ast y) \geq T(\mu(x), \mu(y)) \geq T(\alpha, \alpha) = \alpha,
\]
which implies that \( x \ast y \in U(\mu; \alpha) \). Hence \( U(\mu; \alpha) \) is a subalgebra of \( X \). Since \( x \ast x = 0 \) for all \( x \in X \), we have \( \mu(0) = \mu(x \ast x) \geq T(\mu(x), \mu(x)) = \mu(x) \) for all \( x \in X \). \( \Box \)

Since \( T(1,1) = 1 \), we have the following corollary.

Corollary 4.7. If \( \mu \) is a \( T \)-fuzzy subalgebra of \( X \), then \( U(\mu; 1) \) is either empty or a subalgebra of \( X \).

Theorem 4.8. Let \( \mu \) be a \( T \)-fuzzy subalgebra of \( X \). If there is a sequence \( \{x_n\} \) in \( X \) such that \( \lim_{n \to \infty} T(\mu(x_n), \mu(x_n)) = 1 \), then \( \mu(0) = 1 \).

Proof. Let \( x \in X \). Then \( \mu(0) = \mu(x \ast x) \geq T(\mu(x), \mu(x)) \). Therefore \( \mu(0) \geq T(\mu(x_n), \mu(x_n)) \) for each \( n \in \mathbb{N} \). Since \( 1 \geq \mu(0) \geq \lim_{n \to \infty} T(\mu(x_n), \mu(x_n)) = 1 \), it follows that \( \mu(0) = 1 \), this completes the proof. \( \Box \)

Let \( f : X \to Y \) be a mapping of BCH-algebras. For a fuzzy set \( \mu \) in \( X \), the inverse image of \( \mu \) under \( f \), denoted by \( f^{-1}(\mu) \), is defined by \( f^{-1}(\mu)(x) = \mu(f(x)) \) for all \( x \in X \).

Theorem 4.9. Let \( f : X \to Y \) be a homomorphism of BCH-algebras. If \( \mu \) is a \( T \)-fuzzy subalgebra of \( Y \), then \( f^{-1}(\mu) \) is a \( T \)-fuzzy subalgebra of \( X \).

Proof. For any \( x, y \in X \), we have
\[
f^{-1}(\mu)(x \ast y) = \mu(f(x \ast y)) = \mu(f(x) \ast f(y))
\]
\[
\geq T(\mu(f(x)), \mu(f(y))) \quad \text{[by (T2) and (T3)]}
\]
\[
= T(f^{-1}(\mu)(x), f^{-1}(\mu)(y)).
\]
This completes the proof. \( \Box \)

If \( \mu \) is a fuzzy set in \( X \) and \( f \) is a mapping defined on \( X \). The fuzzy set \( f(\mu) \) in \( f(X) \) defined by \( f(\mu)(y) = \sup \{\mu(x) \mid x \in f^{-1}(y)\} \) for all \( y \in f(X) \) is called the image of \( \mu \) under \( f \). A fuzzy set \( \mu \) in \( X \) is said to have sup property if, for every subset \( T \subseteq X \), there exists \( t_0 \in T \) such that \( \mu(t_0) = \sup \{\mu(t) \mid t \in T\} \).
Theorem 4.10. An onto homomorphic image of a fuzzy subalgebra with sup property is a fuzzy subalgebra.

Proof. Let \( f : X \to Y \) be an onto homomorphism of BCH-algebras and let \( \mu \) be a fuzzy subalgebra of \( X \) with sup property. Given \( u, v \in Y \), let \( x_0 \in f^{-1}(u) \) and \( y_0 \in f^{-1}(v) \) be such that

\[
\mu(x_0) = \sup \{ \mu(t) \mid t \in f^{-1}(u) \}, \quad \mu(y_0) = \sup \{ \mu(t) \mid t \in f^{-1}(v) \},
\]
respectively. Then

\[
f(\mu)(u * v) = \sup \{ \mu(z) \mid z \in f^{-1}(u * v) \}
\geq \min \{ \mu(x_0), \mu(y_0) \}
= \min \{ \sup \{ \mu(t) \mid t \in f^{-1}(u) \}, \sup \{ \mu(t) \mid t \in f^{-1}(v) \} \}
= \min \{ f(\mu)(u), f(\mu)(v) \}.
\]

Hence \( f(\mu) \) is a fuzzy subalgebra of \( Y \). \( \square \)

Theorem 4.10 can be strengthened in the following way. To do this we need the following definition.

Definition 4.11. A \( t \)-norm \( T \) on \([0,1]\) is called a continuous \( t \)-norm if \( T \) is a continuous function from \([0,1] \times [0,1]\) to \([0,1]\) with respect to the usual topology.

Note that the function “min” is a continuous \( t \)-norm.

Theorem 4.12. Let \( T \) be a continuous \( t \)-norm and let \( f : X \to Y \) be an onto homomorphism of BCH-algebras. If \( \mu \) is a \( T \)-fuzzy subalgebra of \( X \), then \( f(\mu) \) is a \( T \)-fuzzy subalgebra of \( Y \).

Proof. Let \( A_1 = f^{-1}(y_1), A_2 = f^{-1}(y_2) \), and \( A_{12} = f^{-1}(y_1 * y_2) \), where \( y_1, y_2 \in Y \). Consider the set

\[
A_1 * A_2 := \{ x \in X \mid x = a_1 * a_2 \text{ for some } a_1 \in A_1, a_2 \in A_2 \}.
\]

If \( x \in A_1 * A_2 \), then \( x = x_1 * x_2 \) for some \( x_1 \in A_1 \) and \( x_2 \in A_2 \) and so

\[
f(x) = f(x_1 * x_2) = f(x_1) * f(x_2) = y_1 * y_2,
\]
that is, \( x \in f^{-1}(y_1 * y_2) = A_{12} \). Thus \( A_1 * A_2 \subseteq A_{12} \). It follows that

\[
f(\mu)(y_1 * y_2) = \sup \{ \mu(x) \mid x \in f^{-1}(y_1 * y_2) \} = \sup \{ \mu(x) \mid x \in A_{12} \}
\geq \sup \{ \mu(x) \mid x \in A_1 * A_2 \}
\geq \sup \{ \mu(x_1 * x_2) \mid x_1 \in A_1, x_2 \in A_2 \}
\geq \sup \{ T(\mu(x_1), \mu(x_2)) \mid x_1 \in A_1, x_2 \in A_2 \}.
\]

Since \( T \) is continuous, for every \( \varepsilon > 0 \) there exists a number \( \delta > 0 \) such that if \( \sup \{ \mu(x_1) \mid x_1 \in A_1 \} - x_1^* \leq \delta \) and \( \sup \{ \mu(x_2) \mid x_2 \in A_2 \} - x_2^* \leq \delta \) then

\[
T(\sup \{ \mu(x_1) \mid x_1 \in A_1 \}, \sup \{ \mu(x_2) \mid x_2 \in A_2 \}) - T(x_1^*, x_2^*) \leq \varepsilon.
\]
Choose \( a_1 \in A_1 \) and \( a_2 \in A_2 \) such that \( \sup \{ \mu(x_1) \mid x_1 \in A_1 \} - \mu(a_1) \leq \delta \) and \( \sup \{ \mu(x_2) \mid x_2 \in A_2 \} - \mu(a_2) \leq \delta \). Then
\[
T(\sup \{ \mu(x_1) \mid x_1 \in A_1 \}, \sup \{ \mu(x_2) \mid x_2 \in A_2 \}) - T(\mu(a_1), \mu(a_2)) \leq \varepsilon. \tag{4.17}
\]
Consequently
\[
f(\mu)(y_1 * y_2) \geq \sup \{ T(\mu(x_1), \mu(x_2)) \mid x_1 \in A_1, x_2 \in A_2 \}
\geq T(\sup \{ \mu(x_1) \mid x_1 \in A_1 \}, \sup \{ \mu(x_2) \mid x_2 \in A_2 \})
= T(f(\mu)(y_1), f(\mu)(y_2)),
\tag{4.18}
\]
which shows that \( f(\mu) \) is a \( T \)-fuzzy subalgebra of \( Y \).

\textbf{Lemma 4.13} (see [1]). For all \( \alpha, \beta, y, \delta \in [0, 1] \),
\[
T(T(\alpha, \beta), T(y, \delta)) = T(T(\alpha, y), T(\beta, \delta)). \tag{4.19}
\]

\textbf{Theorem 4.14.} Let \( X = X_1 \times X_2 \) be the direct product BCH-algebra of BCH-algebras \( X_1 \) and \( X_2 \). If \( \mu_1 \) (resp., \( \mu_2 \)) is a \( T \)-fuzzy subalgebra of \( X_1 \) (resp., \( X_2 \)), then \( \mu = \mu_1 \times \mu_2 \) is a \( T \)-fuzzy subalgebra of \( X \) defined by
\[
\mu(x_1, x_2) = (\mu_1 \times \mu_2)(x_1, x_2) = T(\mu_1(x_1), \mu_2(x_2)), \tag{4.20}
\]
for all \( (x_1, x_2) \in X_1 \times X_2 \).

\textbf{Proof.} Let \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \) be any elements of \( X = X_1 \times X_2 \). Then
\[
\mu(x * y) = \mu((x_1, x_2) * (y_1, y_2)) = \mu(x_1 * y_1, x_2 * y_2)
= T(\mu_1(x_1 * y_1), \mu_2(x_2 * y_2))
\geq T(T(\mu_1(x_1), \mu_1(y_1)), T(\mu_2(x_2), \mu_2(y_2)))
= T(T(\mu_1(x_1), \mu_2(x_2)), T(\mu_1(y_1), \mu_2(y_2)))
= T(\mu(x_1, x_2), \mu(x_2, y_2))
= T(\mu(x), \mu(y)).
\tag{4.21}
\]
Hence \( \mu \) is a \( T \)-fuzzy subalgebra of \( X \). \( \square \)

We will generalize the idea to the product of \( n \) \( T \)-fuzzy subalgebras. We first need to generalize the domain of \( T \) to \( \prod_{i=1}^{n}[0, 1] \) as follows:

\textbf{Definition 4.15} (see [1]). The function \( T_n : \prod_{i=1}^{n}[0, 1] \to [0, 1] \) is defined by
\[
T_n(\alpha_1, \alpha_2, \ldots, \alpha_n) = T(\alpha_i, T_{n-1}(\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n)), \tag{4.22}
\]
for all \( 1 \leq i \leq n \), where \( n \geq 2 \), \( T_2 = T \), and \( T_1 = \text{id} \) (identity).

\textbf{Lemma 4.16} (see [1]). For every \( \alpha_i, \beta_i \in [0, 1] \) where \( 1 \leq i \leq n \) and \( n \geq 2 \),
\[
T_n(T(\alpha_1, \beta_1), T(\alpha_2, \beta_2), \ldots, T(\alpha_n, \beta_n)) = T(T_n(\alpha_1, \alpha_2, \ldots, \alpha_n), T_n(\beta_1, \beta_2, \ldots, \beta_n)). \tag{4.23}
\]
**Theorem 4.17.** Let \( \{X_i\}_{i=1}^n \) be the finite collection of BCH-algebras and \( X = \prod_{i=1}^n X_i \) the direct product BCH-algebra of \( \{X_i\} \). Let \( \mu_i \) be a \( T \)-fuzzy subalgebra of \( X_i \), where \( 1 \leq i \leq n \). Then \( \mu = \prod_{i=1}^n \mu_i \) defined by

\[
\mu(x_1, x_2, \ldots, x_n) = \left( \prod_{i=1}^n \mu_i \right)(x_1, x_2, \ldots, x_n) = T_n(\mu_1(x_1), \mu_2(x_2), \ldots, \mu_n(x_n)),
\]

is a \( T \)-fuzzy subalgebra of the BCH-algebra \( X \).

**Proof.** Let \( x = (x_1, x_2, \ldots, x_n) \) and \( y = (y_1, y_2, \ldots, y_n) \) be any elements of \( X = \prod_{i=1}^n X_i \). Then

\[
\mu(x \star y) = \mu(x_1 \star y_1, x_2 \star y_2, \ldots, x_n \star y_n)
= T_n(\mu_1(x_1 \star y_1), \mu_2(x_2 \star y_2), \ldots, \mu_n(x_n \star y_n))
\geq T_n(T(\mu_1(x_1), \mu_1(y_1)), T(\mu_2(x_2), \mu_2(y_2)), \ldots, T(\mu_n(x_n), \mu_n(y_n)))
= T(T_n(\mu_1(x_1), \mu_2(x_2), \ldots, \mu_n(x_n)), T_n(\mu_1(y_1), \mu_2(y_2), \ldots, \mu_n(y_n)))
= T(\mu(x_1, x_2, \ldots, x_n), \mu(y_1, y_2, \ldots, y_n))
= T(\mu(x), \mu(y)).
\]

Hence \( \mu \) is a \( T \)-fuzzy subalgebra of \( X \).

**Definition 4.18.** Let \( \mu \) and \( \nu \) be fuzzy sets in \( X \). Then the \( T \)-product of \( \mu \) and \( \nu \), written \([\mu \cdot \nu]_T\), is defined by \([\mu \cdot \nu]_T(x) = T(\mu(x), \nu(x))\) for all \( x \in X \).

**Theorem 4.19.** Let \( \mu \) and \( \nu \) be \( T \)-fuzzy subalgebras of \( X \). If \( T^* \) is a \( t \)-norm which dominates \( T \), that is,

\[
T^*(T(\alpha, \beta), T(\gamma, \delta)) \geq T(T^*(\alpha, \gamma), T^*(\beta, \delta)),
\]

for all \( \alpha, \beta, \gamma, \delta \in [0,1] \), then the \( T^* \)-product of \( \mu \) and \( \nu \), \([\mu \cdot \nu]_{T^*}\), is a \( T \)-fuzzy subalgebra of \( X \).

**Proof.** For any \( x, y \in X \) we have

\[
[\mu \cdot \nu]_{T^*}(x \star y) = T^*(\mu(x \star y), \nu(x \star y))
\geq T^*(T(\mu(x), \mu(y)), T(\nu(x), \nu(y)))
\geq T(T^*(\mu(x), \nu(x)), T^*(\mu(y), \nu(y)))
= T([\mu \cdot \nu]_{T^*}(x), [\mu \cdot \nu]_{T^*}(y)).
\]

Hence \([\mu \cdot \nu]_{T^*}\) is a \( T \)-fuzzy subalgebra of \( X \).

Let \( f : X \to Y \) be an onto homomorphism of BCH-algebras. Let \( T \) and \( T^* \) be \( t \)-norms such that \( T^* \) dominates \( T \). If \( \mu \) and \( \nu \) are \( T \)-fuzzy subalgebras of \( Y \), then the \( T^* \)-product of \( \mu \) and \( \nu \), \([\mu \cdot \nu]_{T^*}\), is a \( T \)-fuzzy subalgebra of \( Y \). Since every onto homomorphic inverse image of a \( T \)-fuzzy subalgebra is a \( T \)-fuzzy subalgebra, the
inverse images \( f^{-1}(\mu), f^{-1}(\nu), \) and \( f^{-1}([\mu \cdot \nu]_{T^*}) \) are \( T \)-fuzzy subalgebras of \( X \). The next theorem provides that the relation between \( f^{-1}([\mu \cdot \nu]_{T^*}) \) and the \( T^* \)-product \([f^{-1}(\mu) \cdot f^{-1}(\nu)]_{T^*}\) of \( f^{-1}(\mu) \) and \( f^{-1}(\nu) \).

**Theorem 4.20.** Let \( f : X \to Y \) be an onto homomorphism of BCH-algebras. Let \( T^* \) be a \( t \)-norm such that \( T^* \) dominates \( T \). Let \( \mu \) and \( \nu \) be \( T \)-fuzzy subalgebras of \( Y \). If \([\mu \cdot \nu]_{T^*}\) is the \( T^* \)-product of \( \mu \) and \( \nu \) and \([f^{-1}(\mu) \cdot f^{-1}(\nu)]_{T^*}\) is the \( T^* \)-product of \( f^{-1}(\mu) \) and \( f^{-1}(\nu) \), then

\[
f^{-1}([\mu \cdot \nu]_{T^*}) = [f^{-1}(\mu) \cdot f^{-1}(\nu)]_{T^*}.
\]

**Proof.** For any \( x \in X \) we get

\[
f^{-1}([\mu \cdot \nu]_{T^*})(x) = [\mu \cdot \nu]_{T^*}(f(x))
= T^*(\mu(f(x)), \nu(f(x)))
= T^*(f^{-1}(\mu)(x), f^{-1}(\nu)(x))
= [f^{-1}(\mu) \cdot f^{-1}(\nu)]_{T^*}(x),
\]

This completes the proof.

**Definition 4.21.** A fuzzy set \( \mu \) in \( X \) is called a fuzzy closed ideal of \( X \) under a \( t \)-norm \( T \) (briefly, \( T \)-fuzzy closed ideal of \( X \)) if

(F1) \( \mu(0 \ast x) \geq \mu(x) \) for all \( x \in X \),
(F3) \( \mu(x) \geq T(\mu(x \ast y), \mu(y)) \) for all \( x, y \in X \).

A \( T \)-fuzzy closed ideal of \( X \) is said to be imaginable if it satisfies the imaginable property.

**Example 4.22.** Let \( T_m \) be a \( t \)-norm in Example 4.3. Consider a BCH-algebra \( X = \{0, a, b, c\} \) with Cayley table as follows:

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<th>0</th>
<th>a</th>
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<th>c</th>
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<td>0</td>
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<td>c</td>
<td>c</td>
<td>0</td>
<td>c</td>
<td>0</td>
</tr>
</tbody>
</table>

(1) Define a fuzzy set \( \mu : X \to [0, 1] \) by \( \mu(0) = \mu(c) = 0.8 \) and \( \mu(a) = \mu(b) = 0.3 \). Then \( \mu \) is a \( T_m \)-fuzzy closed ideal of \( X \) which is not imaginable.

(2) Let \( \nu \) be a fuzzy set in \( X \) defined by

\[
\nu(x) = \begin{cases} 
1 & \text{if } x \in \{0, c\}, \\
0 & \text{otherwise.}
\end{cases}
\]

Then \( \nu \) is an imaginable \( T_m \)-fuzzy closed ideal of \( X \).

**Theorem 4.23.** Every imaginable \( T \)-fuzzy subalgebra satisfying (F3) is an imaginable \( T \)-fuzzy closed ideal.

**Proof.** Using Proposition 4.5, it is straightforward.
Proposition 4.24. If \( \mu \) is an imaginable \( T \)-fuzzy closed ideal of \( X \), then \( \mu(0) \geq \mu(x) \) for all \( x \in X \).

**Proof.** Using (F1), (F3), and (T2), we have

\[
\mu(0) \geq T(\mu(0 * x), \mu(x)) \geq T(\mu(x), \mu(x)) = \mu(x) \tag{4.31}
\]

for all \( x \in X \), completing the proof.

Theorem 4.25. Every \( T \)-fuzzy closed ideal is a \( T \)-fuzzy subalgebra.

**Proof.** Let \( \mu \) be a \( T \)-fuzzy closed ideal of \( X \) and let \( x, y \in X \). Then

\[
\mu(x * y) \geq T(\mu((x * y) * x), \mu(x)) \text{ [by (F3)]} \\
= T(\mu(x * x) * y), \mu(x)) \text{ [by (H3)]} \\
= T(\mu(0 * y), \mu(x)) \text{ [by (H1)]} \\
\geq T(\mu(x), \mu(y)) \text{ [by (F1), (T2), and (T3)].} \tag{4.32}
\]

Hence \( \mu \) is a \( T \)-fuzzy subalgebra of \( X \).

The converse of Theorem 4.25 may not be true. For example, the \( T_m \)-fuzzy subalgebra \( \mu \) in Example 4.3(1) is not a \( T_m \)-fuzzy closed ideal of \( X \) since

\[
\mu(a) = 0.09 < 0.9 = T_m(\mu(a * d), \mu(d)). \tag{4.33}
\]

We give a condition for a \( T \)-fuzzy subalgebra to be a \( T \)-fuzzy closed ideal.

**Theorem 4.26.** Let \( \mu \) be a \( T \)-fuzzy subalgebra of \( X \). If \( \mu \) satisfies the imaginable property and the inequality

\[
\mu(x * y) \leq \mu(y * x) \quad \forall x, y \in X, \tag{4.34}
\]

then \( \mu \) is a \( T \)-fuzzy closed ideal of \( X \).

**Proof.** Let \( \mu \) be an imaginable \( T \)-fuzzy subalgebra of \( X \) which satisfies the inequality

\[
\mu(x * y) \leq \mu(y * x) \quad \forall x, y \in X. \tag{4.35}
\]

It follows from Proposition 4.5 that \( \mu(0 * x) \geq \mu(x) \) for all \( x \in X \). Let \( x, y \in X \). Then

\[
\mu(x) = \mu(x * 0) \geq \mu(0 * x) = \mu((y * y) * x) \\
= \mu((y * x) * y) \geq T(\mu(y * x), \mu(y)) \geq T(\mu(x * y), \mu(y)). \tag{4.36}
\]

Hence \( \mu \) is a \( T \)-fuzzy closed ideal of \( X \).

**Proposition 4.27.** Let \( T_m \) be a \( t \)-norm in Example 4.3. Let \( D \) be a closed ideal of \( X \) and let \( \mu \) be a fuzzy set in \( X \) defined by

\[
\mu(x) = \begin{cases} 
\alpha_1 & \text{if } x \in D, \\
\alpha_2 & \text{otherwise},
\end{cases} \tag{4.37}
\]

for all \( x \in X \).
Similarly we have
\[ \mu(x) \neq 0 \quad \text{for all } x, y \in X. \]
Assume that \( x \notin D \). Then \( x \neq 0 \) and \( y \notin D \), that is, \( \mu(x) = 0 \) or \( \mu(y) = 0 \). It follows that
\[ T_m(\mu(x \cdot y), \mu(y)) = 0 = \mu(x). \]
Hence \( \mu(x) \geq T_m(\mu(x \cdot y), \mu(y)) \) for all \( x, y \in X \). Clearly \( \text{Im}(\mu) \subseteq \Delta T_m \).

(ii) Similar to (i), we know that \( \mu \) is a \( T_m \)-fuzzy closed ideal of \( X \). Taking \( \alpha_1 = 0.7 \), then
\[ T_m(\alpha_1, \alpha_1) = T_m(0.7, 0.7) = \max(0.7 + 0.7 - 1, 0) = 0.4 \neq \alpha_1. \]
Hence \( \alpha_1 \notin \Delta T_m \), that is, \( \text{Im}(\mu) \notin \Delta T_m \), and so \( \mu \) is not imaginable.

**Proposition 4.28.** Let \( \mu \) be an imaginable \( T \)-fuzzy closed ideal of \( X \). If \( \mu \) satisfies the inequality \( \mu(x) \geq \mu(0 \cdot x) \) for all \( x \in X \), then it satisfies the equality \( \mu(x \cdot y) = \mu(y \cdot x) \) for all \( x, y \in X \).

**Proof.** Let \( \mu \) be an imaginable \( T \)-fuzzy closed ideal of \( X \) satisfying the inequality \( \mu(x) \geq \mu(0 \cdot x) \) for all \( x \in X \). For every \( x, y \in X \), we have
\[
\mu(y \cdot x) \geq \mu(0 \cdot (y \cdot x)) \quad \text{[by assumption]}
\]
\[
\geq T(\mu((0 \cdot (y \cdot x)) \cdot (x \cdot y)), \mu(x \cdot y)) \quad \text{[by (F3)]}
\]
\[
= T(\mu(((0 \cdot y) \cdot (0 \cdot x)) \cdot (x \cdot y)), \mu(x \cdot y)) \quad \text{[by (P3)]}
\]
\[
= T(\mu(((0 \cdot y) \cdot (x \cdot y)) \cdot (0 \cdot x)), \mu(x \cdot y)) \quad \text{[by (H3)]}
\]
\[
= T(\mu(((0 \cdot (x \cdot y)) \cdot y) \cdot (0 \cdot x)), \mu(x \cdot y)) \quad \text{[by (H3)]}
\]
\[
= T(\mu(((0 \cdot x) \cdot (0 \cdot y)) \cdot y), \mu(x \cdot y)) \quad \text{[by (H3)]}
\]
\[
= T(\mu(((0 \cdot x) \cdot (0 \cdot y)) \cdot (0 \cdot x) \cdot y), \mu(x \cdot y)) \quad \text{[by (H3)]}
\]
\[
= T(\mu((0 \cdot (0 \cdot y)) \cdot y), \mu(x \cdot y)) \quad \text{[by (H1)]}
\]
\[
= T(\mu(0), \mu(x \cdot y)) \quad \text{[by (H3) and (H1)]}
\]
\[
= T(\mu((x \cdot y) \cdot (x \cdot y)), \mu(x \cdot y)) \quad \text{[by (H1)]}
\]
\[
\geq T(T(\mu(x \cdot y), \mu(x \cdot y)), \mu(x \cdot y)) \quad \text{[by Proposition 4.24 and (T2)]}
\]
\[
= \mu(x \cdot y) \quad \text{[since } \mu \text{ is imaginable].}
\]

(4.41)

Similarly we have \( \mu(x \cdot y) \geq \mu(y \cdot x) \) for all \( x, y \in X \), completing the proof. 

\[ \square \]
**Theorem 4.29.** Every imaginable $T$-fuzzy closed ideal is a fuzzy closed ideal.

**Proof.** Let $\mu$ be an imaginable $T$-fuzzy closed ideal of $X$. Then

$$\mu(x) \geq T(\mu(x \ast y), \mu(y)) \quad \forall x, y \in X.$$  \hspace{1cm} (4.42)

Since $\mu$ is imaginable, we have

$$\min (\mu(x \ast y), \mu(y)) = T(\min (\mu(x \ast y), \mu(y)), \min (\mu(x \ast y), \mu(y)))$$

$$\leq T(\mu(x \ast y), \mu(y)) \quad \text{(4.43)}$$

$$\leq \min (\mu(x \ast y), \mu(y)).$$

It follows that $\mu(x) \geq T(\mu(x \ast y), \mu(y)) = \min(\mu(x \ast y), \mu(y))$ so that $\mu$ is a fuzzy closed ideal of $X$.

Combining Theorems 3.3, 4.29, we have the following corollary.

**Corollary 4.30.** If $\mu$ is an imaginable $T$-fuzzy closed ideal of $X$, then the nonempty level set of $\mu$ is a closed ideal of $X$.

Noticing that the fuzzy set $\mu$ in Example 4.22(1) is a fuzzy closed ideal of $X$, we know from Example 4.22(1) that there exists a $t$-norm such that the converse of Theorem 4.29 may not be true.

**Proposition 4.31.** Every imaginable $T$-fuzzy closed ideal is order reversing.

**Proof.** Let $\mu$ be an imaginable $T$-fuzzy closed ideal of $X$ and let $x, y \in X$ be such that $x \leq y$. Using (P4), (T2), Theorem 4.29, Proposition 4.24, and the definition of a fuzzy closed ideal, we get

$$\mu(x) \geq \min \{\mu(x \ast y), \mu(y)\} \geq T(\mu(x \ast y), \mu(y))$$

$$= T(\mu(0), \mu(y)) \geq T(\mu(y), \mu(y)) = \mu(y).$$  \hspace{1cm} (4.44)

This completes the proof.

**Proposition 4.32.** Let $\mu$ be a $T$-fuzzy closed ideal of $X$, where $T$ is a diagonal $t$-norm on $[0, 1]$, that is, $T(\alpha, \alpha) = \alpha$ for all $\alpha \in [0, 1]$. If $(x \ast a) \ast b = 0$ for all $a, b, x \in X$, then $\mu(x) \geq T(\mu(a), \mu(b))$.

**Proof.** Let $a, b, x \in X$ be such that $(x \ast a) \ast b = 0$. Then

$$\mu(x) \geq T(\mu(x \ast a), \mu(a))$$

$$\geq T(T(\mu((x \ast a) \ast b), \mu(b)), \mu(a))$$

$$= T(T(\mu(0), \mu(b)), \mu(a))$$

$$\geq T(T(\mu(b), \mu(b)), \mu(a))$$

$$= T(\mu(a), \mu(b)),$$  \hspace{1cm} (4.45)

completing the proof.
Corollary 4.33. Let $\mu$ be a $T$-fuzzy closed ideal of $X$, where $T$ is a diagonal $t$-norm on $[0,1]$. If $(\cdots ((x \ast a_1) \ast a_2) \ast \cdots) \ast a_n = 0$ for all $x, a_1, a_2, \ldots, a_n \in X$, then
\[
\mu(x) \geq T_n(\mu(a_1), \mu(a_2), \ldots, \mu(a_n)).
\] (4.46)

**Proof.** Using induction on $n$, the proof is straightforward. \qed

**Theorem 4.34.** There exists a $t$-norm $T$ such that every closed ideal of $X$ can be realized as a level closed ideal of a $T$-fuzzy closed ideal of $X$.

**Proof.** Let $D$ be a closed ideal of $X$ and let $\mu$ be a fuzzy set in $X$ defined by
\[
\mu(x) = \begin{cases} 
\alpha & \text{if } x \in D, \\
0 & \text{otherwise},
\end{cases}
\] (4.47)
where $\alpha \in (0,1)$ is fixed. It is clear that $U(\mu; \alpha) = D$. We will prove that $\mu$ is a $T_m$-fuzzy closed ideal of $X$, where $T_m$ is a $t$-norm in Example 4.3. If $x \in D$, then $0 \ast x \in D$ and so $\mu(0 \ast x) = \alpha = \mu(x)$. If $x \notin D$, then clearly $\mu(x) = 0 \leq \mu(0 \ast x)$. Let $x, y \in X$. If $x \in D$, then $\mu(x) = \alpha \geq T_m(\mu(x \ast y), \mu(y))$. If $x \notin D$, then $x \ast y \notin D$ or $y \notin D$. It follows that $\mu(x) = 0 \leq T_m(\mu(x \ast y), \mu(y))$. This completes the proof. \qed

For a family $\{\mu_\alpha \mid \alpha \in \Lambda\}$ of fuzzy sets in $X$, define the join $\vee_{\alpha \in \Lambda} \mu_\alpha$ and the meet $\wedge_{\alpha \in \Lambda} \mu_\alpha$ as follows:
\[
(\vee_{\alpha \in \Lambda} \mu_\alpha)(x) = \sup \{\mu_\alpha(x) \mid \alpha \in \Lambda\}, \quad (\wedge_{\alpha \in \Lambda} \mu_\alpha)(x) = \inf \{\mu_\alpha(x) \mid \alpha \in \Lambda\},
\] (4.48)
for all $x \in X$, where $\Lambda$ is any index set.

**Theorem 4.35.** The family of $T$-fuzzy closed ideals in $X$ is a completely distributive lattice with respect to meet $\wedge$ and the join $\vee$.

**Proof.** Since $[0,1]$ is a completely distributive lattice with respect to the usual ordering in $[0,1]$, it is sufficient to show that $\vee_{\alpha \in \Lambda} \mu_\alpha$ and $\wedge_{\alpha \in \Lambda} \mu_\alpha$ are $T$-fuzzy closed ideals of $X$ for a family of $T$-fuzzy closed ideals $\{\mu_\alpha \mid \alpha \in \Lambda\}$. For any $x \in X$, we have
\[
(\vee_{\alpha \in \Lambda} \mu_\alpha)(0 \ast x) = \sup \{\mu_\alpha(0 \ast x) \mid \alpha \in \Lambda\} \\
\geq \sup \{\mu_\alpha(x) \mid \alpha \in \Lambda\} \\
= (\vee_{\alpha \in \Lambda} \mu_\alpha)(x),
\]
(4.49)
\[
(\wedge_{\alpha \in \Lambda} \mu_\alpha)(0 \ast x) = \inf \{\mu_\alpha(0 \ast x) \mid \alpha \in \Lambda\} \\
\geq \inf \{\mu_\alpha(x) \mid \alpha \in \Lambda\} \\
= (\wedge_{\alpha \in \Lambda} \mu_\alpha)(x).
\]
Let $x, y \in X$. Then
\[
(\vee_{\alpha \in \Lambda} \mu_\alpha)(x) = \sup \{\mu_\alpha(x) \mid \alpha \in \Lambda\} \\
\geq \sup \{T(\mu_\alpha(x \ast y), \mu_\alpha(y)) \mid \alpha \in \Lambda\} \\
\geq T(\sup \{\mu_\alpha(x \ast y) \mid \alpha \in \Lambda\}, \sup \{\mu_\alpha(y) \mid \alpha \in \Lambda\}) \\
= T((\vee_{\alpha \in \Lambda} \mu_\alpha)(x \ast y), (\vee_{\alpha \in \Lambda} \mu_\alpha)(y)).
\]
ON IMAGINABLE $T$-FUZZY SUBALGEBRAS ...

$$(\wedge_{\alpha \in \Lambda} \mu_{\alpha})(x) = \inf \{\mu_{\alpha}(x) \mid \alpha \in \Lambda\}$$

$$\geq \inf \{T(\mu_{\alpha}(x \ast y), \mu_{\alpha}(y)) \mid \alpha \in \Lambda\}$$

$$\geq T(\inf \{\mu_{\alpha}(x \ast y) \mid \alpha \in \Lambda\}, \inf \{\mu_{\alpha}(y) \mid \alpha \in \Lambda\})$$

$$= T((\wedge_{\alpha \in \Lambda} \mu_{\alpha})(x \ast y), (\wedge_{\alpha \in \Lambda} \mu_{\alpha})(y)).$$

(4.50)

Hence $\vee_{\alpha \in \Lambda} \mu_{\alpha}$ and $\wedge_{\alpha \in \Lambda} \mu_{\alpha}$ are $T$-fuzzy closed ideals of $X$, completing the proof. $\square$

5. Conclusions and future works. We inquired into further properties on fuzzy closed ideals in BCH-algebras, and using a $t$-norm $T$, we introduced the notion of (imaginable) $T$-fuzzy subalgebras and (imaginable) $T$-fuzzy closed ideals, and obtained some related results. Moreover, we discussed the direct product and $T$-product of $T$-fuzzy subalgebras. We finally showed that the family of $T$-fuzzy closed ideals is a completely distributive lattice. These ideas enable us to define the notion of (imaginable) $T$-fuzzy filters in BCH-algebras, and to discuss the direct products and $T$-products of $T$-fuzzy filters. It also gives us possible problems to discuss relations among $T$-fuzzy subalgebras, $T$-fuzzy closed ideals and $T$-fuzzy filters, and to construct the normalizations. We may also use these ideas to introduce the notion of interval-valued fuzzy subalgebras/closed ideals.

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REFERENCES


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